Graph Properties of Graph Associahedra

Thibault Manneville (LIX, Polytechnique)

joint work with **Vincent Pilaud** (CNRS)

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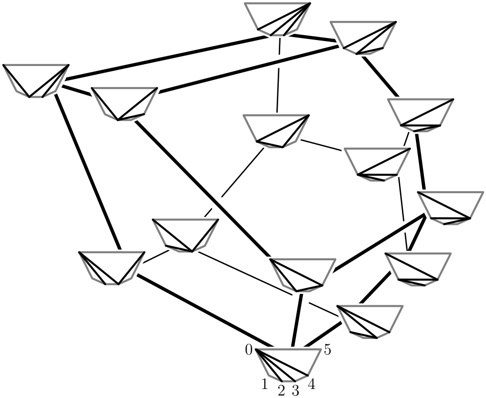
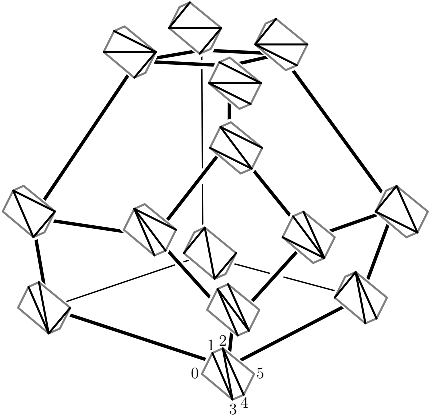
Definition

An ***associahedron*** is a polytope whose face lattice is isomorphic to the lattice of dissections of a convex polygon.



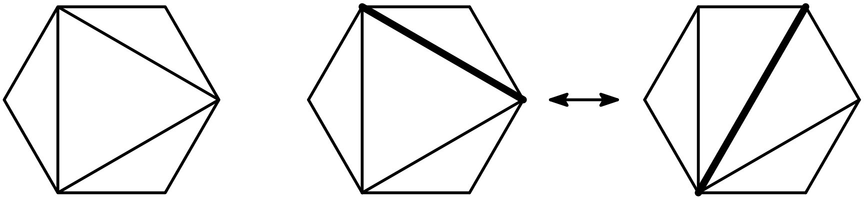


Focus on graphs

Flip graph on the triangulations of the polygon:

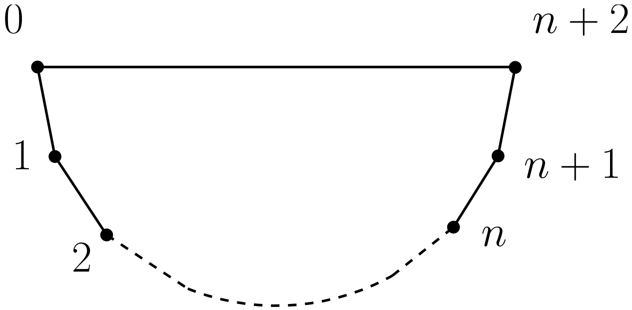
Vertices: *triangulations*

Edges: *flips*



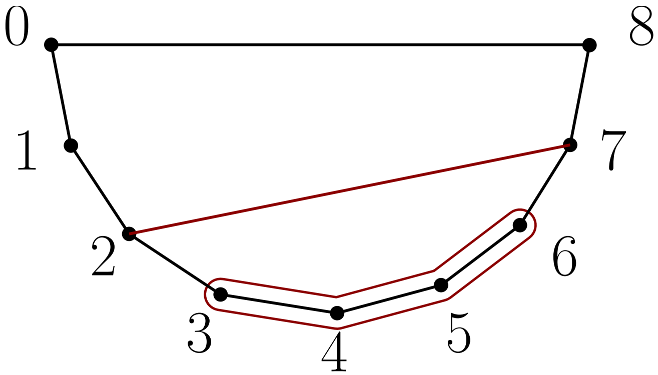
Useful configuration (Loday's)

$$G_{n+3} =$$



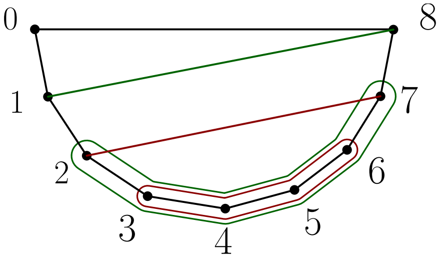
Graph point of view

$$\{\text{diagonals of } G_{n+3}\} \longleftrightarrow \{\text{subpaths of the path } \{1, \ldots, n+1\}\}$$

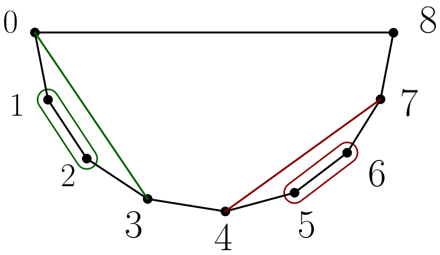


Non-crossing diagonals

Two ways to be non-crossing in Loday's configuration:



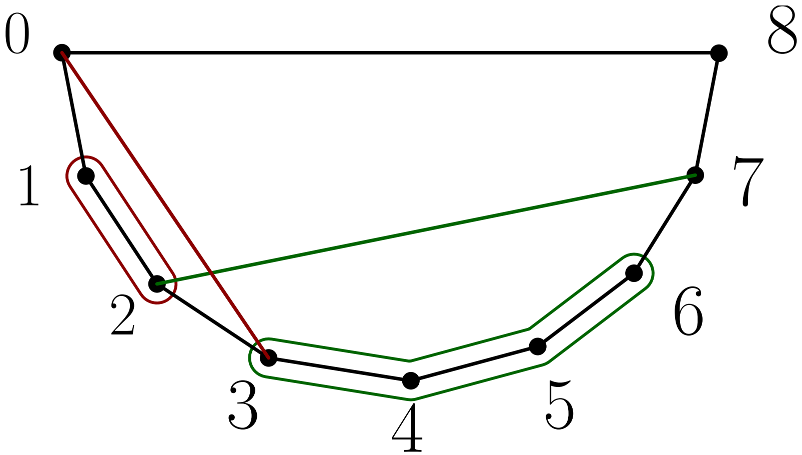
nested subpaths



non-adjacent subpaths

Pay attention to the second case:

The right condition is indeed *non-adjacent*, disjoint is not enough!



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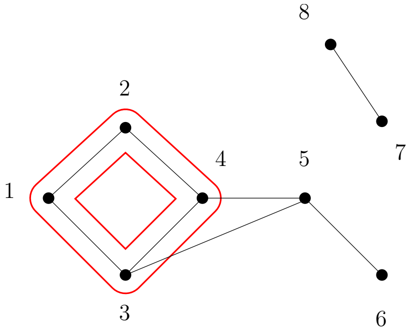
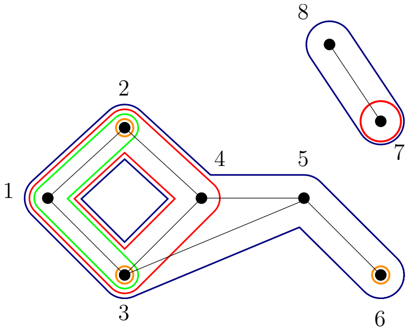
Definition

- A ***tube*** of G is a proper subset $t \subseteq V$ inducing a connected subgraph of G;
- t and t' are ***compatible*** if they are nested or non-adjacent;
- A ***tubing*** of G is a set of pairwise compatible tube of G.



A tube

(generalizes a diagonal)



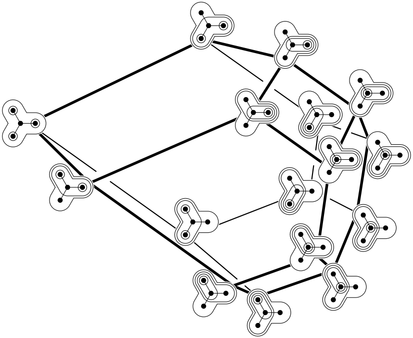
A maximal tubing

(generalizes a triangulation)

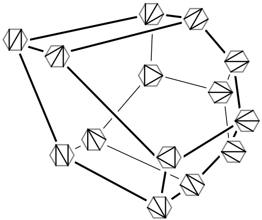
Graph associahedra

Theorem (Carr and Devadoss '06)

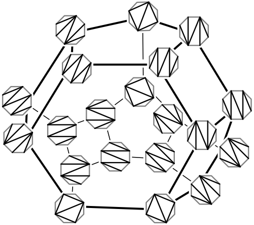
The simplicial complex of tubings of G can be realized as the face lattice of a polytope. Such a polytope is called a ***graph associahedron*** *of G and is denoted* $\mathbf{Asso}_G$.



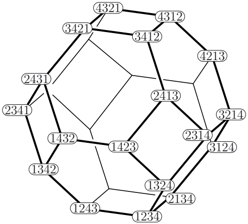
Classical polytopes...



The associahedron

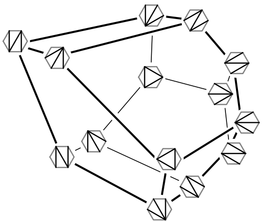


The cyclohedron



The permutahedron

...can be seen as graph associahedra



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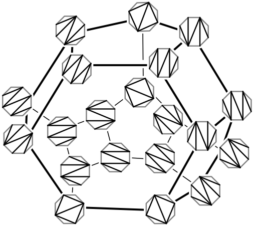

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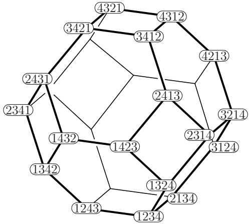

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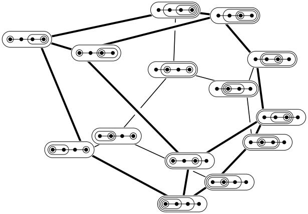

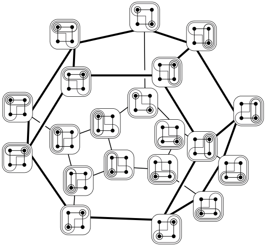

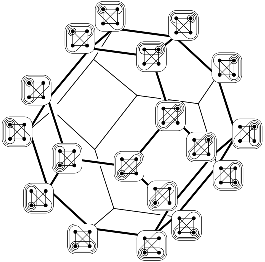

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Theorem (Pournin '12)

The diameter of the n-dimensional associahedron is $2n - 4$ *for* $n \geq 10$.

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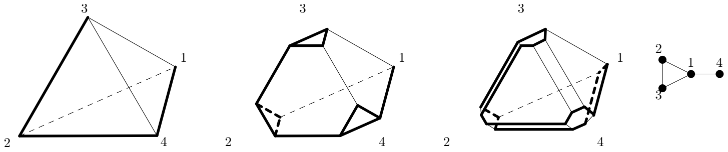

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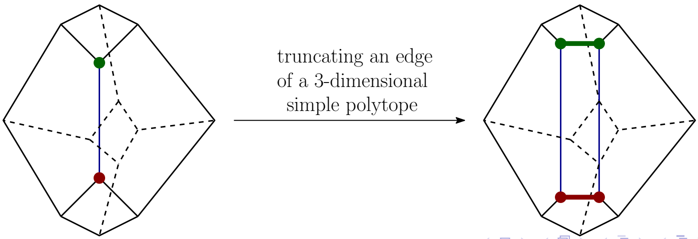
→ Truncating $\iff$ replacing vertices by complete graphs.



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THANK YOU FOR

YOUR ENTHUSIASTIC

ATTENTION !