Lambda Determinants

Robin Langer, Institut Gaspard Monge, Paris Est, CNRS

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Alternating sign matrices

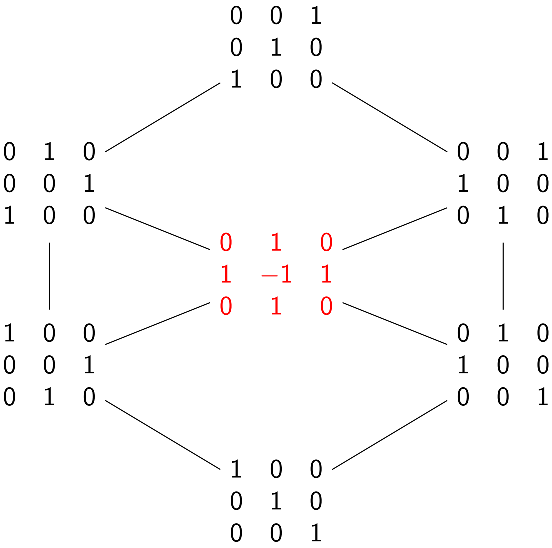
Definition:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

An alternating sign matrix is a square matrix of 0's, 1's, and -1's such that the sum of each row and column is 1 and the nonzero entries in each row and column alternate in sign.

$$A_n = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} \qquad\qquad 1, 1, 2, 7, 42, 429, 7436, \ldots$$

Bruhat order



Corner sum matrices (left)

$\overline{B}$ = sum of the entries lying above and to the left.

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \overline{B} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

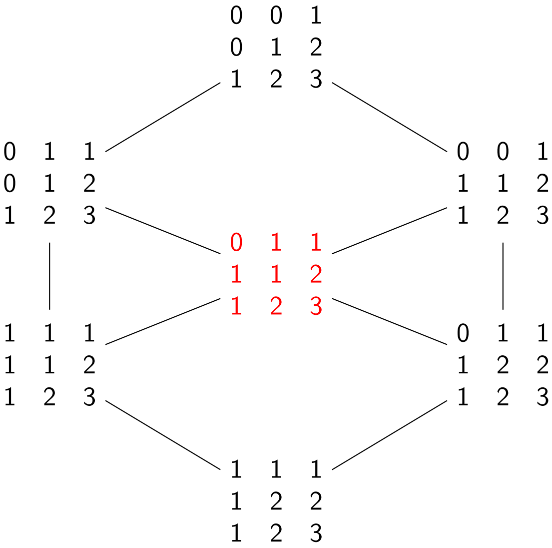
The original alternating sign matrix may be recovered via the formula.

$$B_{ij} = \overline{B}_{ij} + \overline{B}_{i-1,j-1} - \overline{B}_{i,j-1} - \overline{B}_{i-1,j}$$

"Local" matrix:

$$L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Bruhat order (corner sum, left)



Inversions and dual inversions

The alternating sign matrix A has an *inversion* at position (i, j) if:

$$\sum_{k>i} A_{k,j} = 1 = \sum_{k<j} A_{i,k}$$

The alternating sign matrix A has a *dual-inversion* at position (i, j) if:

$$\sum_{k<i} A_{k,j} = 1 = \sum_{k<j} A_{i,k}$$

Our example alternating matrix A has three 3 inversions and 2 dual-inversions:

$$\begin{pmatrix} + & + & 1 & * \\ + & 1 & -1 & 1 \\ 1 & 0 & * & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Duality

The Bruhat order is self dual, and thus doubly graded (by inversion and by dual inversion).

Let J denote the maximum permutation which sends i to $n - i$.

The alternating sign matrix A has an inversion at position i if and only if the alternating sign matrix AJ has a dual-inversion at position $n + 1 - i$.

$$\boxed{\mathrm{Inv}(A) = \mathrm{Dinv}(AJ)J} \tag{1}$$

Corner sum matrices (right)

$\underline{B}$ = sum of the entries lying above and to the right.

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \underline{B} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

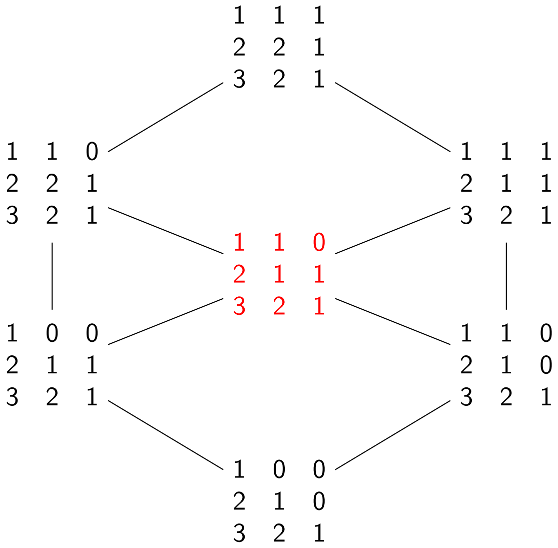
The original alternating sign matrix may be recovered by the formula:

$$B_{ij} = \underline{B}_{ij} + \underline{B}_{i-1,j+1} - \underline{B}_{i,j+1} - \underline{B}_{i-1,j} \qquad (2)$$

Duality:

$$\boxed{\underline{B}J = \overline{B}J}$$

Bruhat order (corner sum, right)



Variable pyramid

For each $k = 0 \ldots n$ let us denote by $x[k]$ the doubly indexed collection of variables $x[k]_{i,j}$ with indices running from $i, j = 1..(n-k+1)$. For example, $n = 4$:

$$x[0] = \begin{bmatrix} x[0]_{11} & x[0]_{12} & x[0]_{13} & x[0]_{14} & x[0]_{15} \\ x[0]_{21} & x[0]_{22} & x[0]_{23} & x[0]_{24} & x[0]_{25} \\ x[0]_{31} & x[0]_{32} & x[0]_{33} & x[0]_{34} & x[0]_{35} \\ x[0]_{41} & x[0]_{42} & x[0]_{43} & x[0]_{44} & x[0]_{45} \\ x[0]_{51} & x[0]_{52} & x[0]_{53} & x[0]_{54} & x[0]_{55} \end{bmatrix} \quad x[3] = \begin{bmatrix} x[3]_{11} & x[3]_{12} \\ x[3]_{21} & x[3]_{22} \end{bmatrix}$$

$$x[1] = \begin{bmatrix} x[1]_{11} & x[1]_{12} & x[1]_{13} & x[1]_{14} \\ x[1]_{21} & x[1]_{22} & x[1]_{23} & x[1]_{24} \\ x[1]_{31} & x[1]_{32} & x[1]_{33} & x[1]_{34} \\ x[1]_{41} & x[1]_{42} & x[1]_{43} & x[1]_{44} \end{bmatrix} \quad x[4] = \begin{bmatrix} x[4]_{11} \end{bmatrix}$$

$$x[2] = \begin{bmatrix} x[2]_{11} & x[2]_{12} & x[2]_{13} \\ x[2]_{21} & x[2]_{22} & x[2]_{23} \\ x[2]_{31} & x[2]_{32} & x[2]_{33} \end{bmatrix}$$

λ-determinant - Robbins and Rumsey

The variables are initialized as follows:

$$x[0]_{i,j} = 1 \text{ for all } i,j = 1..(n+1)$$
$$x[1]_{i,j} = M_{i,j} \text{ for all } i,j = 1..n$$

The remaining variables are calculated via the following *octahedral recurrence*:

$$x[k+1]_{i,j} = \frac{x[k]_{i,j}x[k]_{i+1,j+1} + \lambda\, x[k]_{i,j+1}x[k]_{i+1,j}}{x[k-1]_{i+1,j+1}} \tag{3}$$

Laurent Phenomenon

A *Laurent polynomial* in a single variable is a function of the form:

$$f(z) = \sum_{k=-n}^{m} a_k z^k$$

In general the composition of two Laurent polynomials is not a Laurent polynomial.

For example if $\alpha(z) = 1 + z$ and $\beta(z) = 1/z$ then $\beta(\alpha(z)) = 1/(1+z)$ which is not a Laurent polynomial.

λ-determinant - Robbins and Rumsey

The end result is that:

$$x[n]_{1,1} = \sum_{B \in \mathfrak{A}_n} \lambda^{\mathrm{inv}(B)}(1+\lambda)^{N(B)} M^B \tag{4}$$

We are making use of the notation:

$$M^B = \prod_{i,j} M_{i,j}^{B_{i,j}}$$

Lambda determinant

In our generalization we consider multi-parameter analog of the recurrence (with the same initial conditions):

Definition

$$x[k+1]_{i,j} = \frac{\mu_{i,n-k+1-j}x[k]_{i,j}x[k]_{i+1,j+1} + \lambda_{i,j}x[k]_{i,j+1}x[k]_{i+1,j}}{x[k-1]_{i+1,j+1}} \quad (5)$$

The following special case of this recurrence was previously given by Di Francesco:

$$x[k+1]_{i,j} = \frac{x[k]_{i,j}x[k]_{i+1,j+1} + \lambda_{j-i}x[k]_{i,j+1}x[k]_{i+1,j}}{x[k-1]_{i+1,j+1}} \quad (6)$$

Special case of main result

A special case of our main result is the following:

Theorem

$$x[n]_{1,1} = \sum_{B \in \mathfrak{A}_n} M^B \left(\lambda^{\mathrm{Inv}(B)} \mu^{\mathrm{Dinv}(B) J} \prod_{B_{i,j}=-1} (\mu_{i,n+1-j} + \lambda_{i,j}) \right) \qquad (7)$$

Example (n=3)

Let us calculate the multivariable lambda determinant of the 3 by 3 matrix M whose entries are all equal to one:

$$x[0] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$x[1] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$x[2] = \begin{bmatrix} \mu_{1,2} + \lambda_{1,1} & \mu_{1,1} + \lambda_{1,2} \\ \mu_{2,2} + \lambda_{2,1} & \mu_{2,1} + \lambda_{2,2} \end{bmatrix}$$

$$x[3] = \mu_{1,1}(\mu_{1,2} + \lambda_{1,1})(\mu_{2,1} + \lambda_{2,2}) + \lambda_{1,1}(\mu_{1,1} + \lambda_{1,2})(\mu_{2,2} + \lambda_{2,1})$$

Example (n=3)

$$x[3] = \mu_{1,1}(\mu_{1,2} + \lambda_{1,1})(\mu_{2,1} + \lambda_{2,2}) + \lambda_{1,1}(\mu_{1,1} + \lambda_{1,2})(\mu_{2,2} + \lambda_{2,1})$$

This should be a weighted sum over all alternating sign matrices. The alternating sign matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

contributed the weight $\lambda_{11}\mu_{11}(\mu_{22} + \lambda_{22})$. The six 3 by 3 permutation matrices contribute to the remaining six terms each of degree 3.

$$x[n]_{1,1} = \sum_{B \in \mathfrak{A}} M^B \left(\lambda^{\mathrm{Inv}(B)} \mu^{\mathrm{Dinv}(B)J} \prod_{B_{i,j}=-1} (\mu_{i,n+1-j} + \lambda_{i,j}) \right)$$

Example $n = 4$

$$x[2] = \begin{bmatrix} \mu_{1,3} + \lambda_{1,1} & \mu_{1,2} + \lambda_{1,2} & \mu_{1,1} + \lambda_{1,3} \\ \mu_{2,3} + \lambda_{2,1} & \mu_{2,2} + \lambda_{2,2} & \mu_{2,1} + \lambda_{2,3} \\ \mu_{3,3} + \lambda_{3,1} & \mu_{3,2} + \lambda_{3,2} & \mu_{3,3} + \lambda_{3,3} \end{bmatrix}$$

$$\begin{aligned} x_{1,1}[3] &= \mu_{1,2}(\mu_{1,3} + \lambda_{1,1})(\mu_{2,2} + \lambda_{2,2}) + \lambda_{1,1}(\mu_{1,2} + \lambda_{1,2})(\mu_{2,3} + \lambda_{2,1}) \\ x_{1,2}[3] &= \mu_{1,1}(\mu_{1,2} + \lambda_{1,2})(\mu_{2,1} + \lambda_{2,3}) + \lambda_{1,2}(\mu_{1,1} + \lambda_{1,3})(\mu_{2,2} + \lambda_{2,2}) \\ x_{2,1}[3] &= \mu_{2,2}(\mu_{2,3} + \lambda_{2,1})(\mu_{3,2} + \lambda_{3,2}) + \lambda_{2,1}(\mu_{2,2} + \lambda_{2,2})(\mu_{3,3} + \lambda_{3,1}) \\ x_{2,2}[3] &= \mu_{1,1}(\mu_{2,2} + \lambda_{2,2})(\mu_{3,3} + \lambda_{3,3}) + \lambda_{2,2}(\mu_{2,1} + \lambda_{2,3})(\mu_{3,2} + \lambda_{3,2}) \end{aligned}$$

$$x_{1,1}[4] = \frac{\mu_{1,1} x_{1,1}[3] x_{2,2}[3] + \lambda_{1,1} x_{1,2}[3] x_{2,1}[3]}{(\mu_{2,2} + \lambda_{2,2})}$$

Interlacing matrices (left)

$\overline{A}$ is of dimensions n by n and $\overline{B}$ is of dimension $n+1$ by $n+1$

$$\begin{pmatrix} \overline{B}_{1,1} & & \overline{B}_{1,2} & & \overline{B}_{1,3} & & \overline{B}_{1,4} \\ & \overline{A}_{1,1} & & \overline{A}_{1,2} & & \overline{A}_{1,3} & \\ \overline{B}_{2,1} & & \overline{B}_{2,2} & & \overline{B}_{2,3} & & \overline{B}_{2,4} \\ & \overline{A}_{2,1} & & \overline{A}_{2,2} & & \overline{A}_{2,3} & \\ \overline{B}_{3,1} & & \overline{B}_{3,2} & & \overline{B}_{3,3} & & \overline{B}_{3,4} \\ & \overline{A}_{3,1} & & \overline{A}_{3,2} & & \overline{A}_{3,3} & \\ \overline{B}_{4,1} & & \overline{B}_{4,2} & & \overline{B}_{4,3} & & \overline{B}_{4,4} \end{pmatrix}$$

For all elements x, y, z, w of $\overline{B}$ and and all elements a of $\overline{A}$ which are arranged in the following configuration:

$$\begin{pmatrix} x & & y \\ & a & \\ z & & w \end{pmatrix} \qquad \boxed{x, w-1 \leq a \leq y, z}$$

Example

$$\begin{pmatrix} 0 & & 1 & & 1 & & 1 \\ & \{0,1\} & & 1 & & 1 & \\ 1 & & 1 & & 2 & & 2 \\ & 1 & & \{1,2\} & & 2 & \\ 1 & & 2 & & 2 & & 3 \\ & 1 & & 2 & & 3 & \\ 1 & & 2 & & 3 & & 4 \end{pmatrix}$$

Above and to the left of a -1 in the alternating sign matrix B there are two possible choices for the corresponding value of the left cumulant matrix $\overline{A}$. At all other positions there is a single choice.

$$\begin{pmatrix} x & & y \\ & a & \\ z & & w \end{pmatrix} \qquad \boxed{x, w-1 \leq a \leq y, z}$$

Interlacing matrices (right)

$\underline{A}$ is of dimension n by n and $\underline{B}$ is of dimension $n+1$ by $n+1$.

$$\begin{pmatrix} \underline{B}_{1,1} & & \underline{B}_{1,2} & & \underline{B}_{1,3} & & \underline{B}_{1,4} \\ & \color{red}{\underline{A}_{1,1}} & & \color{red}{\underline{A}_{1,2}} & & \color{red}{\underline{A}_{1,3}} & \\ \underline{B}_{2,1} & & \underline{B}_{2,2} & & \underline{B}_{2,3} & & \underline{B}_{2,4} \\ & \color{red}{\underline{A}_{2,1}} & & \color{red}{\underline{A}_{2,2}} & & \color{red}{\underline{A}_{2,3}} & \\ \underline{B}_{3,1} & & \underline{B}_{3,2} & & \underline{B}_{3,3} & & \underline{B}_{3,4} \\ & \color{red}{\underline{A}_{3,1}} & & \color{red}{\underline{A}_{3,2}} & & \color{red}{\underline{A}_{3,3}} & \\ \underline{B}_{4,1} & & \underline{B}_{4,2} & & \underline{B}_{4,3} & & \underline{B}_{4,4} \end{pmatrix}$$

For all elements x, y, z, w of $\underline{B}$ and and all elements $\color{red}{a}$ of $\color{red}{\underline{A}}$ which are arranged in the following configuration:

$$\begin{pmatrix} x & & y \\ & \color{red}{a} & \\ z & & w \end{pmatrix} \qquad \boxed{y, z-1 \leq \color{red}{a} \leq x, w}$$

Example

$$\begin{pmatrix} 1 & & 1 & & 0 & & 0 \\ & \color{red}{1} & & \color{red}{\{0,1\}} & & \color{red}{0} & \\ 2 & & 1 & & 1 & & 0 \\ & \color{red}{2} & & \color{red}{1} & & \color{red}{\{0,1\}} & \\ 3 & & 2 & & 1 & & 1 \\ & \color{red}{3} & & \color{red}{2} & & \color{red}{1} & \\ 4 & & 3 & & 2 & & 1 \end{pmatrix}$$

Above and to the *right* of a -1 in the alternating sign matrix B there are two possible choices for the corresponding value of the right cumulant matrix $\underline{A}$.

$$\begin{pmatrix} x & & y \\ & \color{red}{a} & \\ z & & w \end{pmatrix} \qquad \boxed{y, z-1 \leq \color{red}{a} \leq x, w}$$

Duality

Let B be an alternating sign matrix with r negative ones. Let us fix some order on the negative ones.

The set of alternating sign matrices which are left interlacing with a given alternating sign matrix B may be indexed by a binary string of length r.

Similarly, the set of alternating sign matrices which is right interlacing with a given alternating sign matrix B may be indexed by a binary string of length r.

Theorem

For any binary string π let $\widehat{\pi}$ denote the binary string obtained from π by interchanging the zeros and the ones.

$$A^{\pi} = A_{\widehat{\pi}}$$

Example

Left interlacing corner sum matrix:

$$\overline{A}^{01} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \mapsto A^{01} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Right interlacing corner sum matrix:

$$\underline{A}_{01} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \mapsto A_{10} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

λ-weights

We shall be working with Laurent polynomials in the doubly indexed sets of variables $\{\lambda_{i,j}\}$ for $i, j = 1, 2, \ldots n$.

Definition

Let X be a k by k alternating sign matrix.

$$F_\lambda(X) = \prod_{i,j=1}^{k} \lambda_{i,j}^{\min(i,j) - \overline{X}_{i,j}} \tag{8}$$

The operators s is defined by:

$$\boxed{s(\lambda_{i,j}) = \lambda_{i+1,j+1}}$$

Special case

It is not too hard to show that:

$$\frac{F_\lambda(B)}{s(F_\lambda(A^{\mathrm{min}}))} = \lambda^{\mathrm{Inv}(B)} \tag{9}$$

Example:

$$\begin{pmatrix}
0 & & 1 & & 1 & & 1 \\
& \{0,1\} & & 1 & & 1 & \\
1 & & 1 & & 2 & & 2 \\
& 1 & & \{1,2\} & & 2 & \\
1 & & 2 & & 2 & & 3 \\
& 1 & & 2 & & 3 & \\
1 & & 2 & & 3 & & 4
\end{pmatrix}$$

μ-weights

Now let us work also with the doubly indexed variable set $\{\mu_{i,j}\}$ for $i, j = 1 \ldots n$.

Definition

For X a k by k alternating sign matrix we have:

$$G_\mu^n(X) = \prod_{i,j=1}^{k} \mu_{i,n+1-j}^{\min(i,k+1-j)-\underline{X}_{i,j}} \tag{10}$$

Note that G_μ^n is a monomial in the variable set:

$$\{\mu_{i,j}\}_{i=1\ldots k, \quad j=n-k+1\ldots n}$$

The operator t is defined by:

$$\boxed{t(\mu_{i,j}) = \mu_{i+1,j}}$$

Special case

It follows by duality that if B is an n by n alternating sign matrix, then:

$$\frac{G_\mu^n(B)}{t(G_\mu^n(A_{\min}))} = \mu^{\mathrm{Dinv}(B)J} \tag{11}$$

Example:

$$\begin{pmatrix}
1 & & 1 & & 0 & & 0 \\
 & 1 & & \{0,1\} & & 0 & \\
2 & & 1 & & 1 & & 0 \\
 & 2 & & 1 & & \{0,1\} & \\
3 & & 2 & & 1 & & 1 \\
 & 3 & & 2 & & 1 & \\
4 & & 3 & & 2 & & 1
\end{pmatrix}$$

Main Theorem

Theorem

$$x[k+1]_{1,1} = \sum_{\substack{(A,B) \\ |B|=k, |A|=k-1}} \frac{F_\lambda(B)}{s(F_\lambda(A))} \frac{G_\mu^n(B)}{t(G_\mu^n(A))} x[1]^B s(x[0])^{-A} \qquad (12)$$

The sum is over all pairs of interlacing matrices (either left or right).

Special case

Initial conditions:

$$x[0]_{i,j} = 1$$
$$x[1]_{i,j} = M_{i,j}$$

We have:

$$\begin{aligned}
x[n]_{1,1} &= \sum_{\substack{(A,B)\\|B|=n,|A|=n-1}} \frac{F_\lambda(B)}{s(F_\lambda(A))} \frac{G_\mu^n(B)}{t(G_\mu^n(A))} M^B \\
&= \sum_B M^B \sum_\pi \frac{F_\lambda(B)}{s(F_\lambda(A^\pi))} \frac{G_\mu^n(B)}{t(G_\mu^n(A_{\widehat{\pi}}))} \\
&= \sum_B M^B \sum_\pi \frac{F_\lambda(B)}{s(F_\lambda(A^{\min}))} \frac{G_\mu^n(B)}{t(G_\mu^n(A_{\min}))} \prod_{B_{i,j}=-1} (\mu_{i,n+1-j} + \lambda_{i,j}) \\
&= \sum_B M^B \sum_\pi \lambda^{\mathrm{Inv}(B)} \mu^{\mathrm{Dinv}(B)J} \prod_{B_{i,j}=-1} (\mu_{i,n+1-j} + \lambda_{i,j})
\end{aligned}$$

Merci!