Bijection for plane graphs

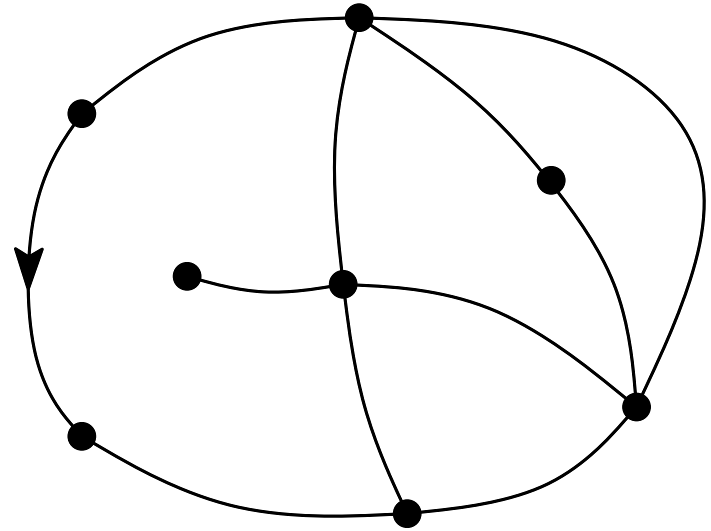
Olivier Bernardi, **Gwendal Collet**, Éric Fusy

LIX – december 17^{th} 2013

Bijection for plane graphs

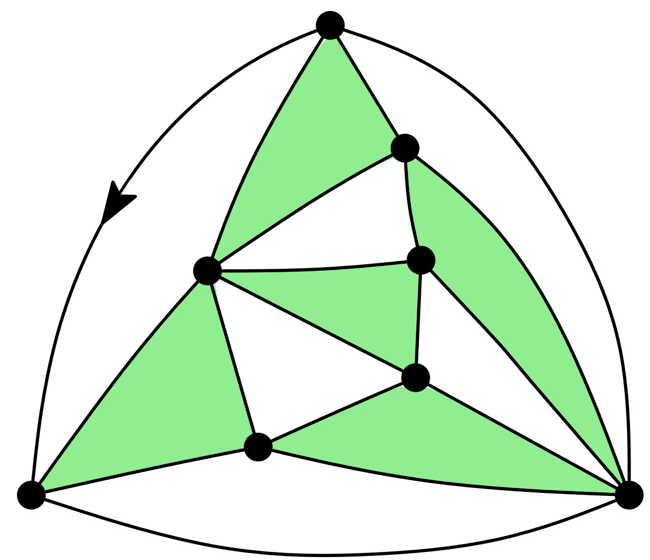
Rooted plane graph:

- no loops
- no multiple edges
- $M(z)$ generating series according edges



Rooted eulerian triangulation:

- vertices have even degree
- green and white triangles
- $T(z)$ generating series according green triangles

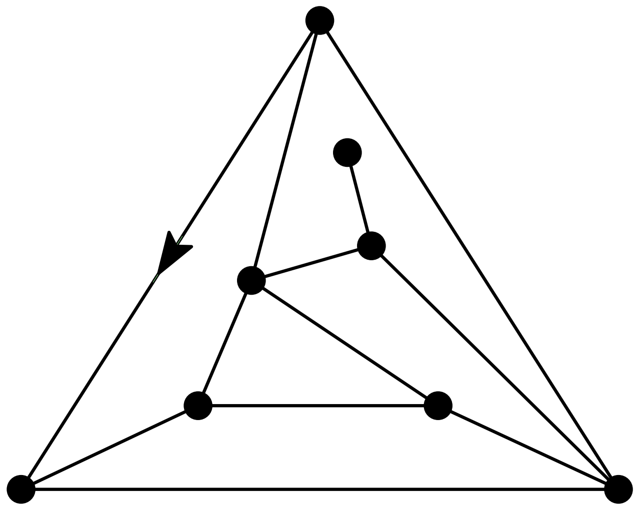


[M. Noy] $M(z) = \frac{z(1+T(z)/z^2)}{1-z(1+T(z)/z^2)}$

Bijection for plane graphs

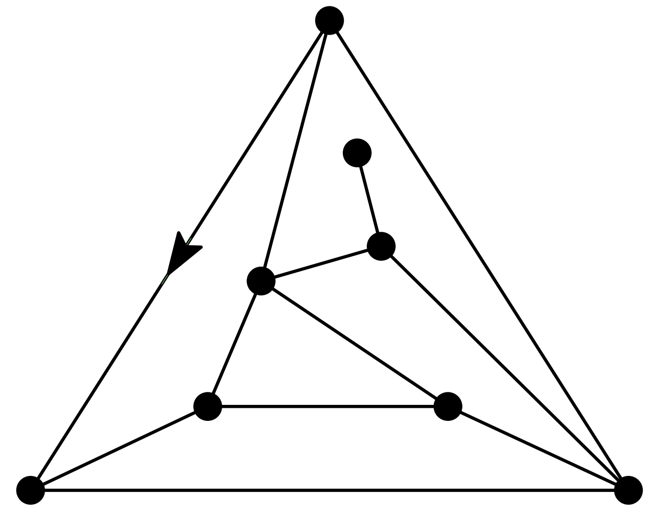
Outertriangular
rooted plane graph

$C(z)$ generating series
according edges



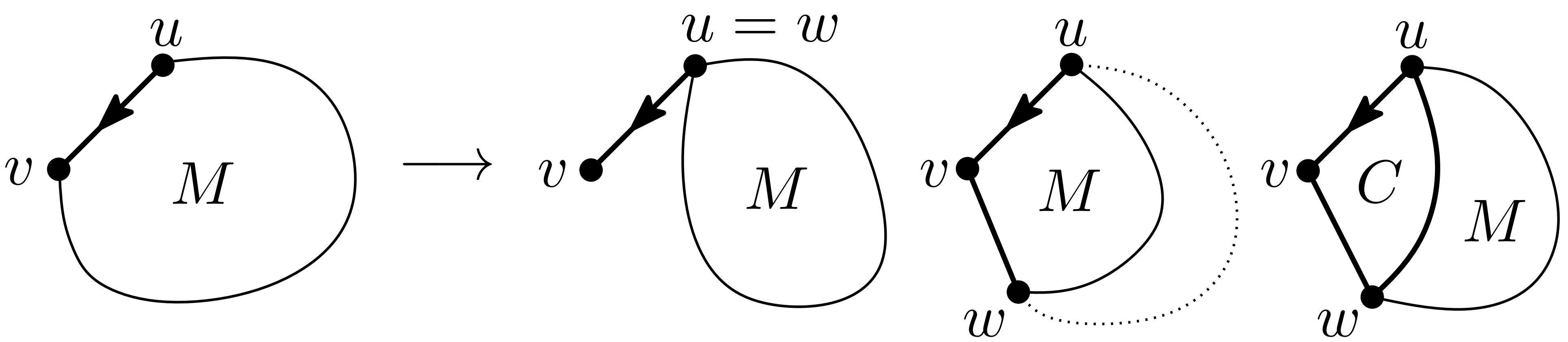
Bijection for plane graphs

Outertriangular
rooted plane graph
$C(z)$ generating series according edges



Plane graph:

3 cases:

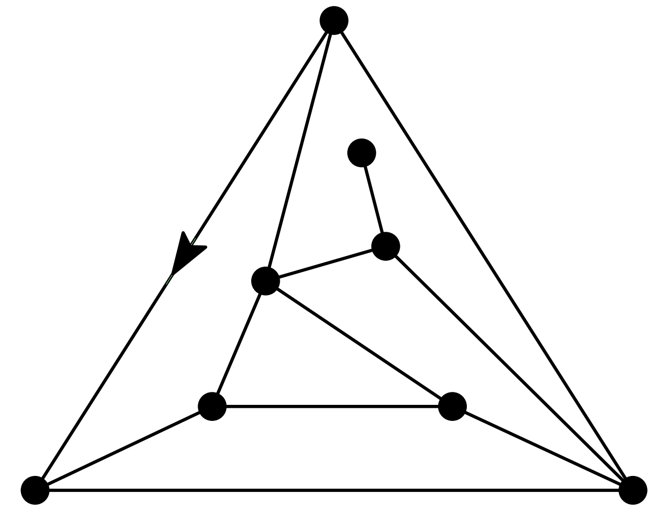


Bijection for plane graphs

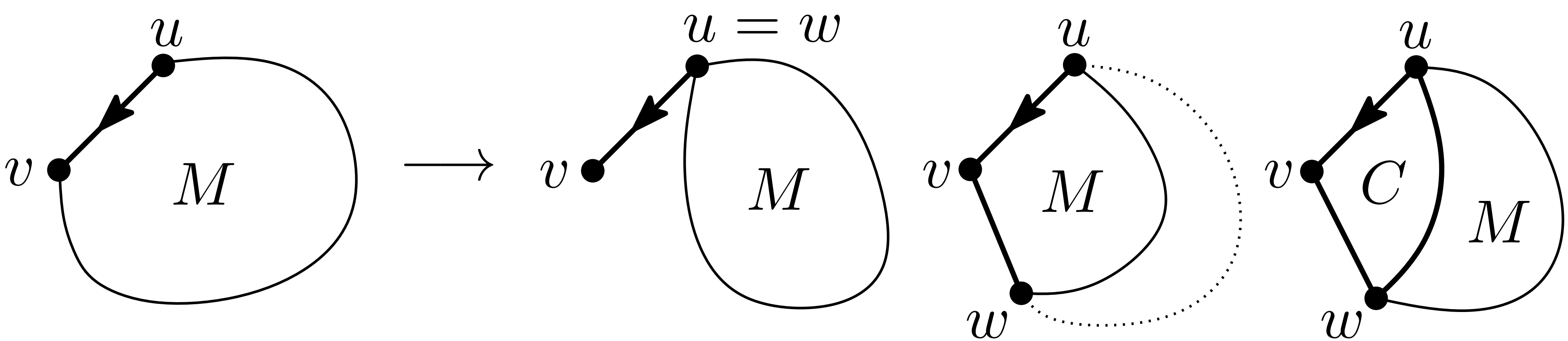
Outertriangular

rooted plane graph

$C(z)$ generating series according edges



Plane graph: 3 cases:

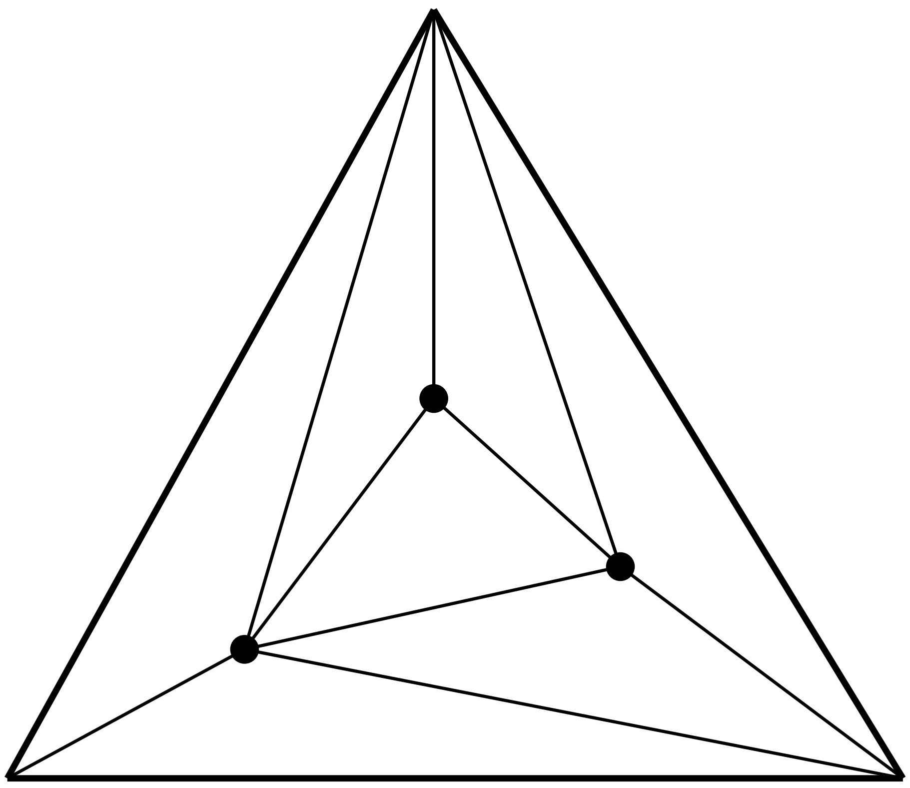


$$\Rightarrow M(z) \quad = \quad z(1 + M(z)) \quad + \quad \frac{C(z)}{z} \quad + \quad \frac{C(z) \cdot M(z)}{z}$$

$$\Rightarrow M(z) = \frac{zB(z)}{1 - zB(z)} \text{ where } B(z) = 1 + \frac{C(z)}{z^2}$$

Bijection for plane graphs

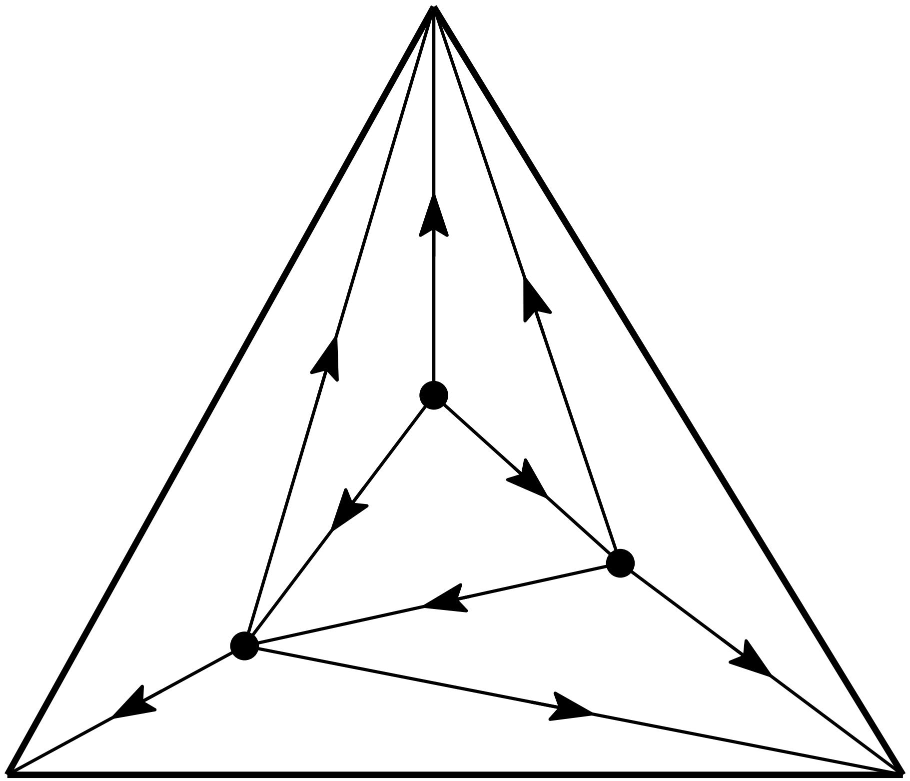
From simple triangulations to eulerian triangulations



Bijection for plane graphs

From simple triangulations to eulerian triangulations

[Schnyder'89, Brehm'02]

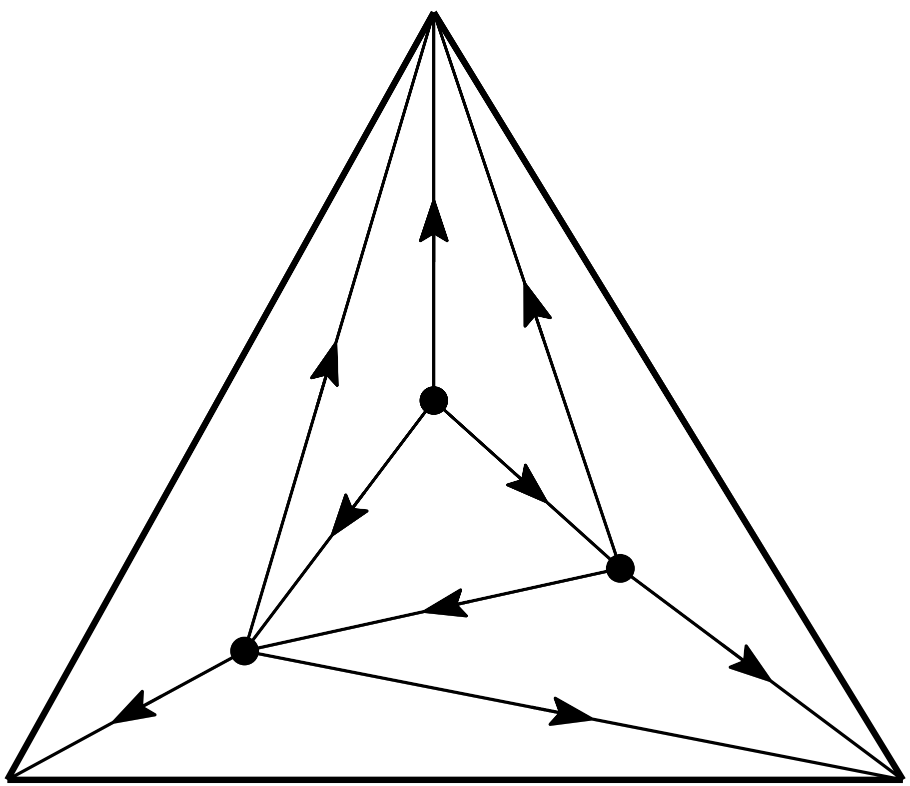


Canonical orientation:

3 outgoing edges at vertices
outer-accessibility
no clockwise circuit

Bijection for plane graphs

From simple triangulations to eulerian triangulations



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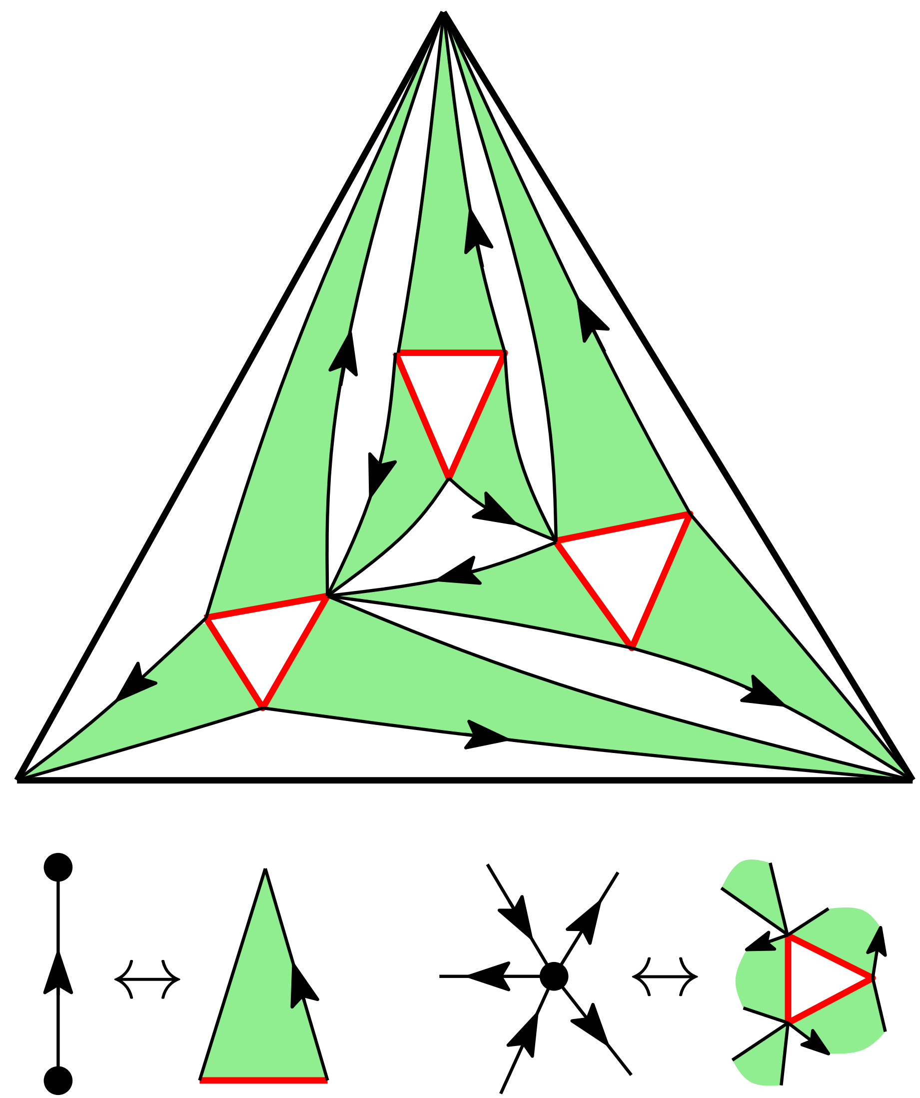

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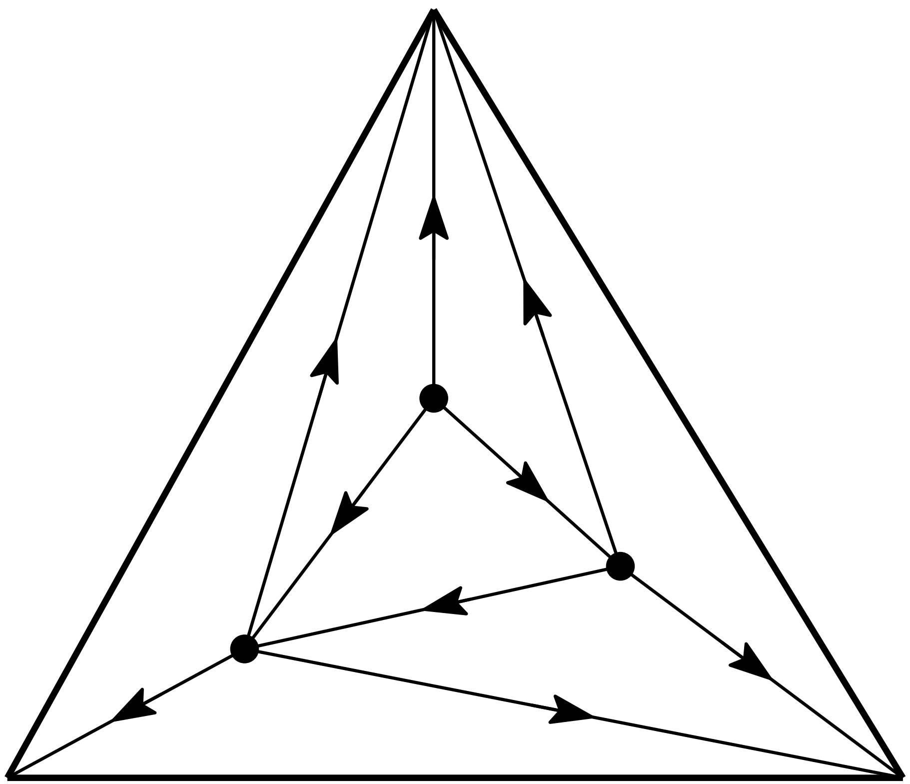

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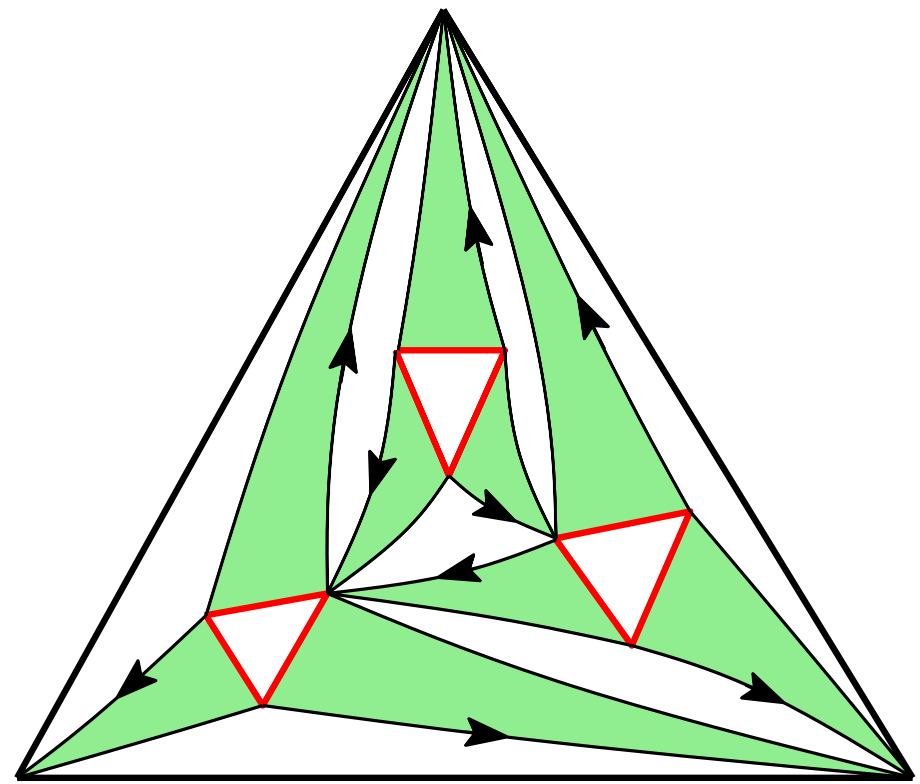
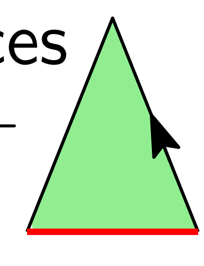


Canonical orientation:

1 outgoing edge at vertices
outer-accessibility +
no clockwise circuit



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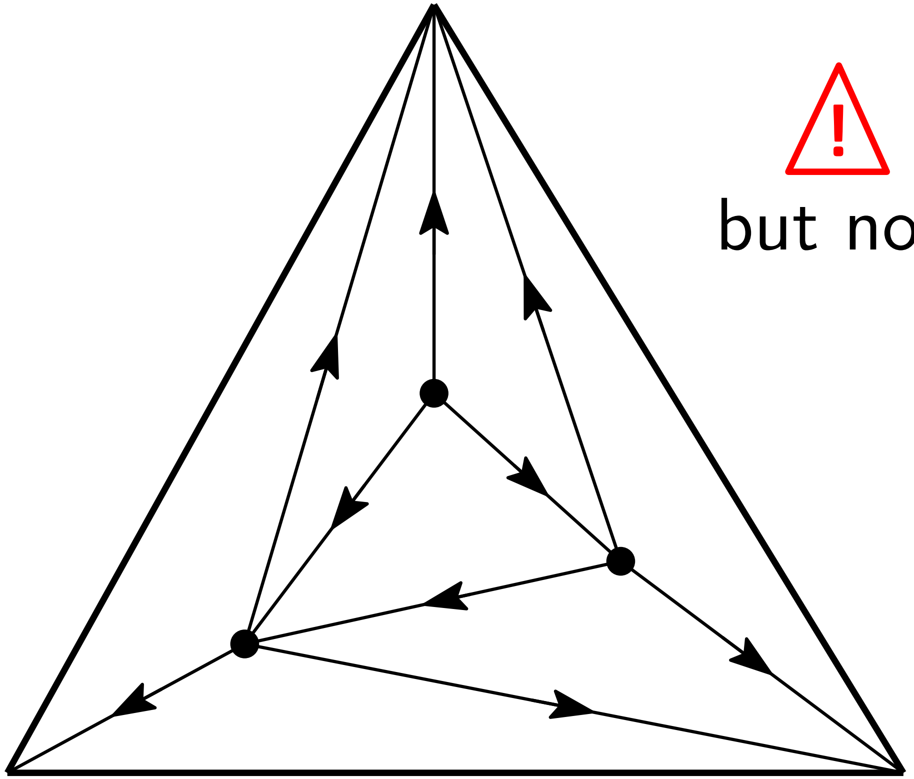
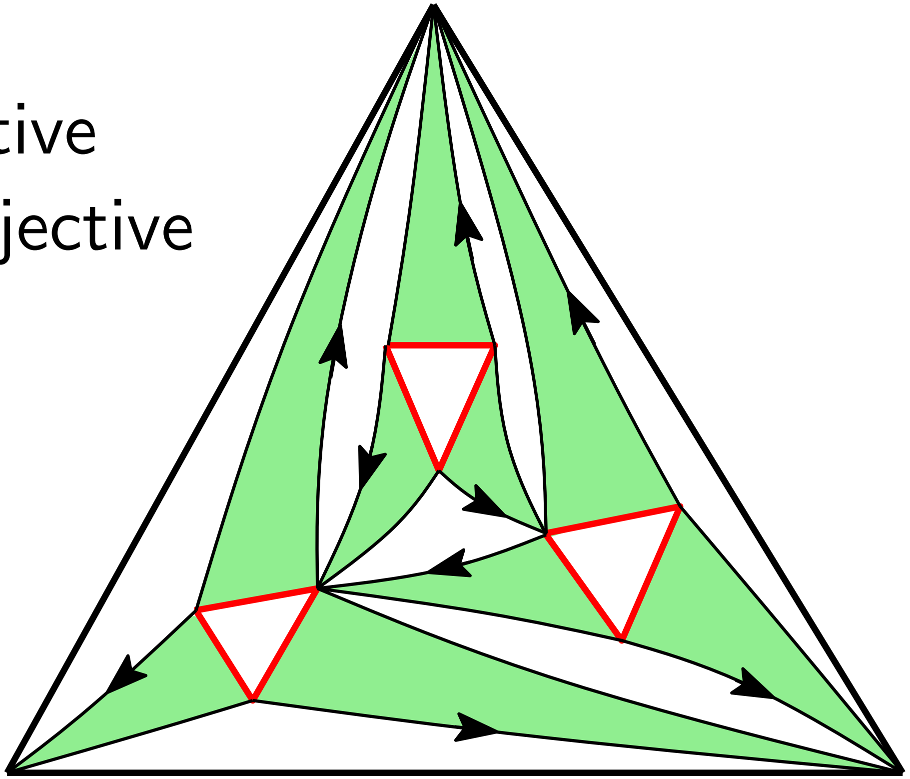


Injective
but not surjective



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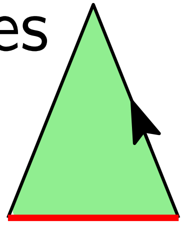
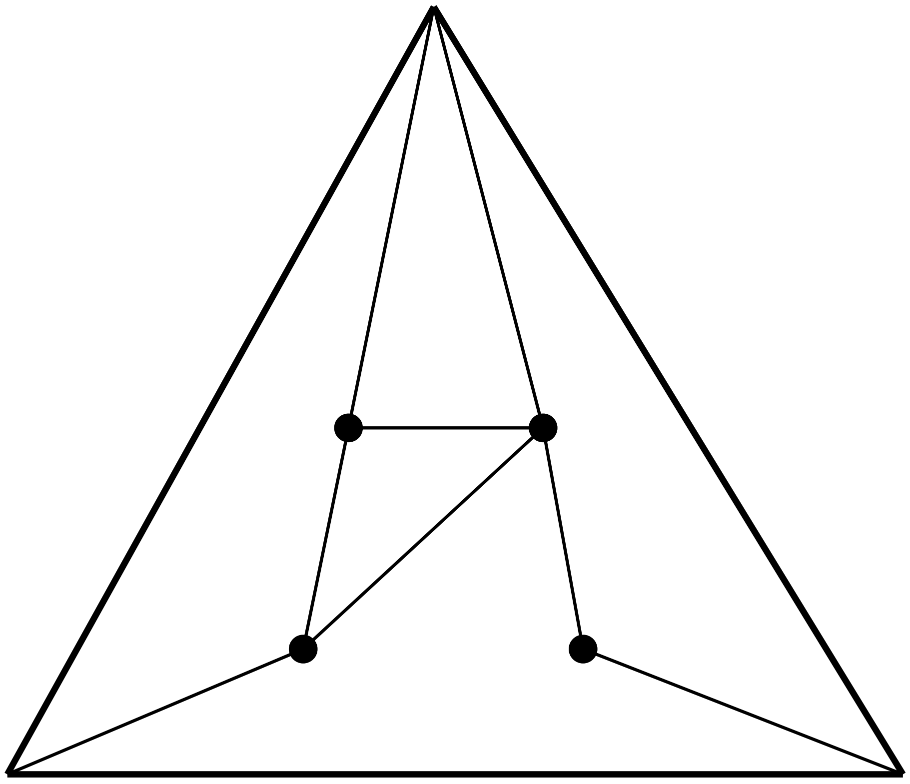
+ 

Bijection for plane graphs

From outer-triangular plane graphs to eulerian triangulations



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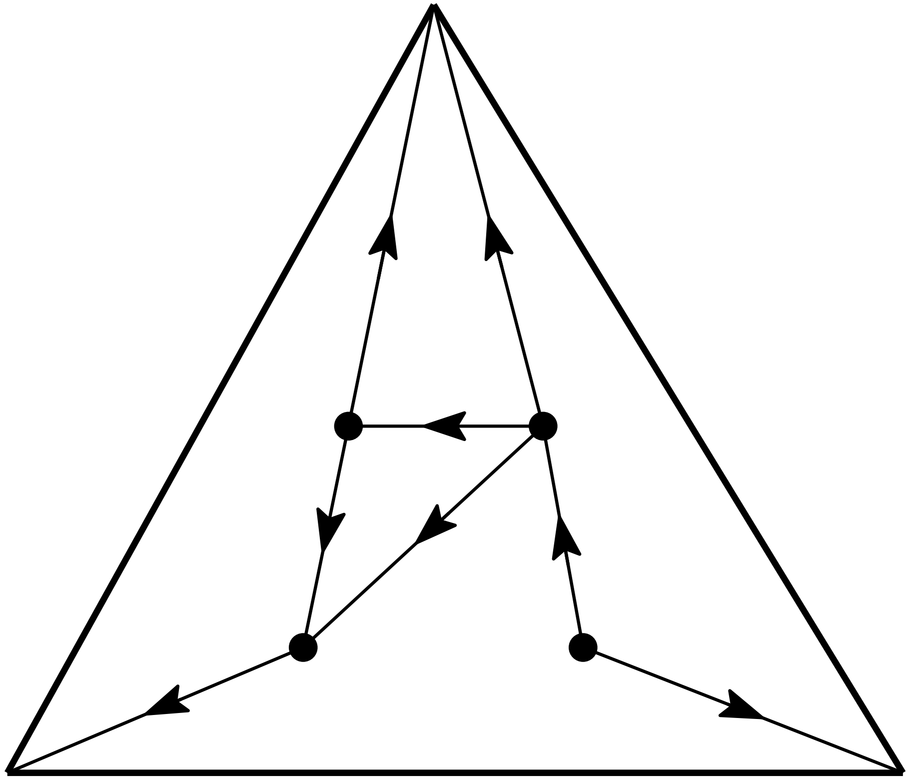
From outer-triangular plane graphs to eulerian triangulations

[Bernardi, Fusy]



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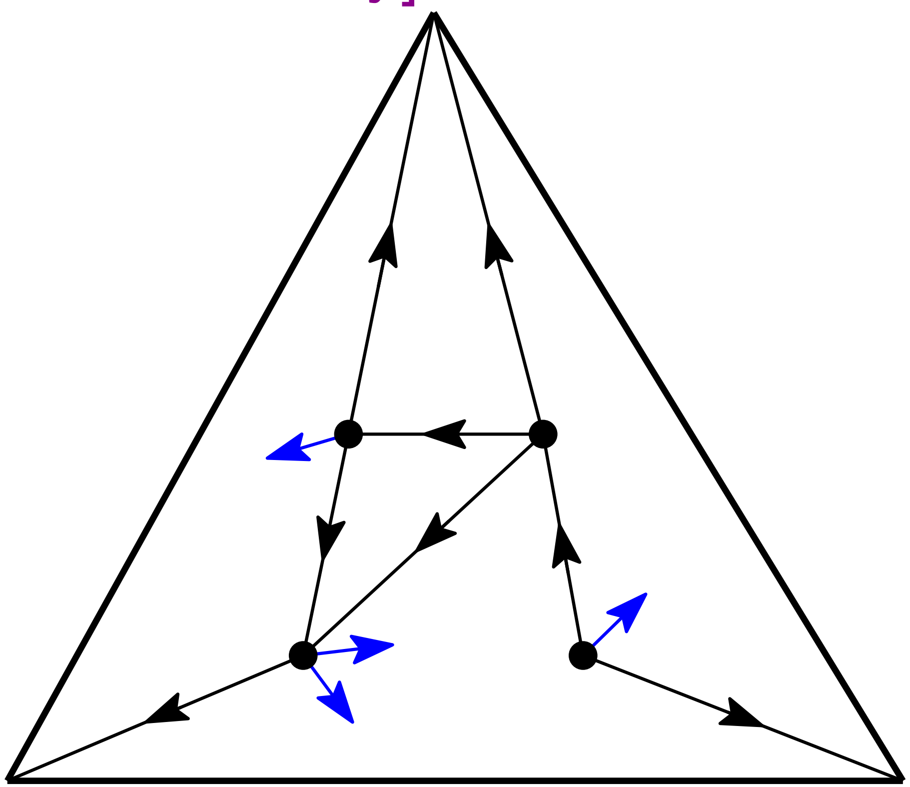

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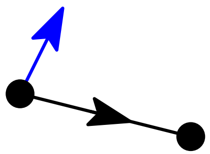
+



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From outer-triangular plane graphs to eulerian triangulations

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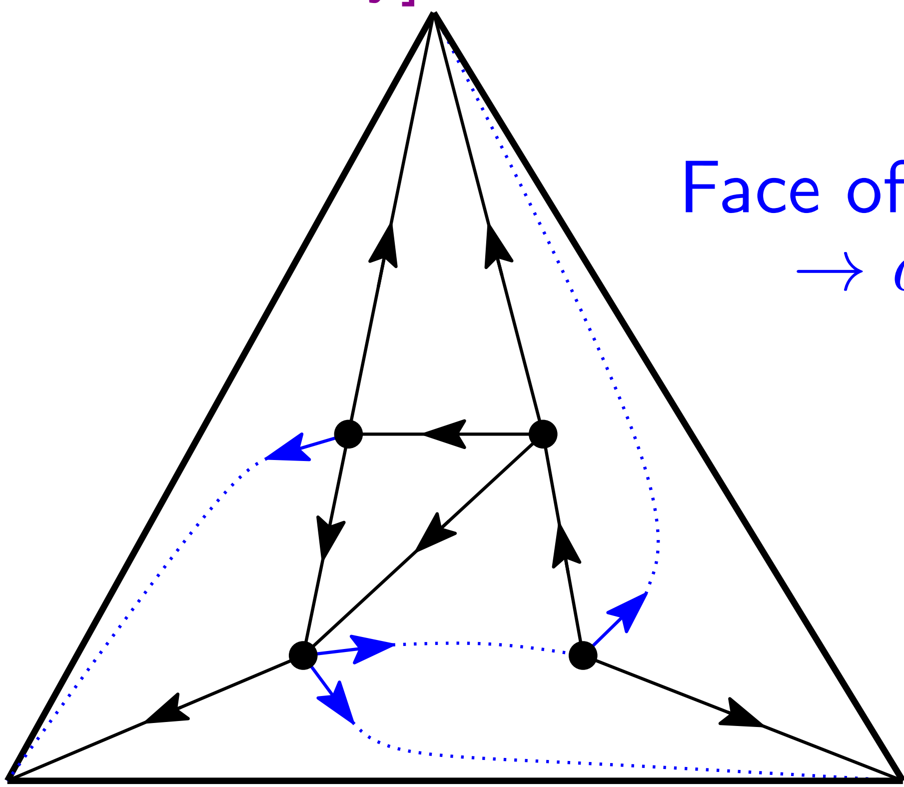

Face of degree $3 + d$
$\to d$ arrows

Canonical orientation:

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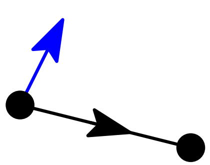
$+$ 

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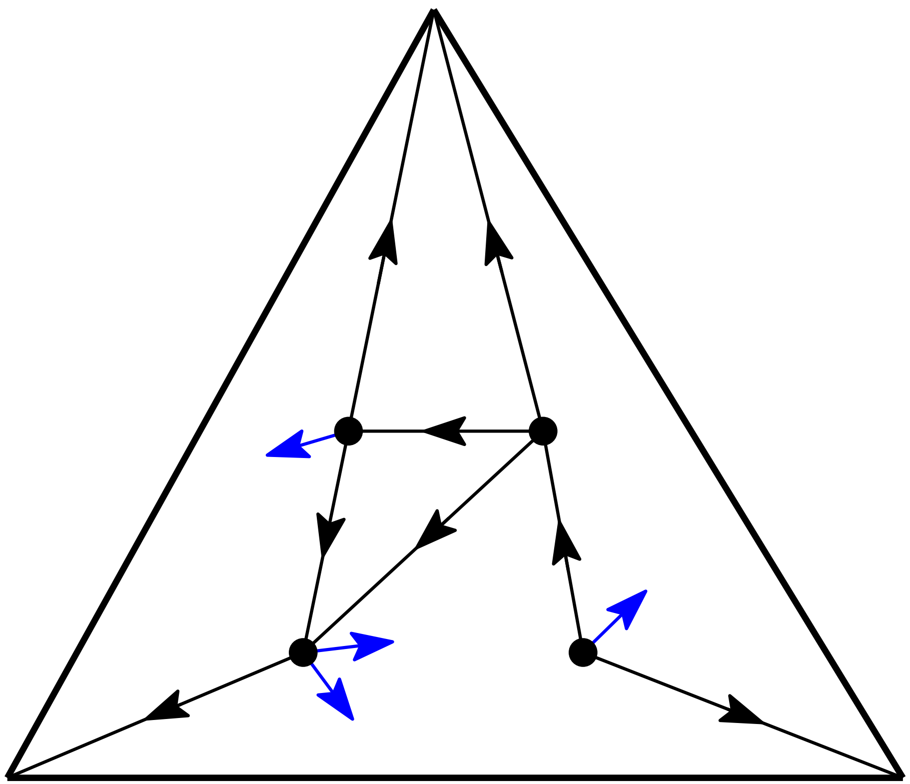

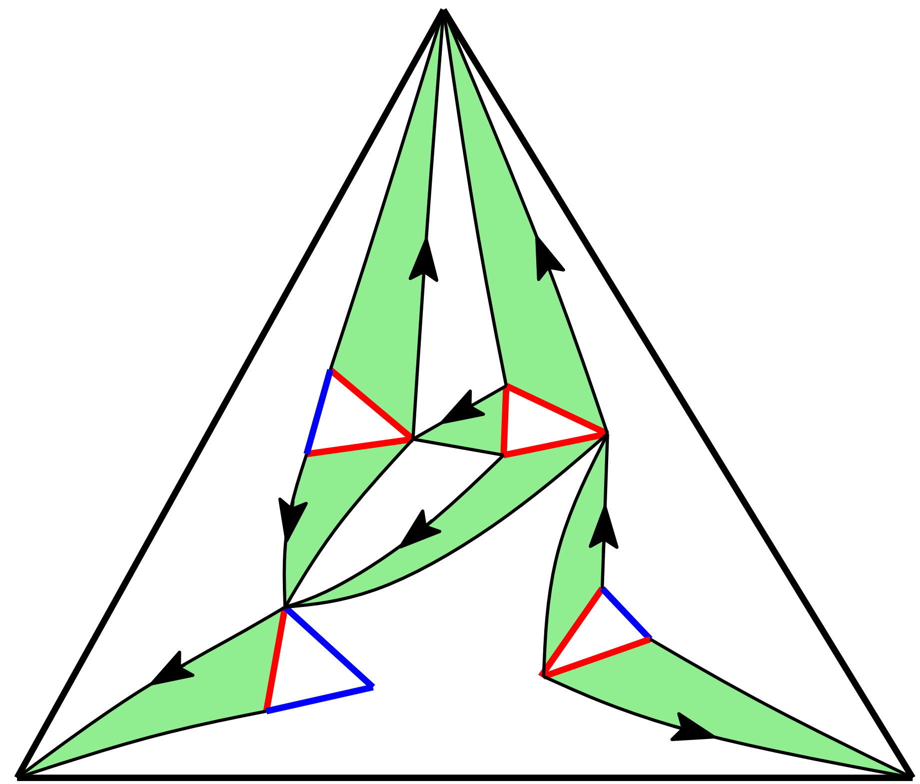

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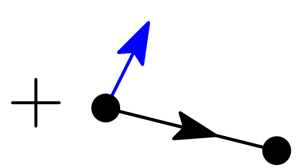

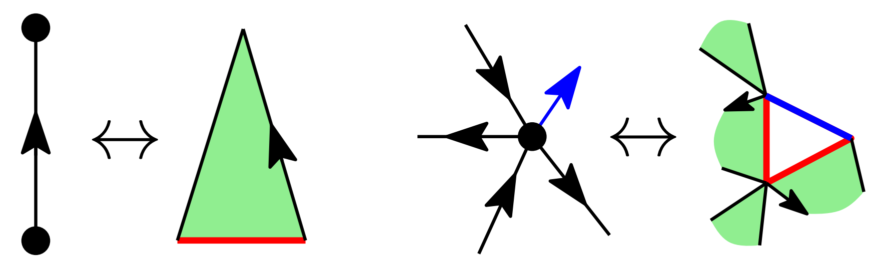

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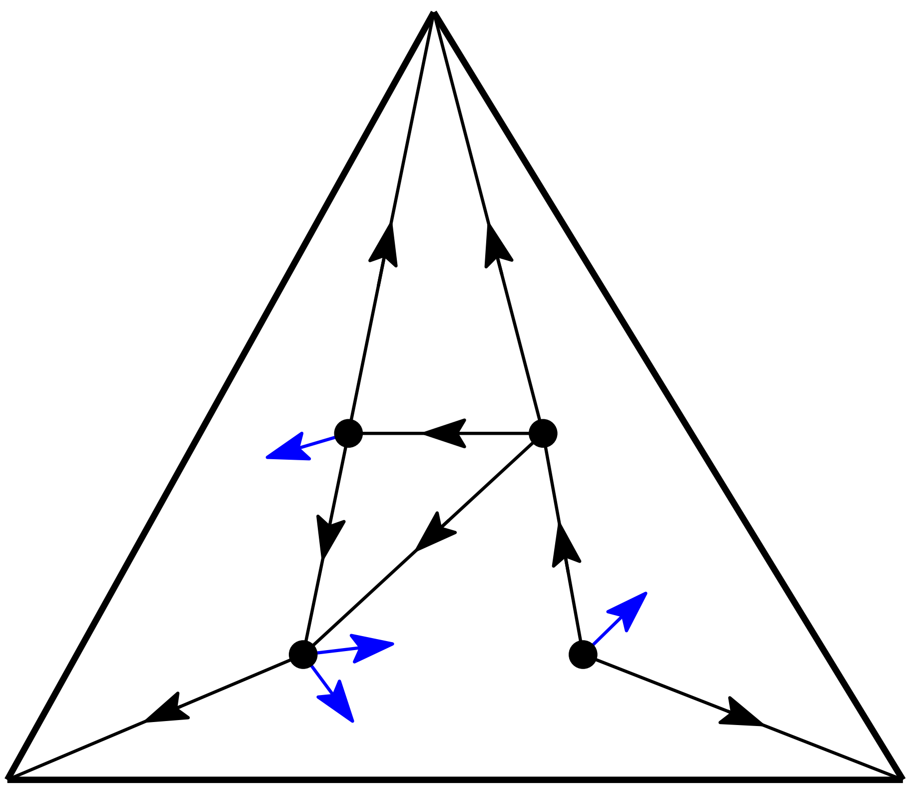

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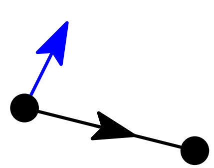
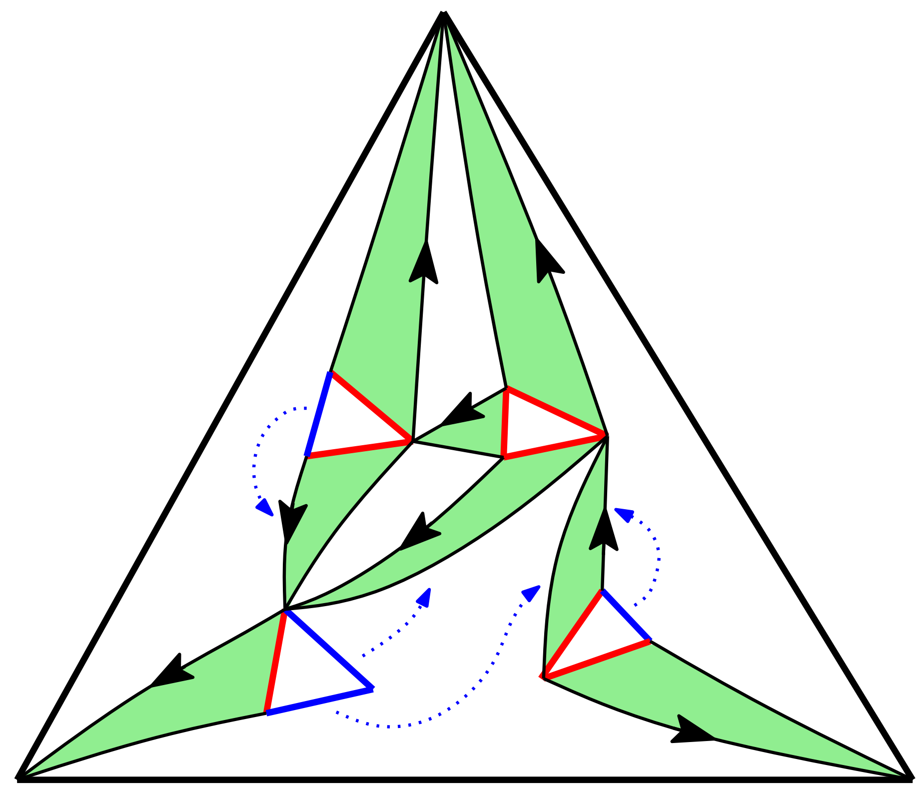
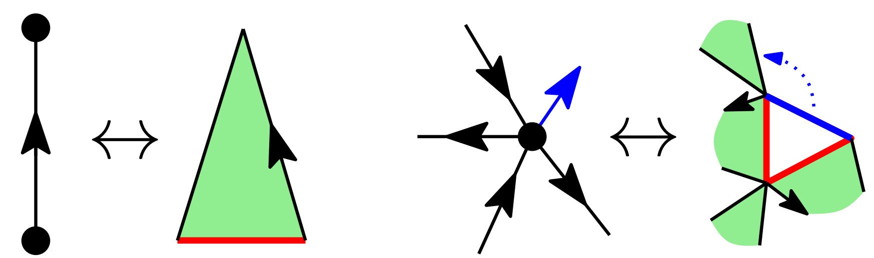
\+ 





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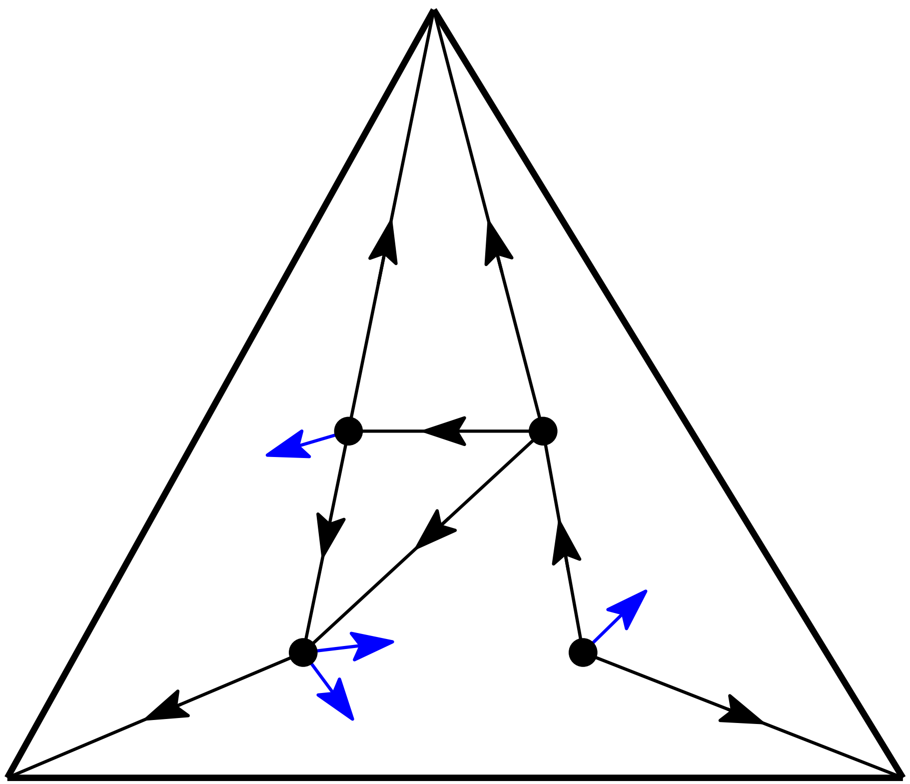

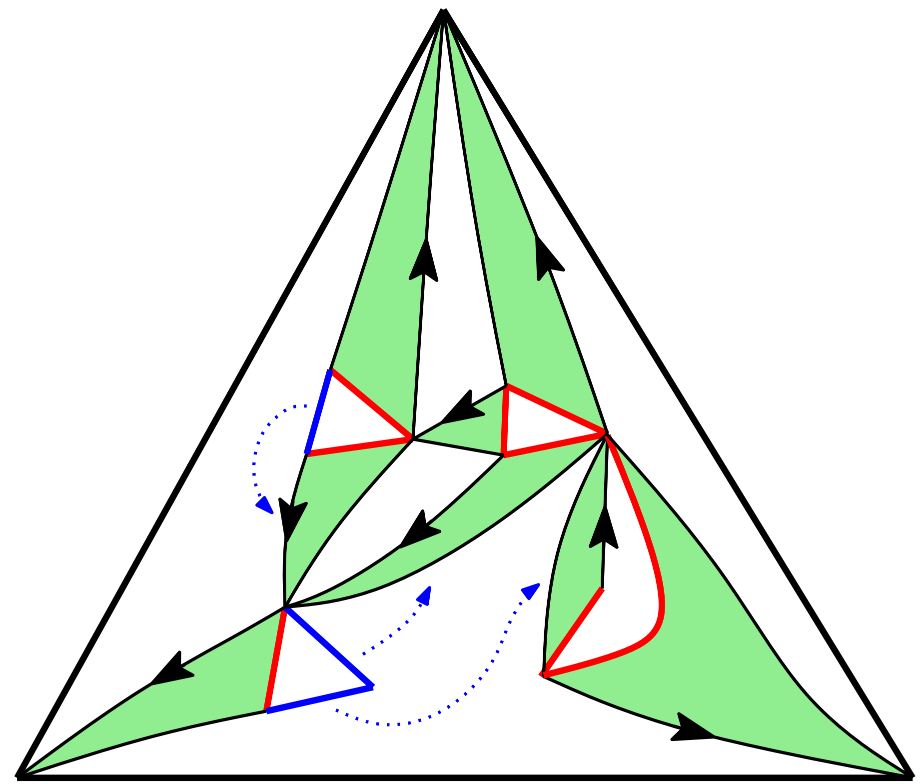

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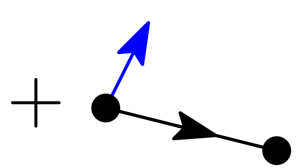

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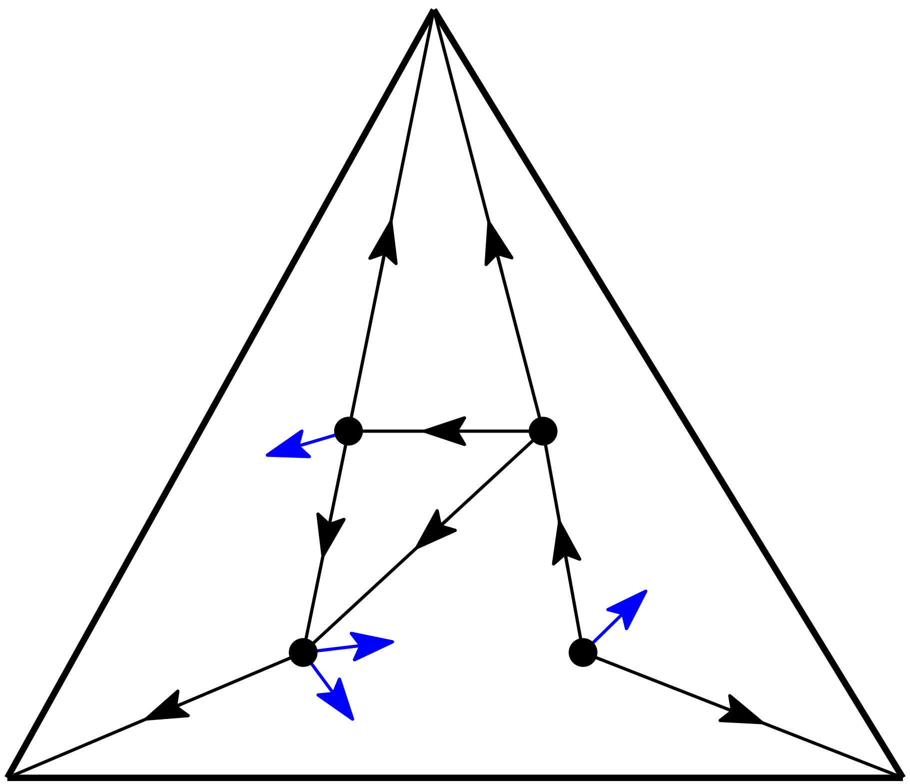

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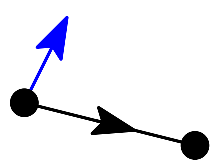
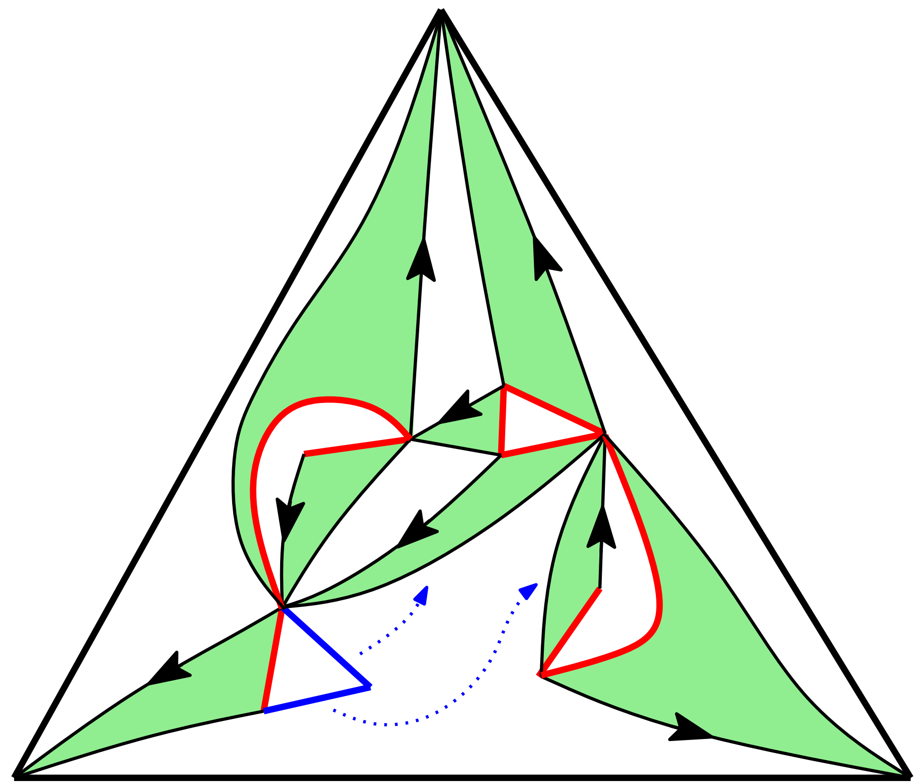
\+ 



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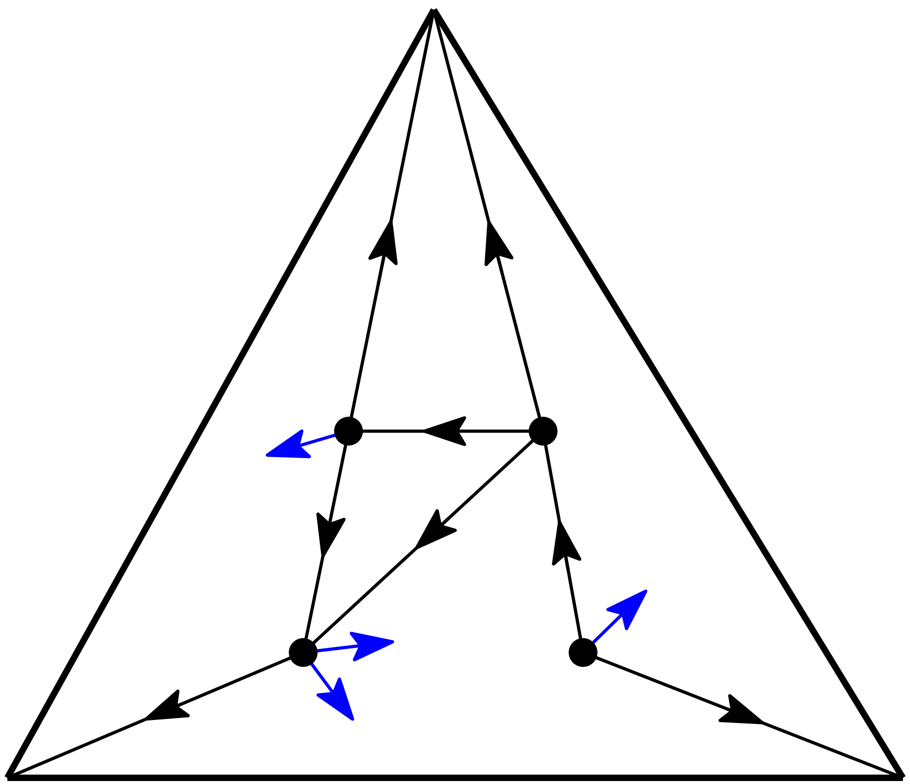

Canonical orientation:

3 outgoing edges

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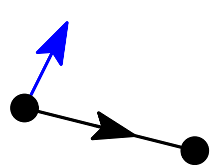
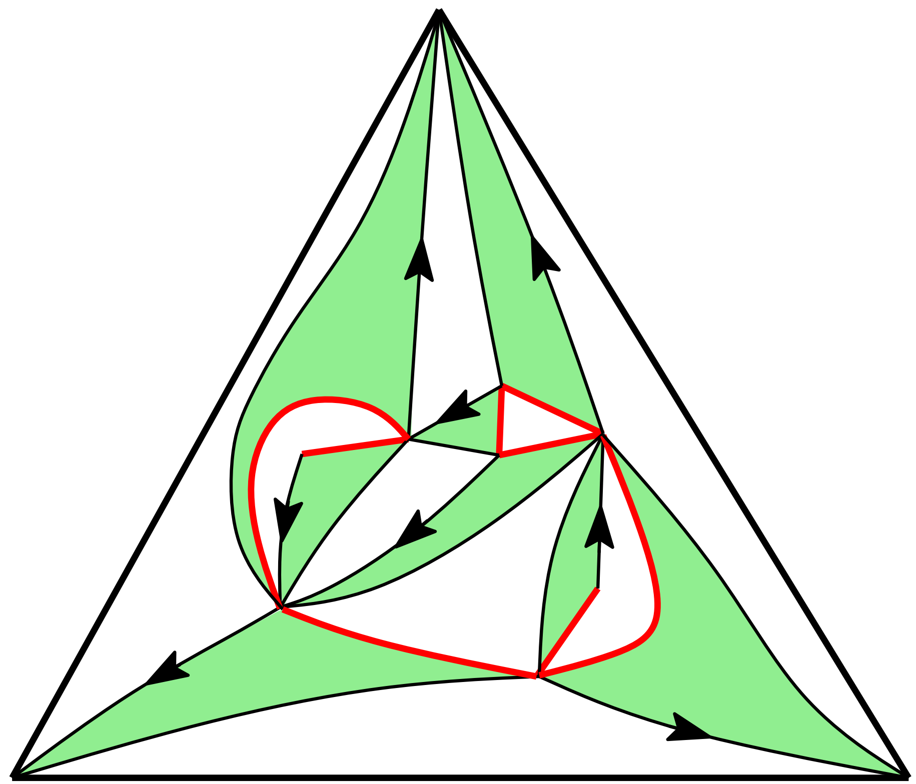
no clockwise circuit

$+$ 



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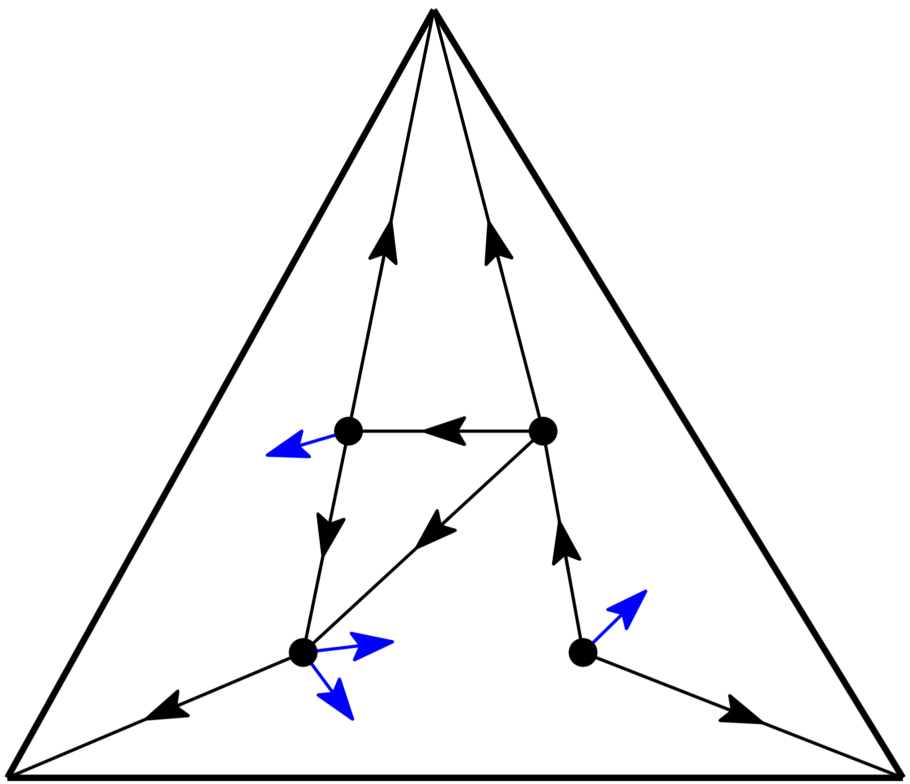

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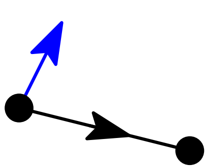
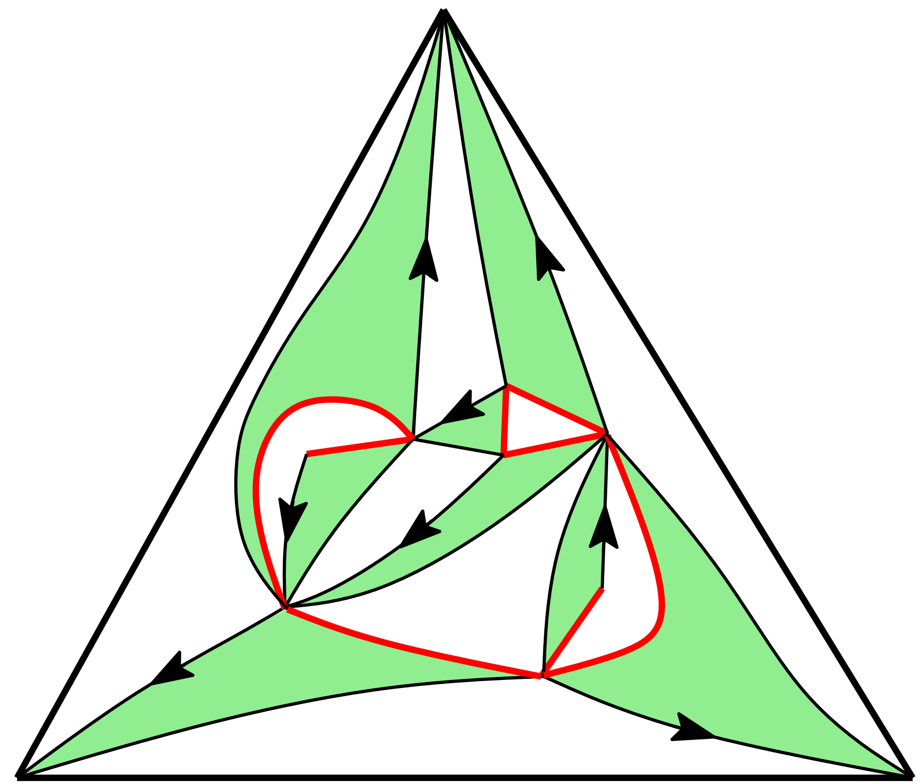
\+ 



Canonical orientation:

1 outgoing edge
outer-accessibility
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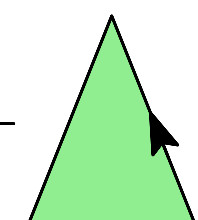
\+ 

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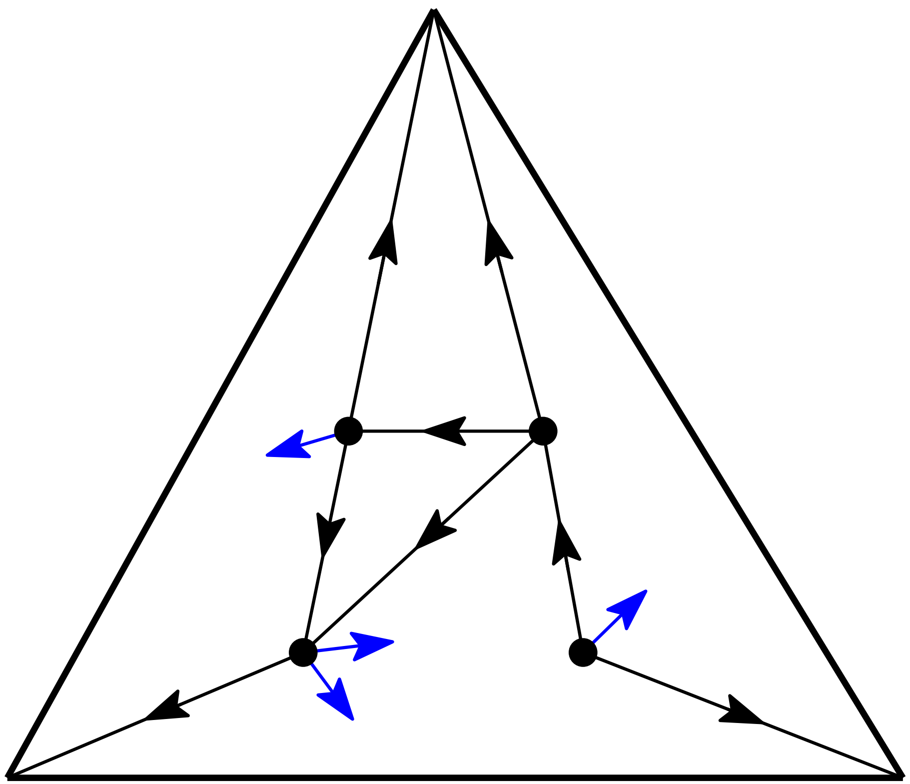
From outer-triangular plane graphs to eulerian triangulations



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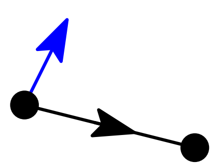
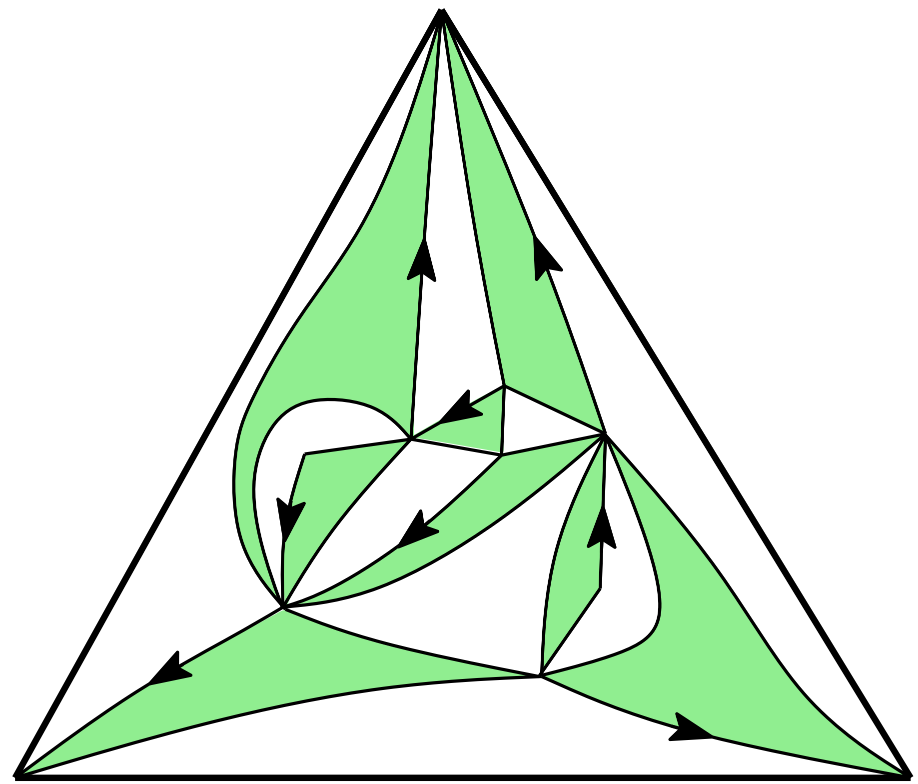
\+ 



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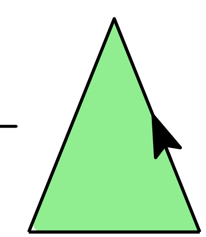
\+ 

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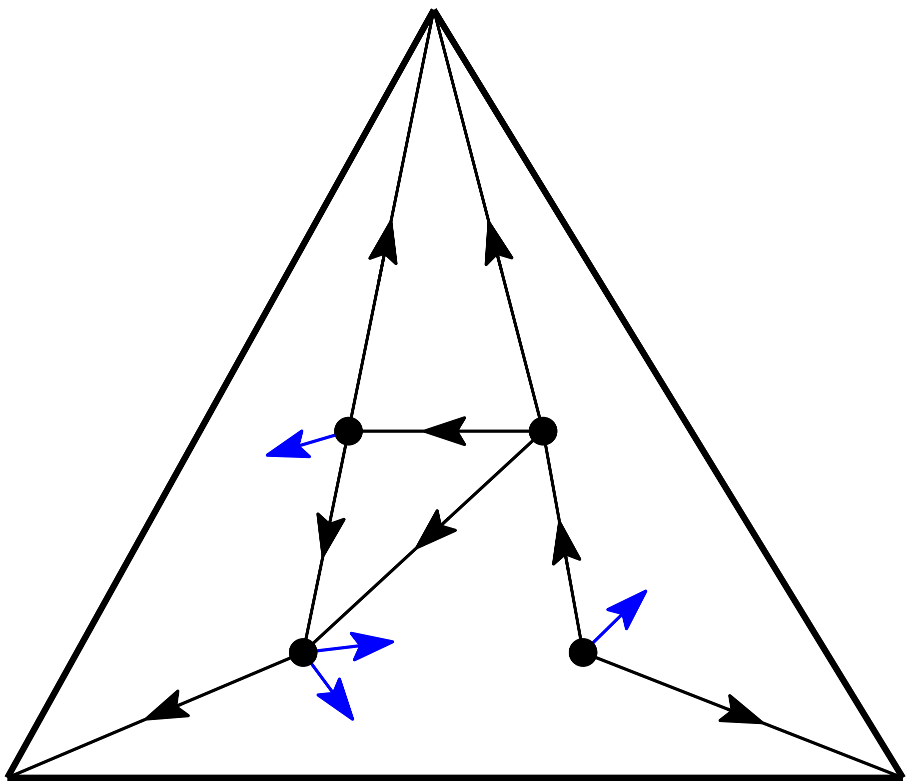
From outer-triangular plane graphs to eulerian triangulations



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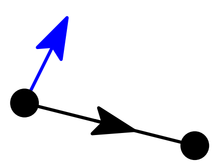
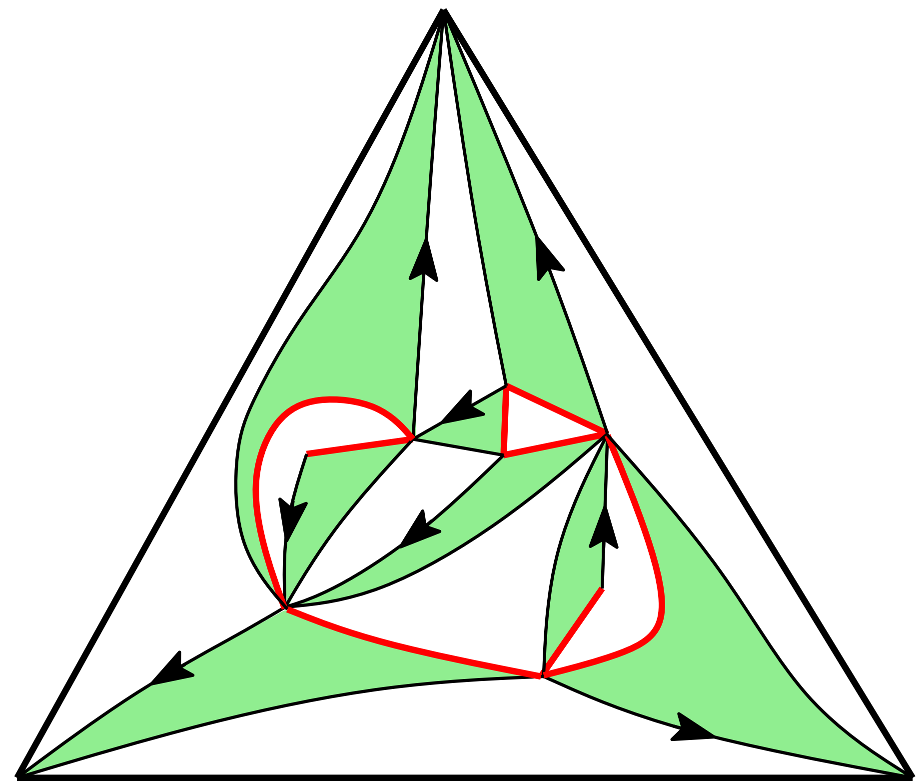
+ 



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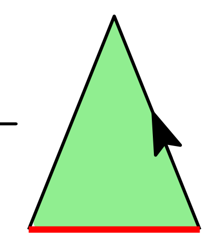
+ 

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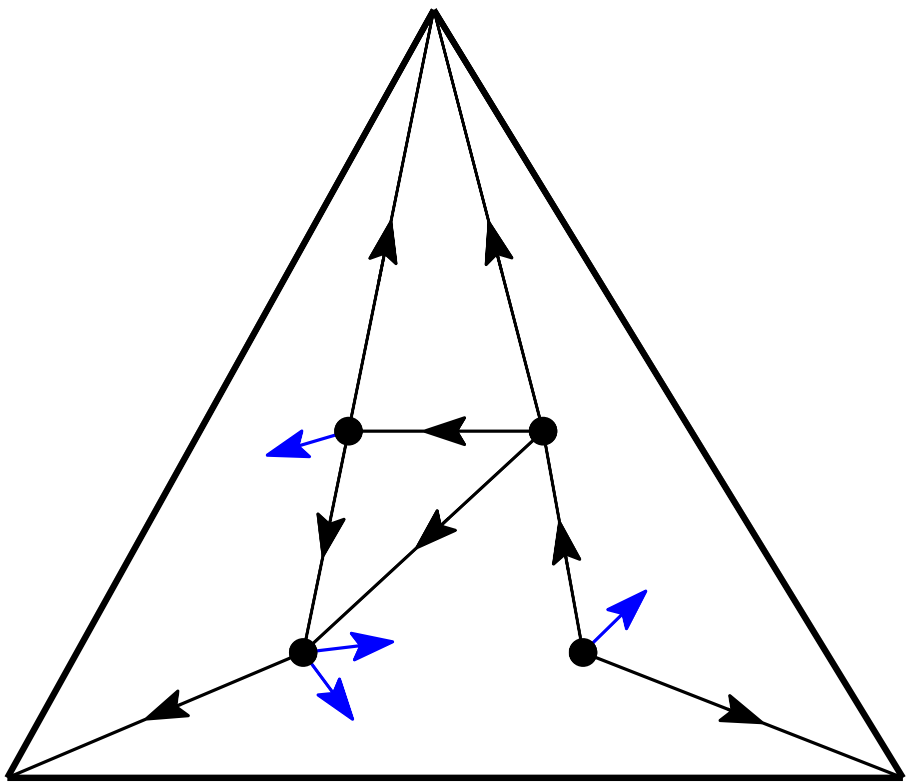

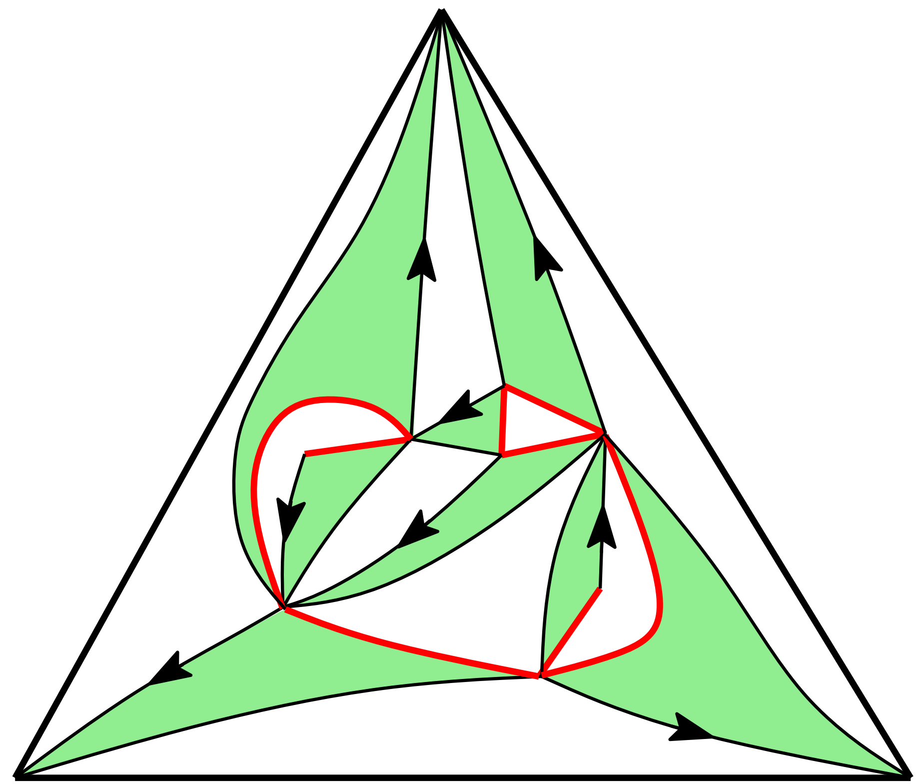

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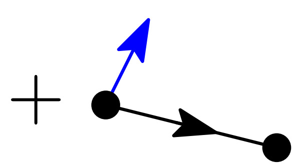
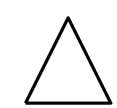
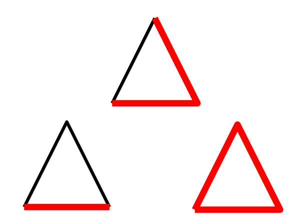


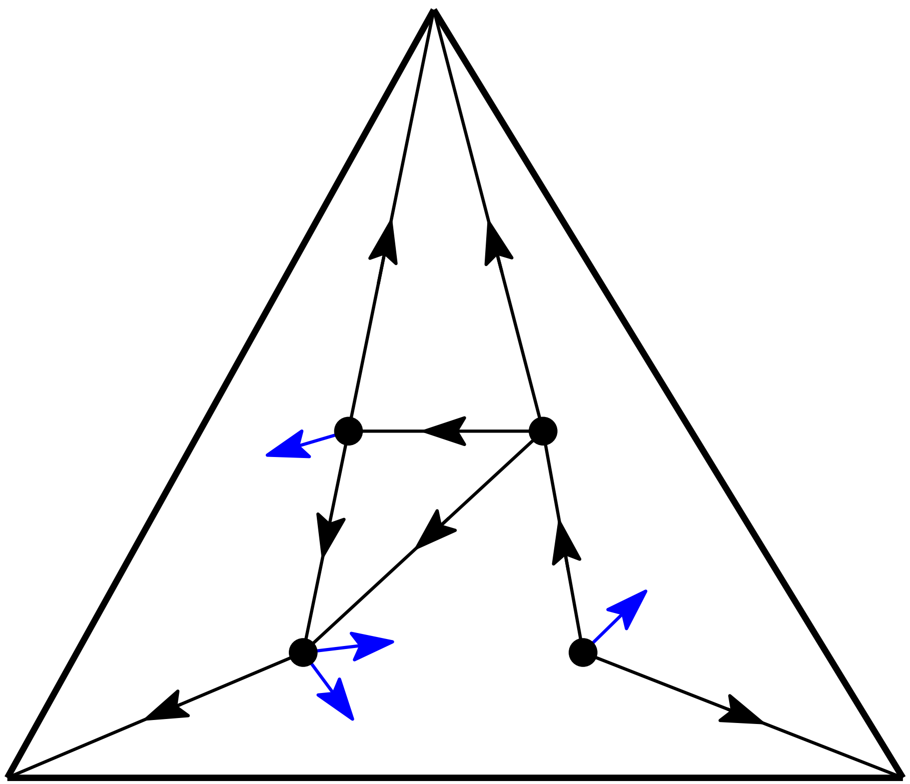
Face of the plane graph: 

Inner node of:
the plane graph



Bijection for plane graphs

From outer-triangular plane graphs to eulerian triangulations



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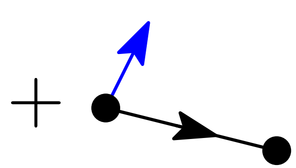

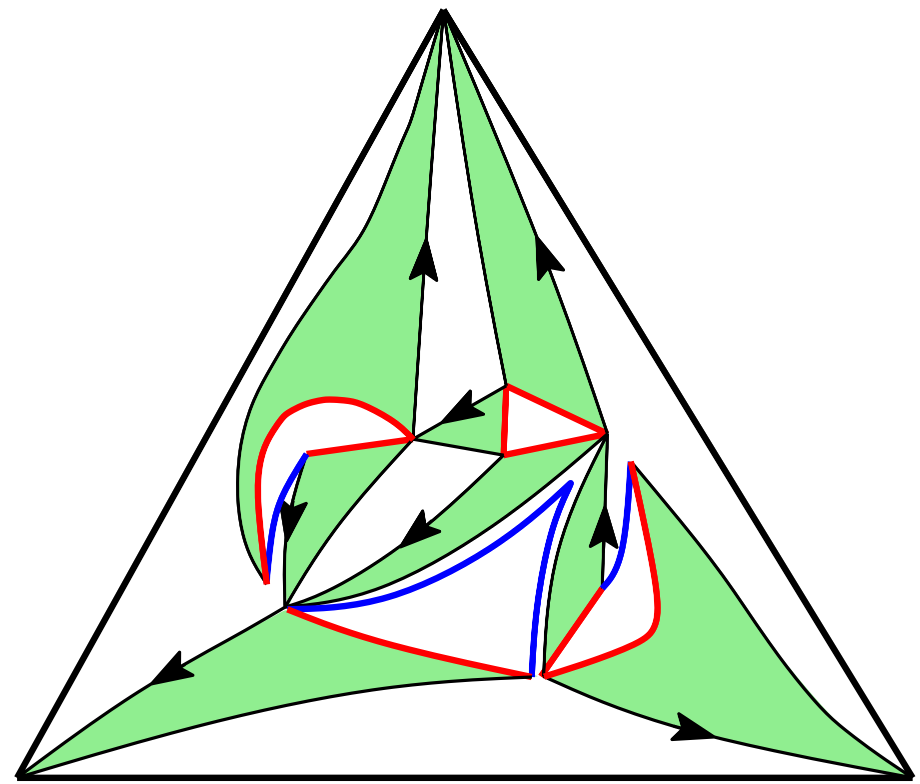

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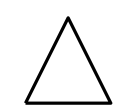

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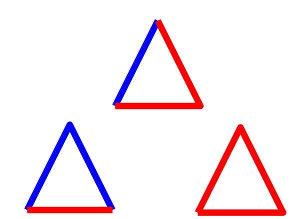

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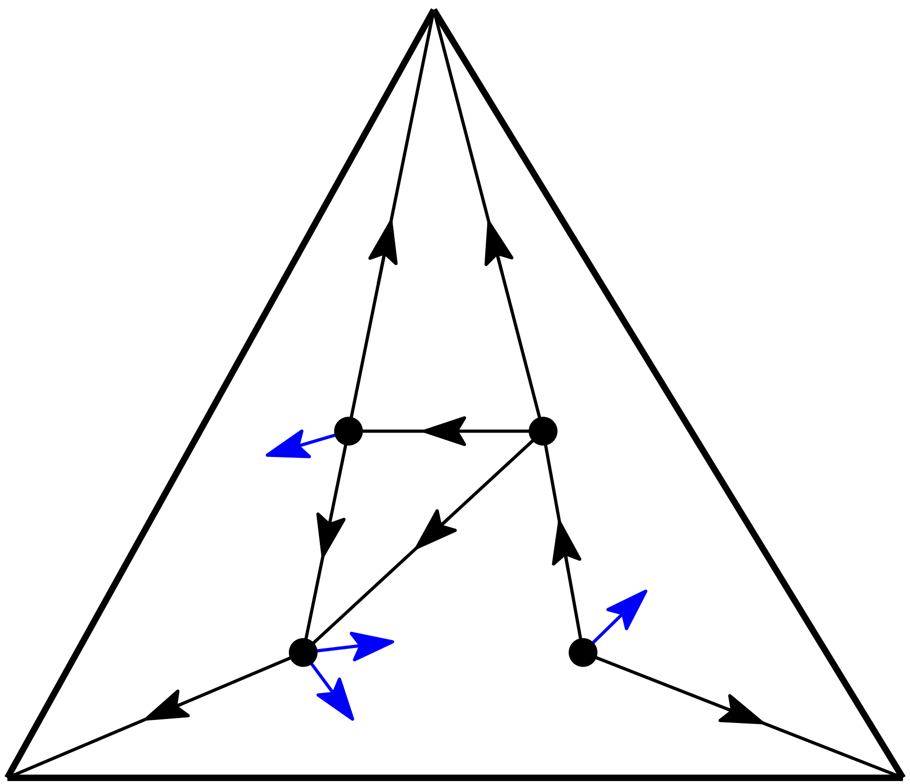

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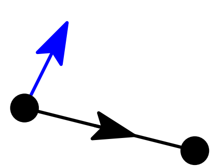
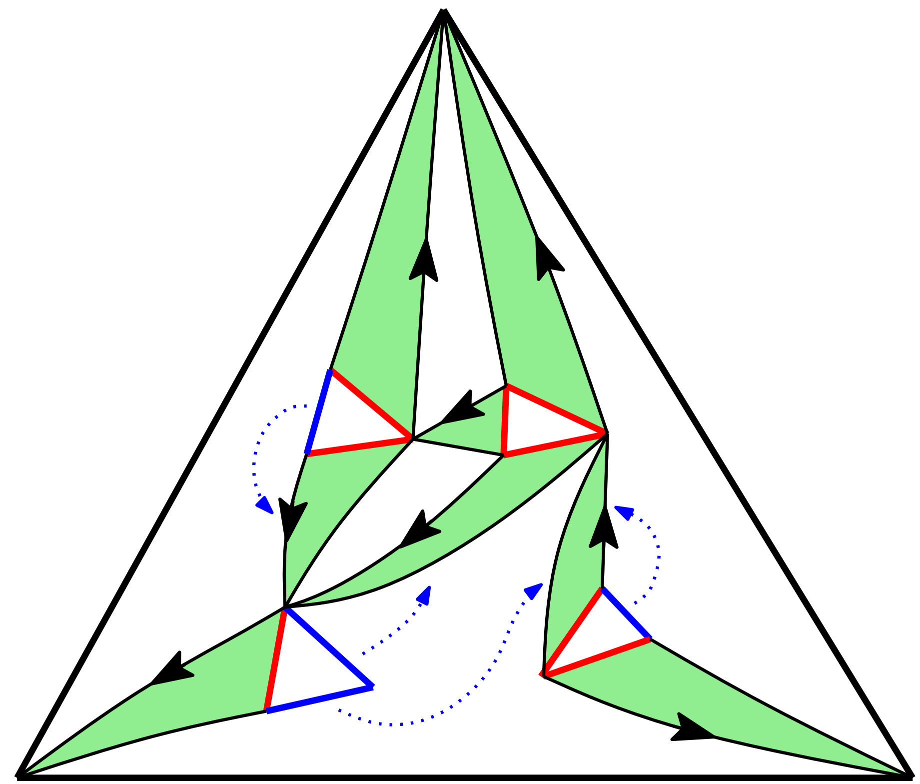
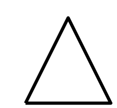
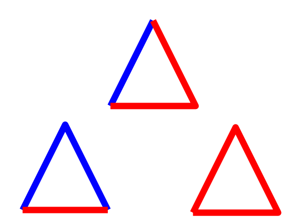
no clockwise circuit

+ 



Face of the plane graph: 

Inner node of:
the plane graph



Bijection for plane graphs

Theorem [Bernardi, C., Fusy'13] :

There is a bijection between outertriangular plane graphs with $n+2$ edges and face-pointed eulerian triangulations with $2n$ faces.

Remark:

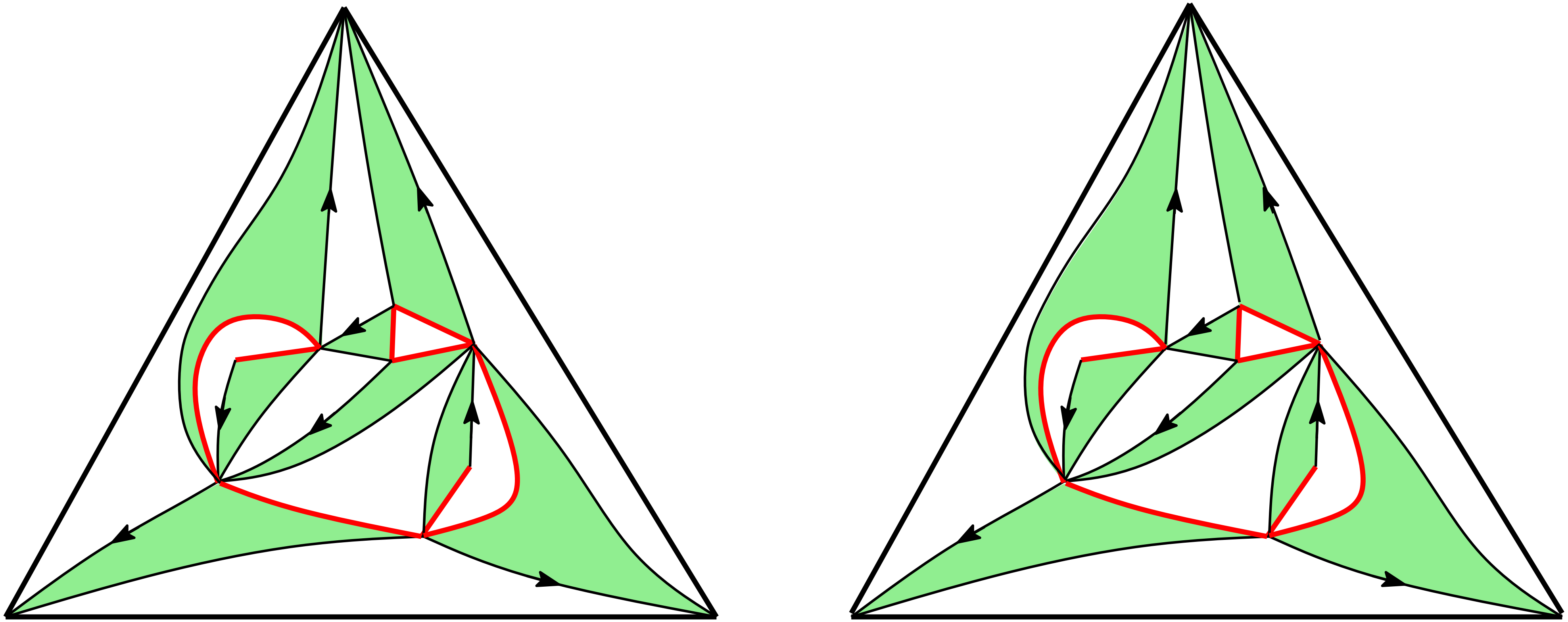
$\rightarrow$ We obtain M. Noy's formula.

$\rightarrow$ Eulerian triangulations that arise are much easier to enumerate and generate.

Bijection for plane graphs

From eulerian triangulations to oriented binary trees

[Bousquet-Mélou, Schaeffer'00]



Bijection for plane graphs

From eulerian triangulations to oriented binary trees

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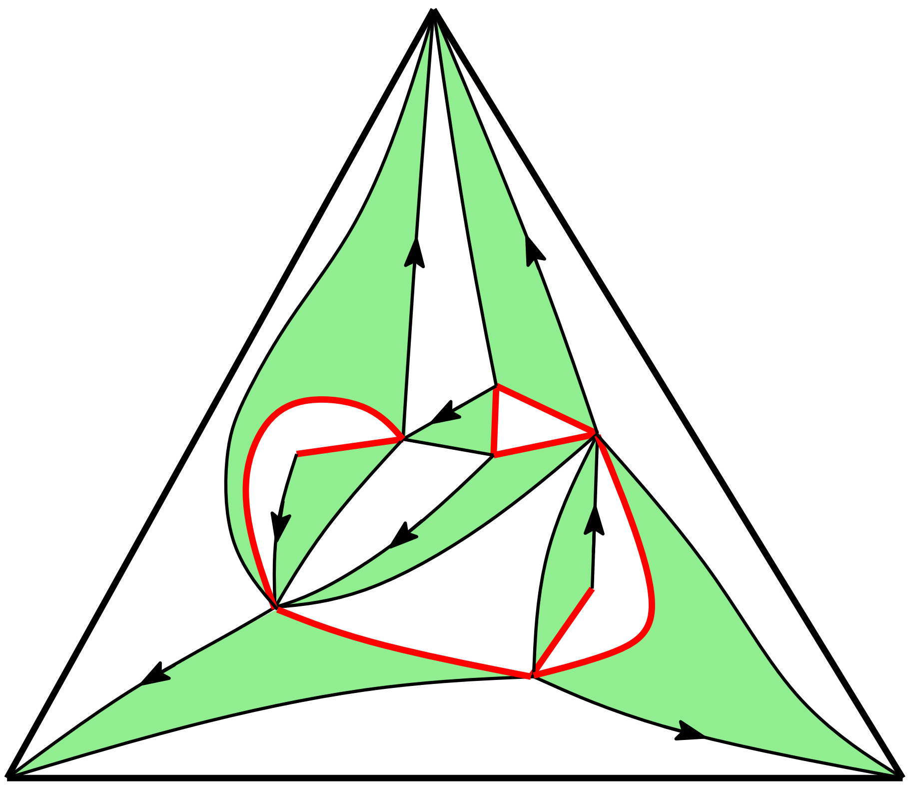
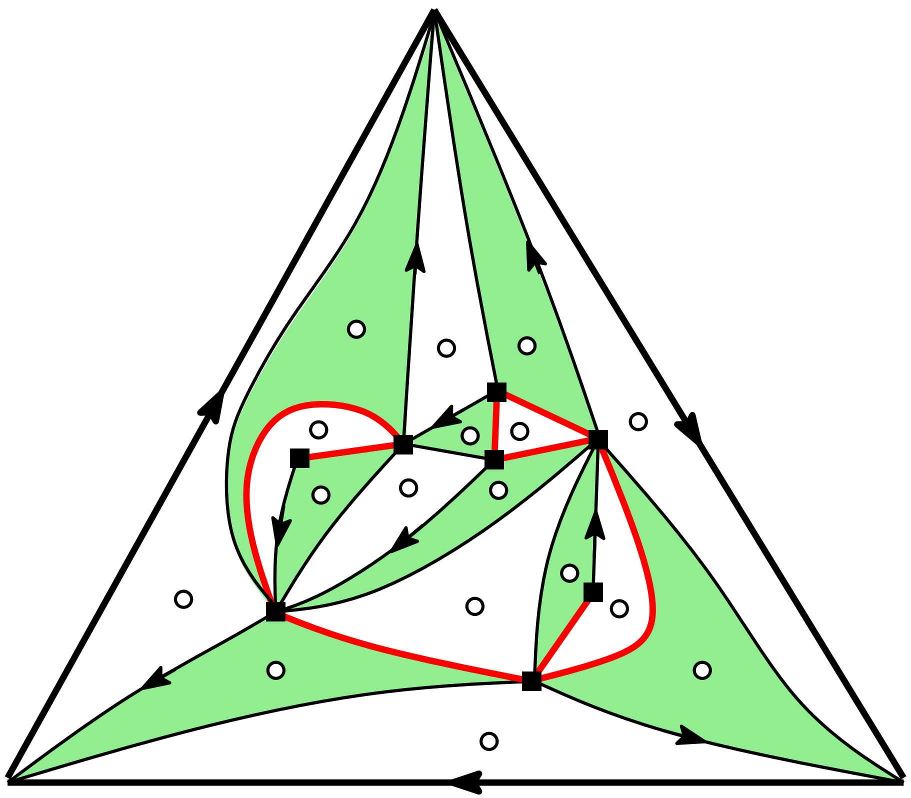
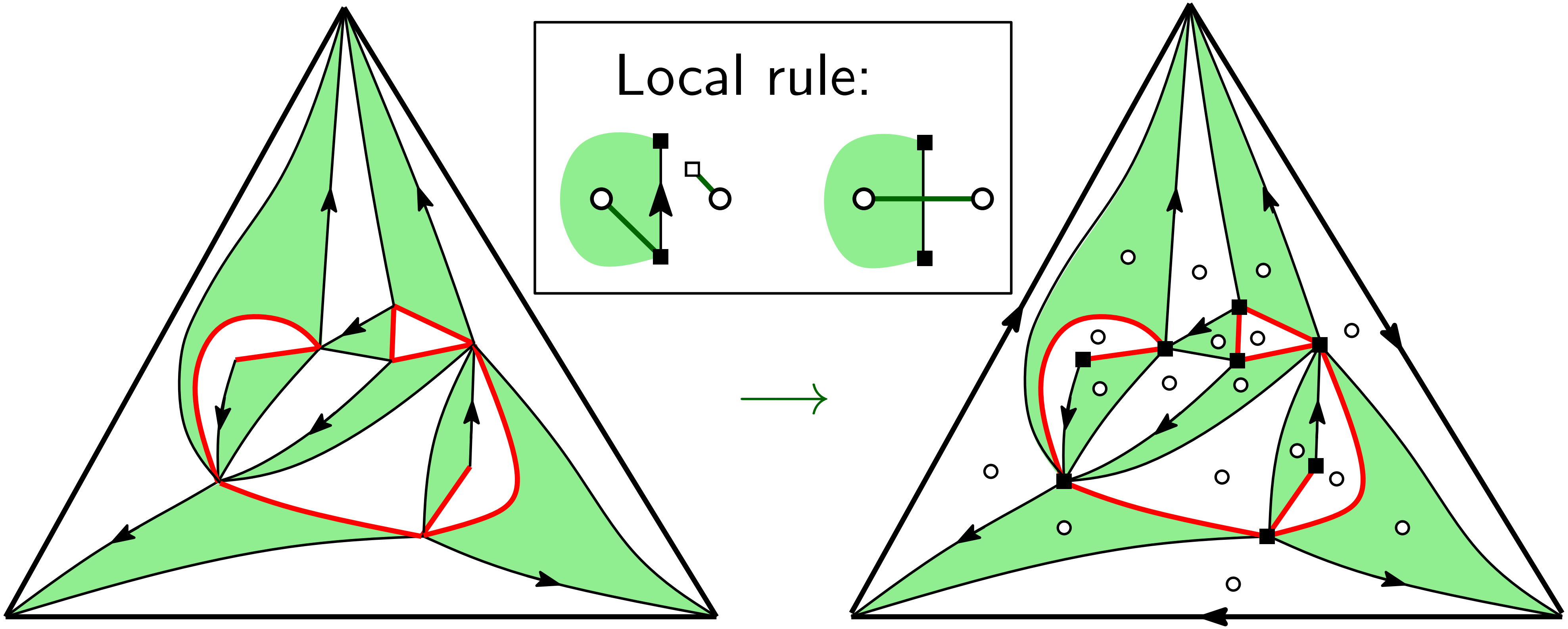




Bijection for plane graphs

From eulerian triangulations to oriented binary trees

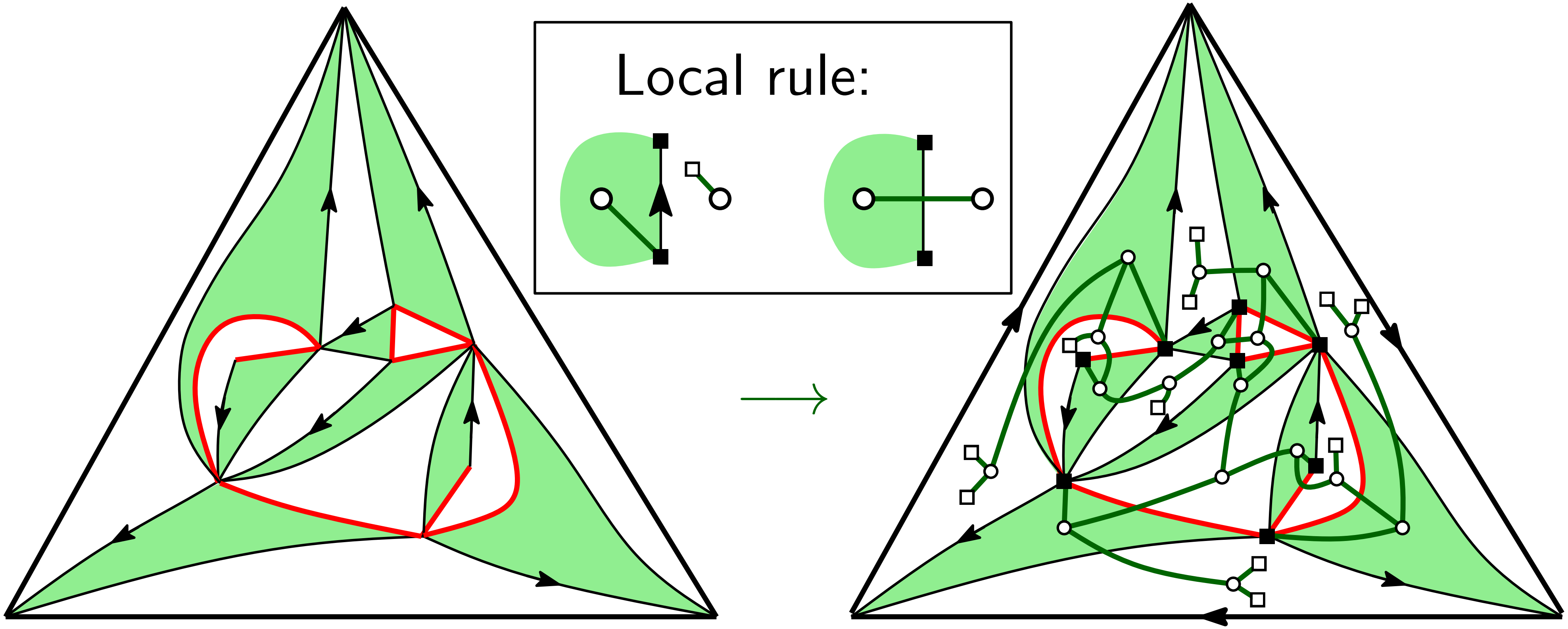
[Bousquet-Mélou, Schaeffer'00]



Bijection for plane graphs

From eulerian triangulations to oriented binary trees

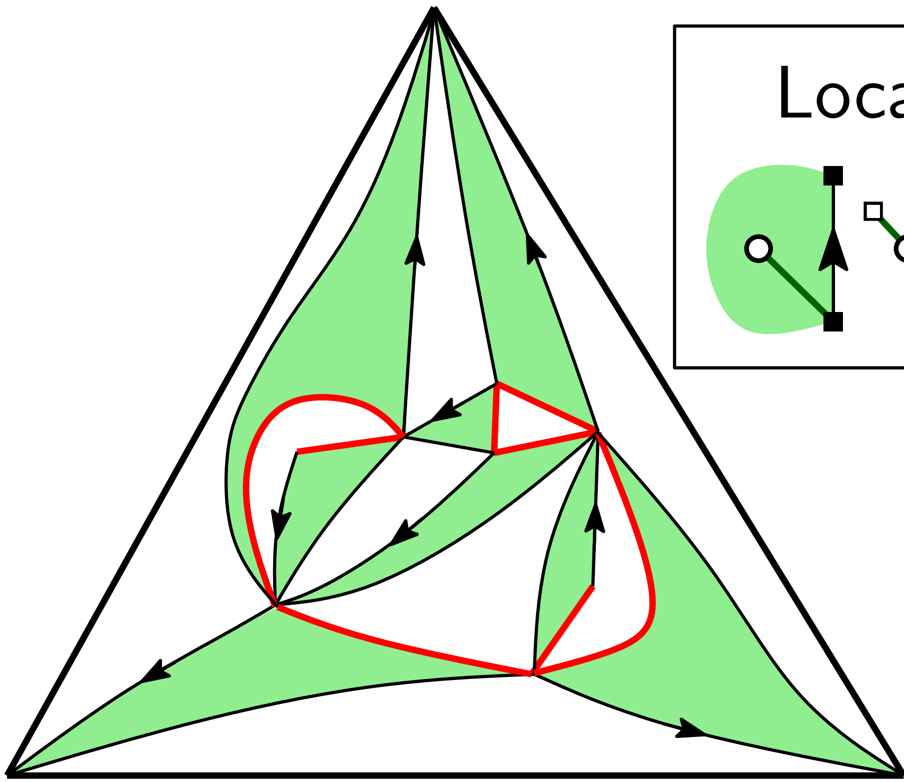
[Bousquet-Mélou, Schaeffer'00]

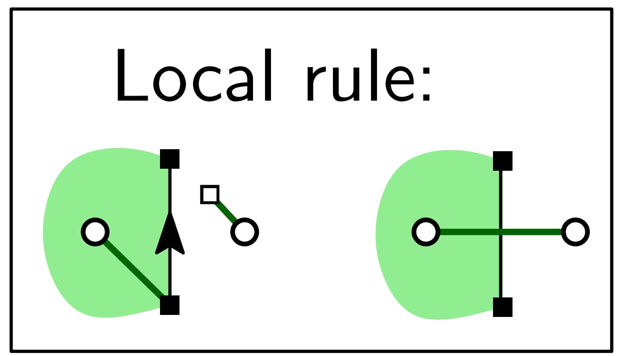


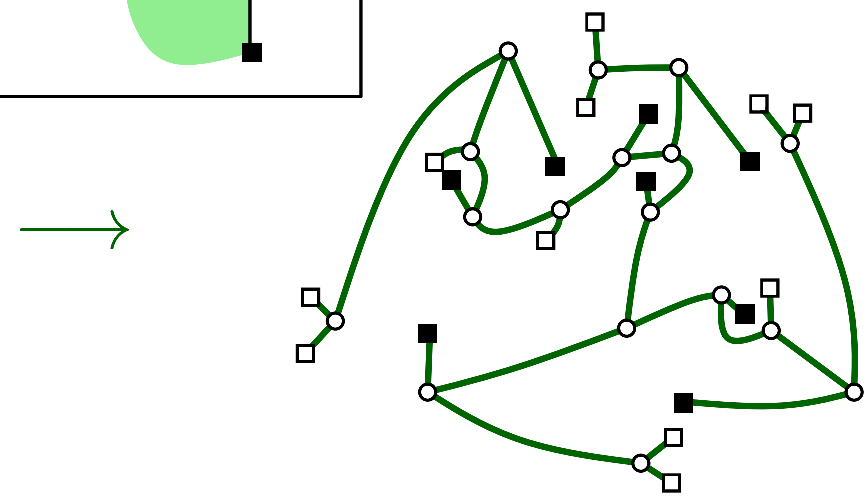
Bijection for plane graphs

From eulerian triangulations to oriented binary trees

[Bousquet-Mélou, Schaeffer'00]





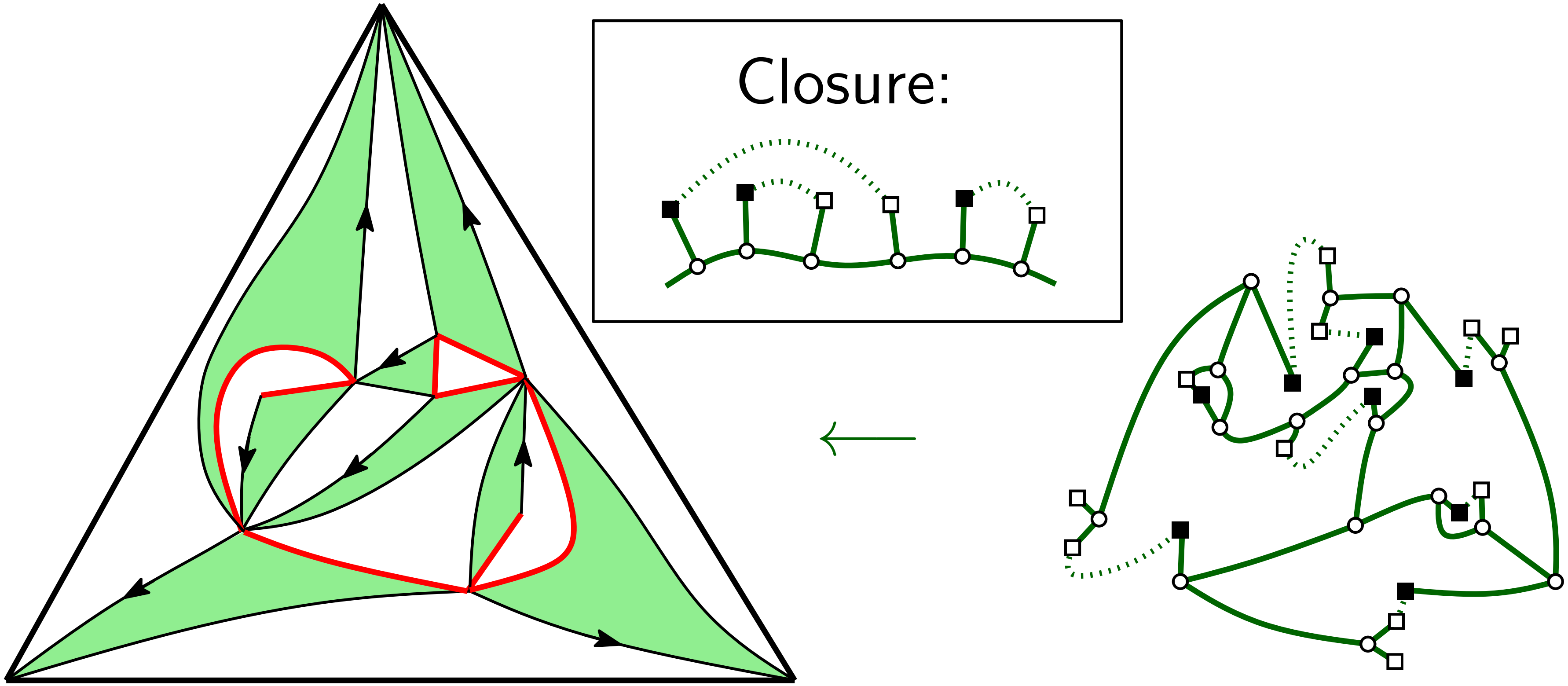


Binary tree with bicolored leaves ■ and □

Bijection for plane graphs

From eulerian triangulations to oriented binary trees

[Bousquet-Mélou, Schaeffer'00]



Binary tree with bicolored leaves ■ and □

Bijection for plane graphs

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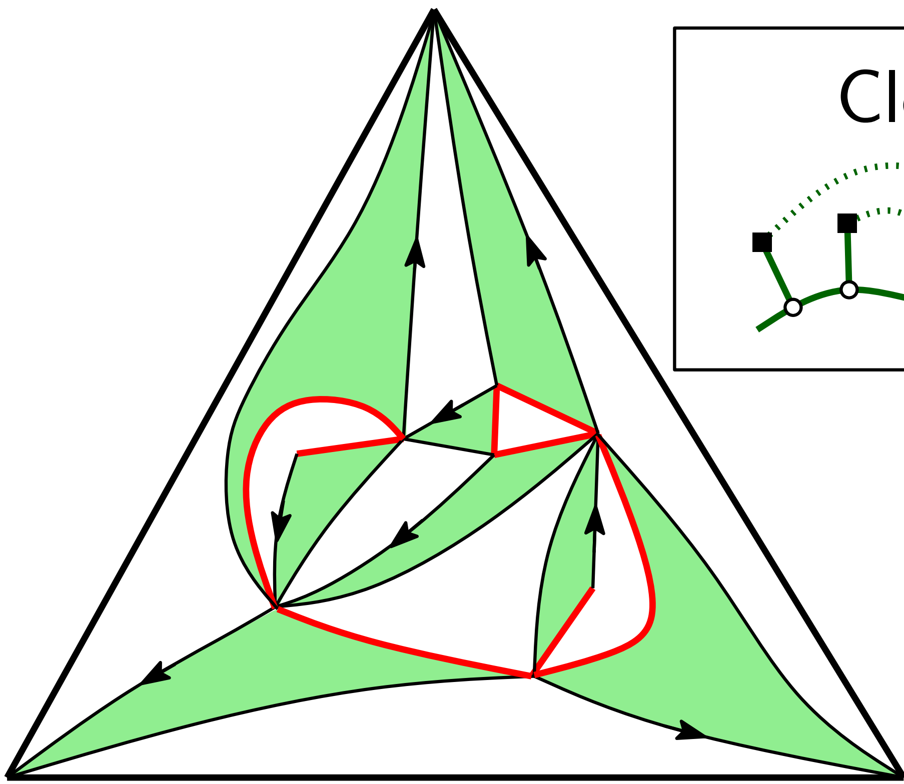
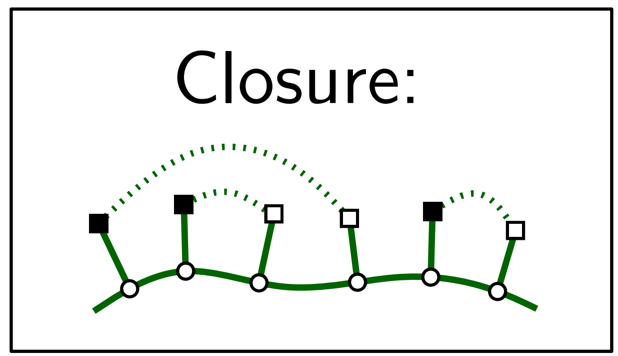
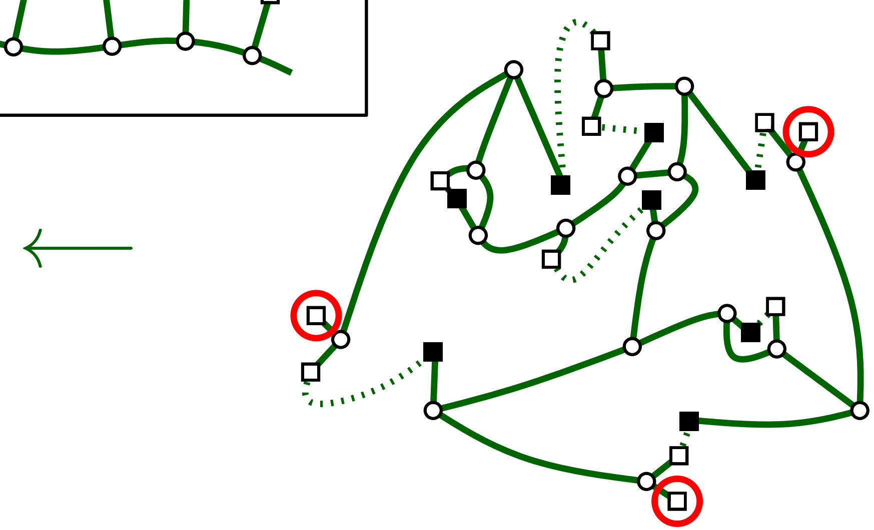






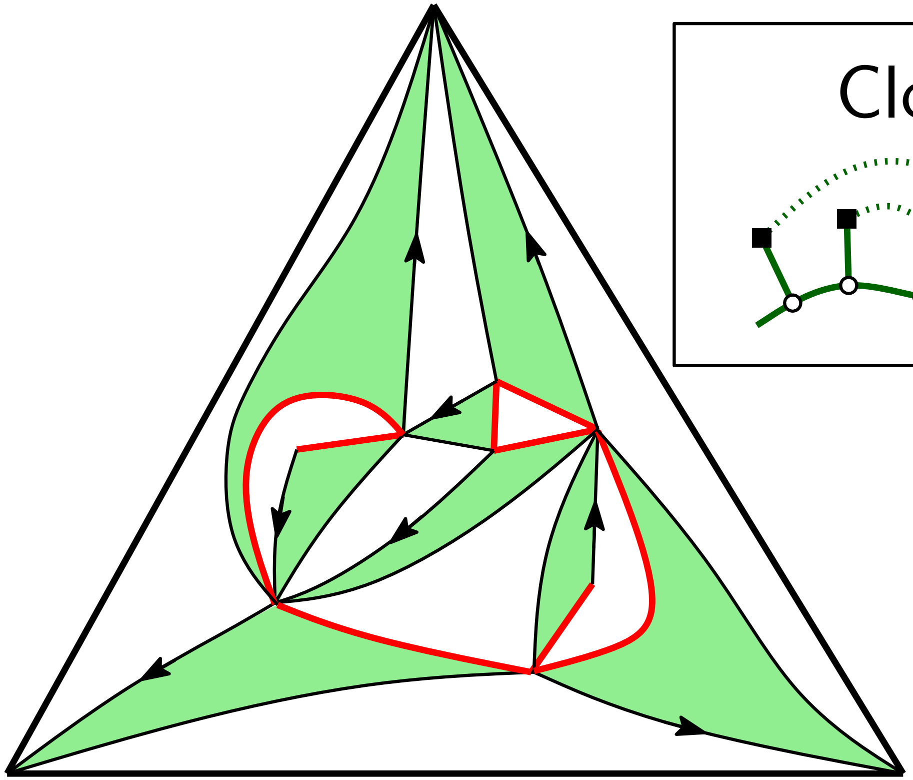
Binary tree with bicolored leaves ■ and □

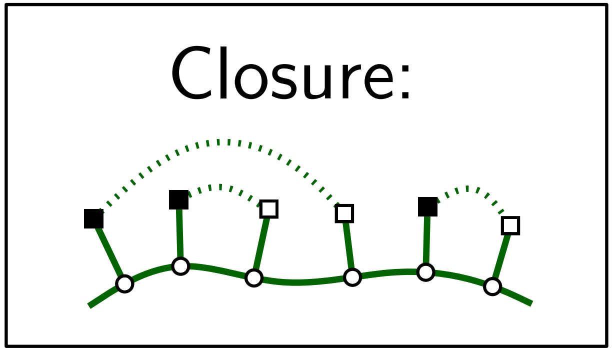
3 unmatched leaves □

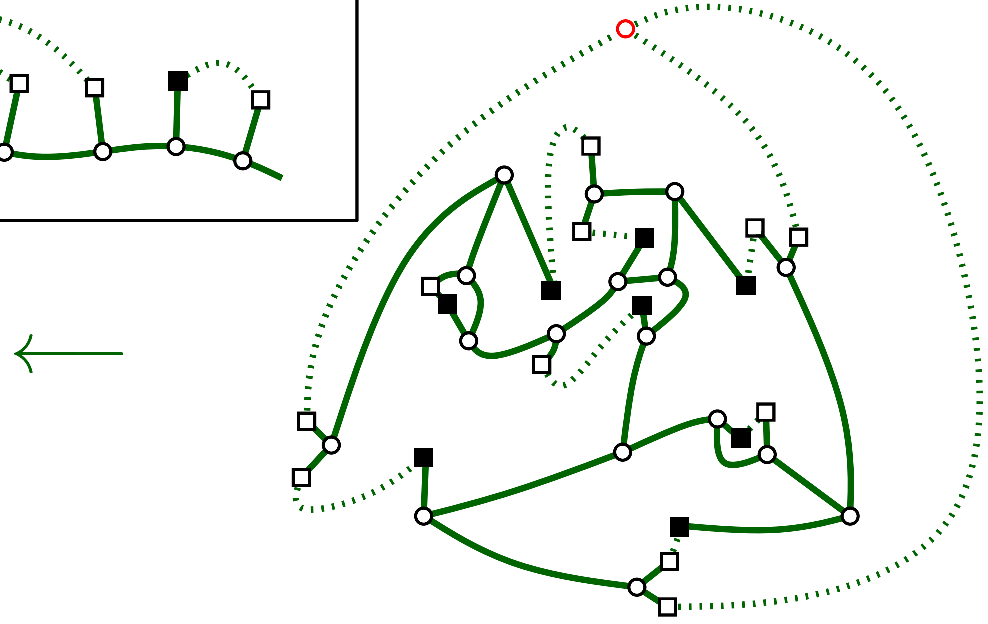
Bijection for plane graphs

From eulerian triangulations to oriented binary trees

[Bousquet-Mélou, Schaeffer'00]







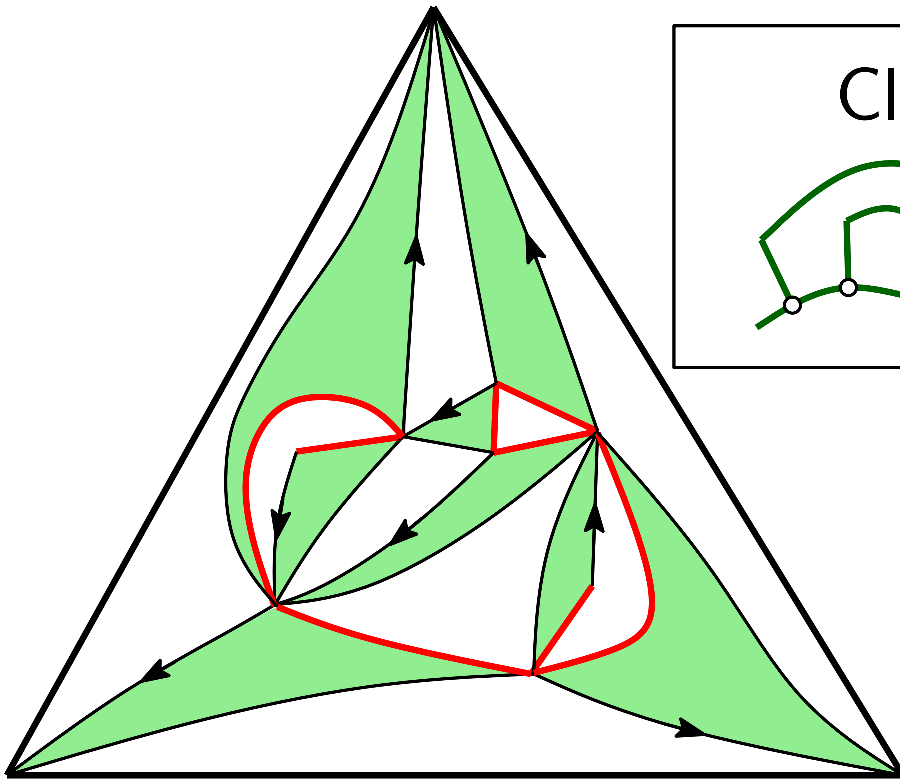
Binary tree with bicolored leaves ■ and □

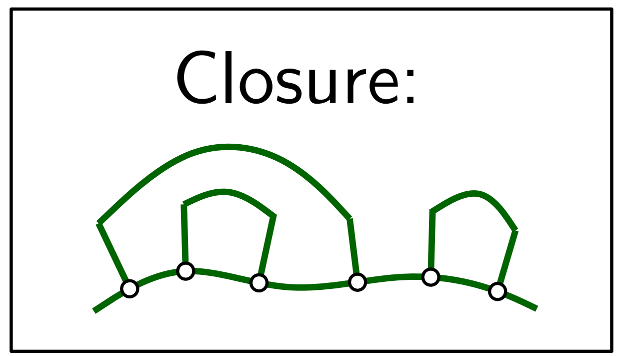
3 unmatched leaves □

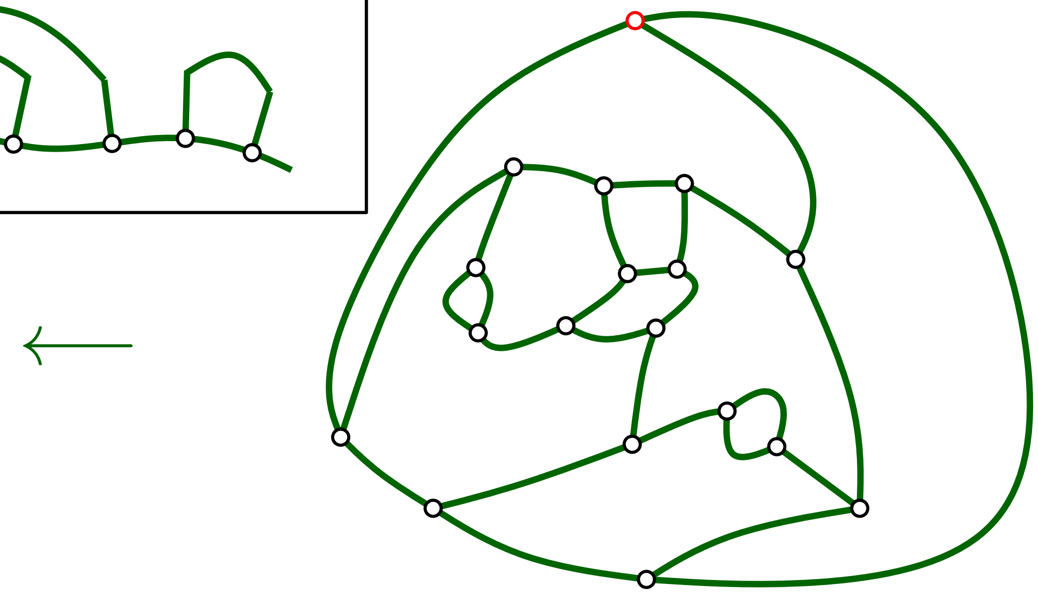
Bijection for plane graphs

From eulerian triangulations to oriented binary trees

[Bousquet-Mélou, Schaeffer'00]







Eulerian triangulation

1 distinguished face

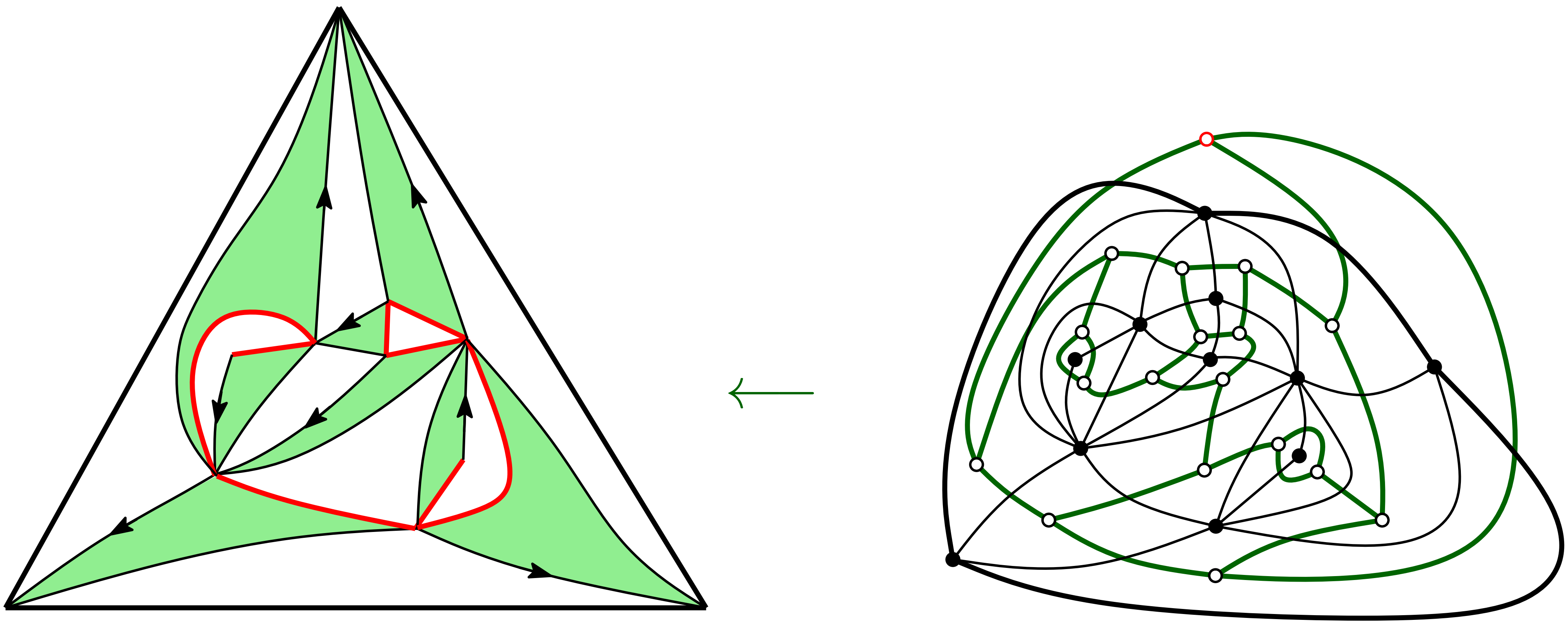
Dual bicubic map

1 distinguished vertex

Bijection for plane graphs

From eulerian triangulations to oriented binary trees

[Bousquet-Mélou, Schaeffer'00]



Eulerian triangulation

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Bijection for plane graphs

From eulerian triangulations to oriented binary trees

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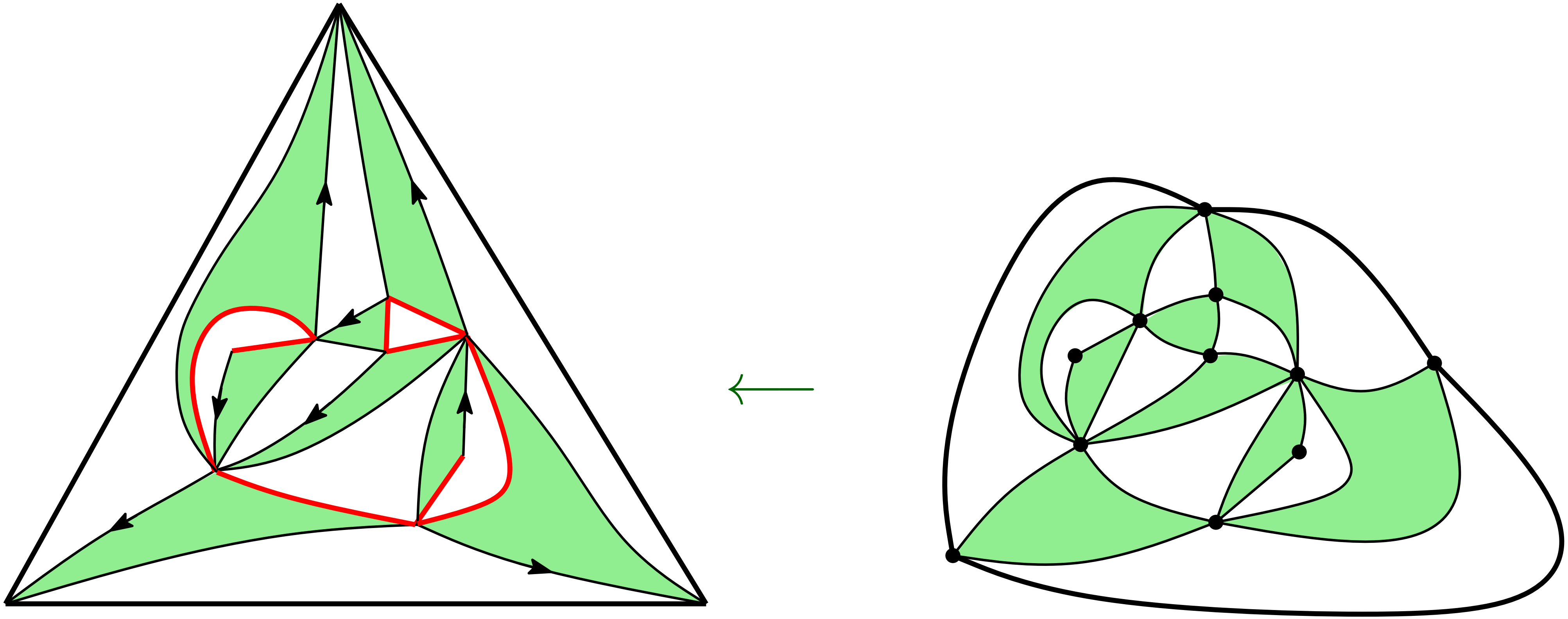


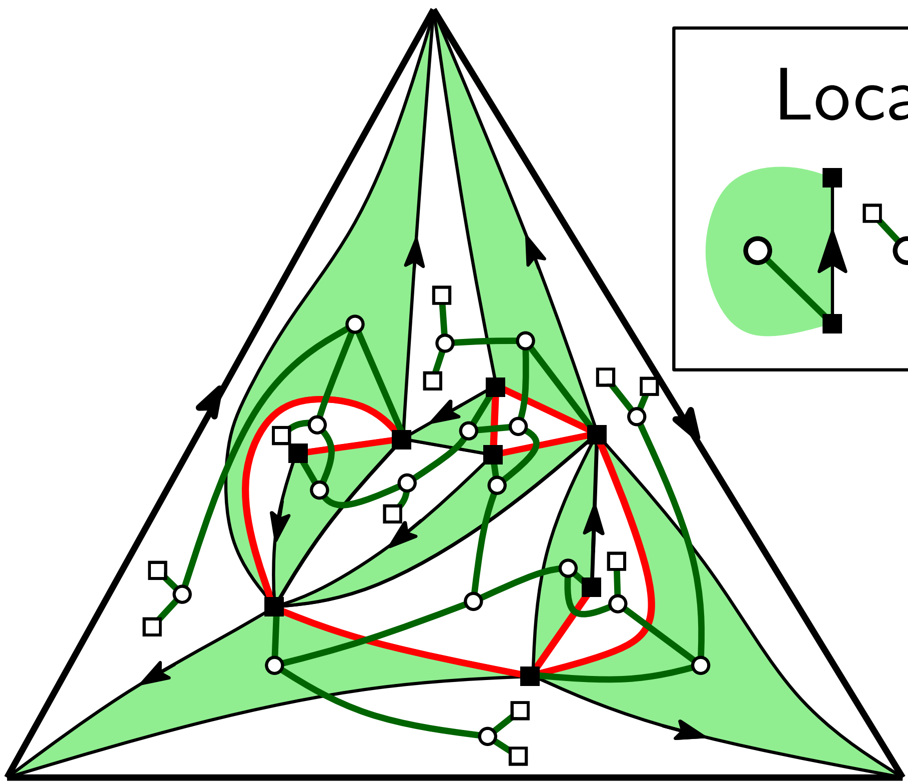
Remark:

The operations of closure and duality can be done in linear time according the number of edges in the tree.

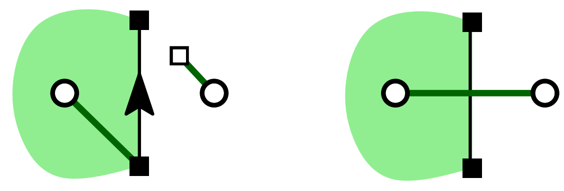
Bijection for plane graphs

From eulerian triangulations to oriented binary trees

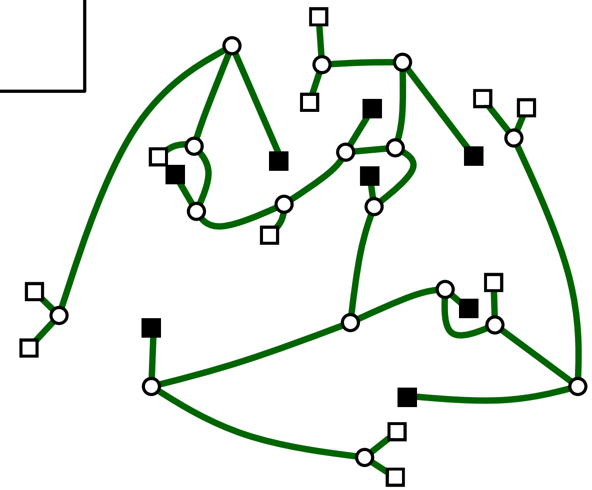
[Bousquet-Mélou, Schaeffer'00]



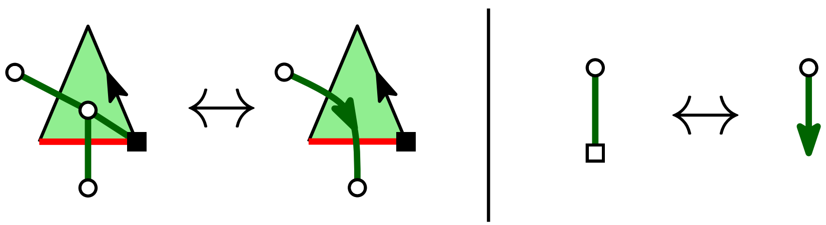
Local rule:



⟶



Reformulation with orientation :

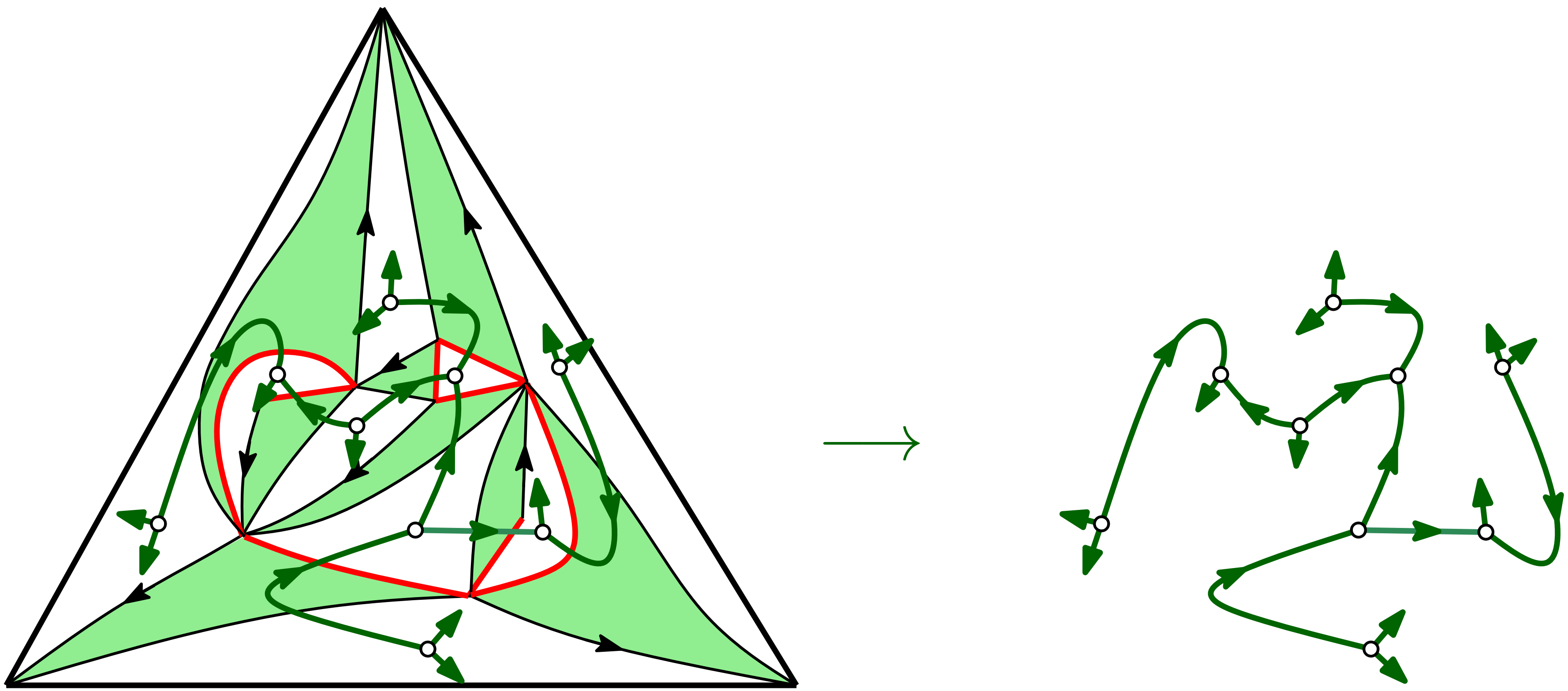


Binary tree with bicolored leaves ■ and □

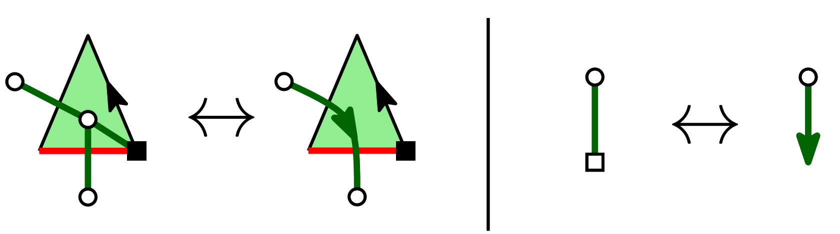
Bijection for plane graphs

From eulerian triangulations to oriented binary trees

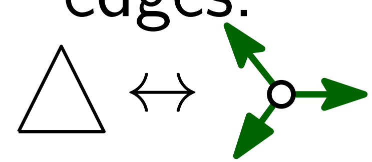
[Bousquet-Mélou, Schaeffer'00]



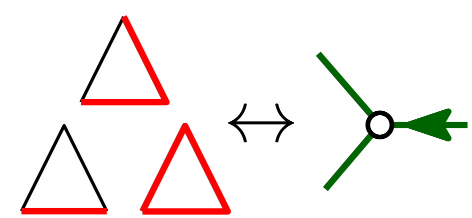
Reformulation with orientation :



Binary tree with oriented edges:



3 outgoing



≥ 1 outgoing

Bijection for plane graphs

Corollaire [Bernardi, C., Fusy'13] :

There is a bijection between outertriangular plane graphs with $n+2$ edges and oriented binary trees with n inner nodes.

Remark:

$\rightarrow$ One can track vertices and faces of the associated plane graph on the binary tree:

face $\leftrightarrow$ source vertex   vertex $\leftrightarrow$ non-source vertex

$\rightarrow$ $$C_{n+2} = \frac{3}{n+2} 2^{n-1} Cat(n)$$

rooting on an unmatched leaf

orientation of the $n-1$ inner edges

Random generation

Random sampler for plane graphs [Bernardi, C., Fusy'13]

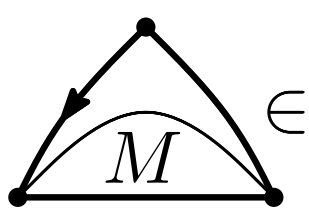
One can build a random sampler in exact size for the rooted plane graphs according to edges in linear time.

Method:

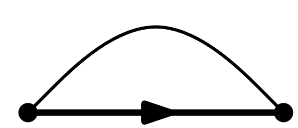
→ Sample a binary tree in exact size and orient inner edges

→ Obtain the outertriangular plane graph by tree closure and duality.

→ If the root vertex has degree 2, return the plane graph obtained by removing the root vertex.



$\in \mathcal{C}_{n+2} \longrightarrow$



$\in \mathcal{M}_n \;\Rightarrow\; \mathcal{M}_n \subset \mathcal{C}_{n+2}$

Random generation

Random sampler for plane graphs [Bernardi, C., Fusy'13]

One can build a random sampler in exact size for the rooted plane graphs according to edges in linear time.

Method:

→ Sample a binary tree in exact size and orient inner edges

→ Obtain the outertriangular plane graph by tree closure and duality.

→ If the root vertex has degree 2, return the plane graph obtained by removing the root vertex.

→ Otherwise reject and repeat:
Probability of success $= \frac{M_n}{C_{n+2}} \longrightarrow_{n\to\infty} 0,177$

Random generation

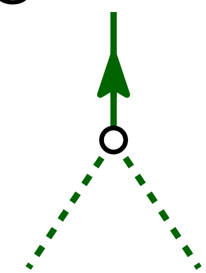
Random sampler for plane graphs [Bernardi, C., Fusy'13]

One can build a random Boltzmann sampler for rooted plane graphs according to edges and vertices in approximate size, in linear time.

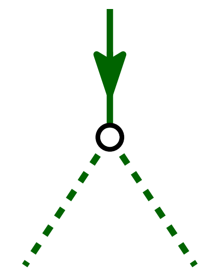
Method:

$\rightarrow$ Use the bivariate series $U(x,z)$ and $V(x,z)$ of rooted oriented trees, according to edges and non-source vertices:

U :



V :



x : non-source vertices

z : edges

$$\Rightarrow \begin{cases} U(x,z) & = & (z+V)^2 + x(2U(z+V)+U^2) \\ V(x,z) & = & x(z+U+V)^2 \end{cases}$$

$$\Rightarrow M(x,z) = \frac{x^2 z + x^3 U(1-V/z)}{1-xz-xU(1-V/z)}$$

Profile of distances in the plane graph

Let G be a plane graph rooted at e_0 uniform with n edges
$\forall e \in E_G : d(e) =$ length of the shortest path from e_0 to e

Profile : $(f_k)_{k\geq 1}$, where $f_k = \frac{1}{n}\#\{e \in E_G : d(e) \leq k\}$
Radius : $r(G) = max(d(e), e \in E)$

Conjecture : $f_{k/n^{1/4}} \xrightarrow{(d)}$ profile of the ISE
$r/n^{1/4} \xrightarrow{(d)}$ width of the ISE

Sketch of the proof:
• Chassaing–Schaeffer for eulerian triangulations.
• Bousquet-Mélou–Schaeffer: canonical path of eulerian triangulation = geodesic (with green triangles lying at the left).
• Our bijection preserves canonical paths for edges.
• Addario-Berry–Albenque for plane graphs: canonical path are near-geodesic.