Central limit theorem for random Young diagrams with respect to Jack measure

(joint work with Valentin Féray)

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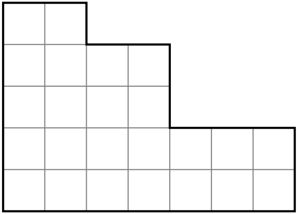
04 XII 2013

Young diagrams

Definition

A partition λ is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. It can be represented by a Young diagram λ.



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A generalized Young diagram is a broken line going from a point $(0, y)$ on the y-axis to a point $(x, 0)$ on the x-axis such that every piece is either a horizontal segment from left to right or a vertical segment from top to bottom.

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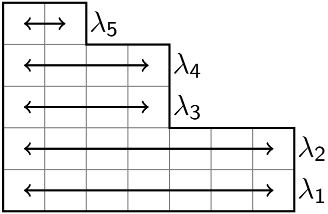

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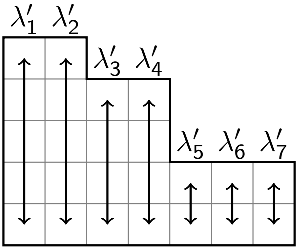

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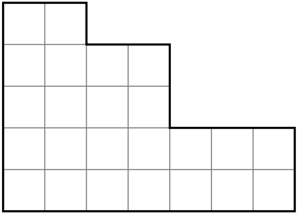

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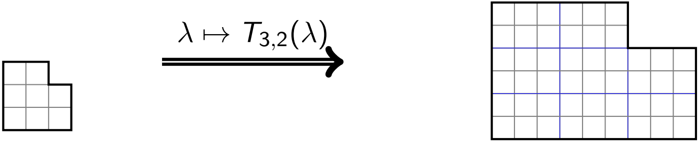
Simple operations on generalized Young diagrams

- Dilation:
  $T_{s,t}(\lambda)$ - generalized Young diagram obtained by stretching λ horizontally by a factor s and vertically by a factor t, where $s, t \in \mathbb{R}_+$.



Examples

Special cases:

- Blowing of Young diagram: $D_s(\lambda) := T_{s,s}(\lambda)$, for $s \in \mathbb{R}_+$;
- α-anisotropic Young diagram: $\lambda^{(\alpha)} := T_{\sqrt{\alpha},\sqrt{\alpha}^{-1}}(\lambda)$ for $\alpha \in \mathbb{R}_+$;

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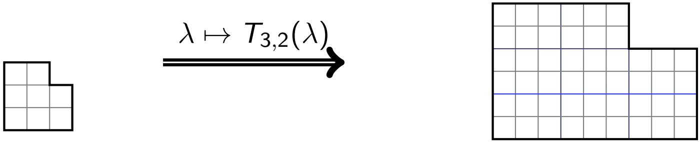

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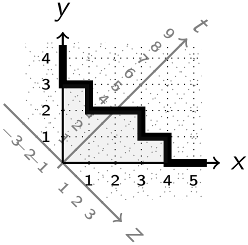
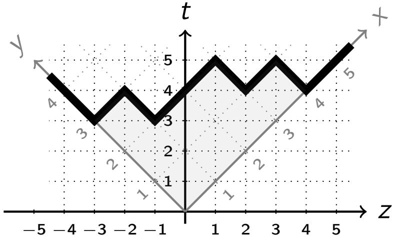
Two conventions of drawing Young diagrams

Conventions of drawing Young diagrams:

- French convention:



- Russian convention:



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A profile of a generalized Young diagram λ is a function $\omega(\lambda) : \mathbb{R} \to \mathbb{R}_+$ such that its graph is a profile of λ drawn in Russian convention.

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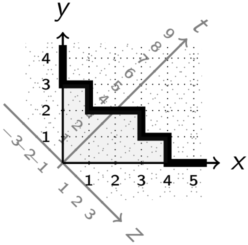

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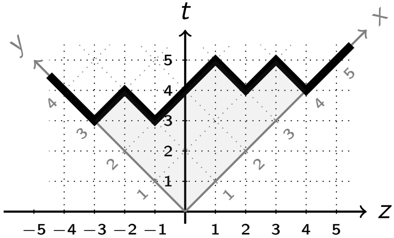

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Problem

Definition

A continuous Young diagram is a function $\omega : \mathbb{R} \to \mathbb{R}_+$ such that

- $\omega(x) - |x|$ has compact support;
- $|\omega(x_1) - \omega(x_2)| \leq |x_1 - x_2|$ for any $x_1, x_2 \in \mathbb{R}$.

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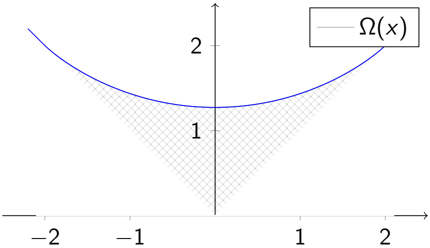
Problem

- $\mathbb{Y}_n$ - *the set of Young diagrams of size n* *($|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$);*
- $\mathbb{P}_n$ - *probability measure defined on the set* $\mathbb{Y}_n$.

Let $\lambda_{(n)}$ *be a sequence of Young diagrams of size n. Does exist some continuous Young diagram* ω *such that, as* $n \to \infty$, *in probability*

$$\left\| \omega(D_{\sqrt{n}^{-1}}(\lambda_{(n)})) - \omega \right\| \to 0?$$

Vershik-Kerov, Logan-Shepp limit shape



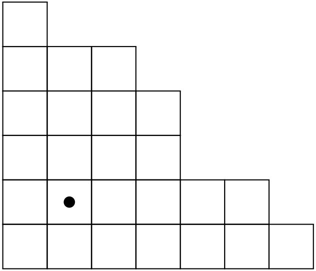
$$\Omega(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi}\left(x \cdot \arcsin\frac{x}{2} + \sqrt{4 - x^2}\right) & \text{otherwise.} \end{cases}$$

Theorem (Vershik-Kerov, Logan-Shepp '77)

Let $\lambda_{(n)}$ be a random Young diagram of size n distributed with Plancherel measure $\mathbb{P}_n^{(1)}$. Then, in probability, as $n \to \infty$

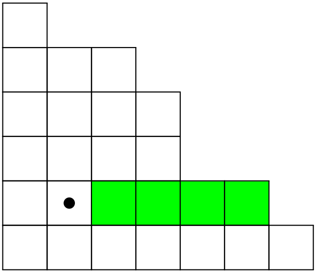
$$\left\| \omega\left(D_{1/\sqrt{n}}(\lambda_{(n)})\right) - \Omega \right\| \to 0.$$

Plancherel measure



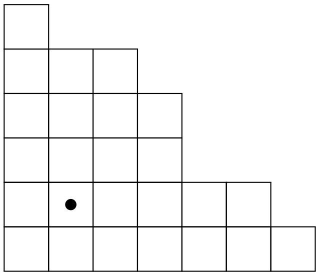
$a(\bullet) =$ number of boxes to the right of the given box

Plancherel measure



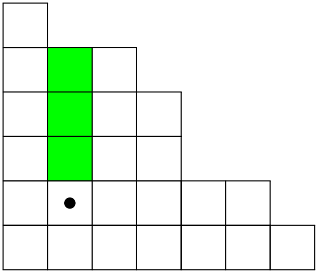
$a(\bullet) = 4$

Plancherel measure



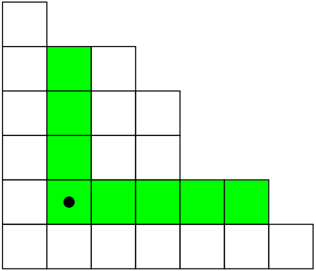
$\ell(\bullet)$ = number of boxes above the given box

Plancherel measure



$\ell(\bullet) = 3$

Plancherel measure

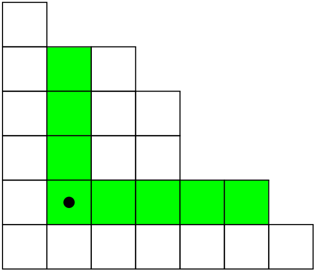


$$\mathbb{P}_n^{(1)}(\lambda) = \frac{\dim(\lambda)^2}{n!},$$

where (hook formula:)

$$\dim(\lambda) = \frac{n!}{\prod_{\square\in\lambda}(a(\square)+\ell(\square)+1)}.$$

Plancherel measure



$$\mathbb{P}_n^{(1)}(\lambda) = \frac{\dim(\lambda)^2}{n!},$$

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$$\mathbb{P}_n^{(1)}(\lambda) = \frac{n!}{\prod_{\square \in \lambda} (a(\square) + \ell(\square) + 1)^2}.$$

Jack measure

- Jack measure is a probability measure on the set $\mathbb{Y}_n$ defined by

$$\mathbb{P}_n^{(\alpha)}(\lambda) := \frac{\alpha^n n!}{\prod_{\square\in\lambda}(\alpha a(\square) + \ell(\square) + 1)(\alpha a(\square)) + \ell(\square) + \alpha)},$$

  where $\alpha \in \mathbb{R}_+$;
- for $\alpha = 1$ Jack measure $\equiv$ Plancherel measure.

Theorem (D., Féray)

Let $\lambda_{(n)}$ be a random Young diagram of size n distributed with Jack measure $\mathbb{P}_n^{(\alpha)}$. Then, in probability, as $n \to \infty$

$$\left\|\omega\left(D_{1/\sqrt{n}}(\lambda_{(n)}^{(\alpha)})\right) - \Omega\right\| \to 0.$$

Central limit theorem

- $\Delta(\lambda)(x) := \sqrt{n}\frac{\omega\left(D_{1/\sqrt{n}}(\lambda)\right)(x)-\Omega(x)}{2}$;
- $u_k(x) = U_k(x/2) = \sum_{0\leq j\leq \lfloor k/2\rfloor}(-1)^j\binom{k-j}{j}x^{k-2j}$;
- $u_k(2\cos(\theta)) = \frac{\sin((k+1)\theta)}{\sin(\theta)}$;
- $u_k(\lambda) = \int_{\mathbb{R}} u_k(x)\Delta(\lambda)(x)\,dx$.

Theorem (Kerov, 1993)

Choose a sequence $(\Xi_k)_{k=2,3,\ldots}$ of independent standard Gaussian random variables and let $\lambda_{(n)}$ be a random Young diagram of size n distributed with Plancherel measure. As $n\to\infty$, we have:

$$\left(u_k(\lambda_{(n)})\right)_{k=1,2,\ldots} \xrightarrow{d} \left(\frac{\Xi_{k+1}}{\sqrt{k+1}}\right)_{k=1,2,\ldots}.$$

Central limit theorem

Theorem (D. Féray)

Choose a sequence $(\Xi_k)_{k=2,3,\dots}$ of independent standard Gaussian random variables and let $\lambda_{(n)}$ be a random Young diagram of size n distributed with Jack measure. As $n \to \infty$, we have:

$$\left(u_k^{(\alpha)}(\lambda_{(n)})\right)_{k=1,2,\dots} \xrightarrow{d} \left(\frac{\Xi_{k+1}}{\sqrt{k+1}} - \frac{\gamma}{k+1}[k \text{ is odd}]\right)_{k=1,2,\dots},$$

where $u_k^{(\alpha)}(\lambda) = \int_{\mathbb{R}} u_k(x)\Delta(\lambda^{(\alpha)})(x)\, dx$, $\gamma := \sqrt{\alpha} - \sqrt{\alpha}^{-1}$, and we use the usual notation [condition] for the indicator function of the corresponding condition.

Symmetric vs shifted-symmetric functions

Symmetric functions:	Shifted symmetric functions:
$f = (f_1, f_2, \dots)$ such that $f_i \in R[x_1, \dots, x_i]$;	$f = (f_1, f_2, \dots)$ such that $f_i \in R[x_1, \dots, x_i]$;
$f_{i+1}(x_1, \dots, x_i, 0) = f_i(x_1, \dots, x_i)$;	$f_{i+1}(x_1, \dots, x_i, 0) = f_i(x_1, \dots, x_i)$;
$f_i(x_1, \dots, x_i)$ is symmetric in $x_1, \dots, x_i$;	$f_i(x_1 - 1/\alpha, x_2 - 2/\alpha, \dots, x_i - i/\alpha)$ is symmetric in $x_1, \dots, x_i$;
$\left(J_\mu^{(\alpha)}\right)_\mu$ - linear basis of Jack symmetric functions	$\left(\mathrm{Ch}_\mu^{(\alpha)}\right)_\mu$ - linear basis of Jack characters

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• $f_i(x_1, \dots, x_i)$ is symmetric in $x_1, \dots, x_i$;	• $f_i(x_1 - 1/\alpha, x_2 - 2/\alpha, \dots, x_i - i/\alpha)$ is symmetric in $x_1, \dots, x_i$;
• $\left(J_\mu^{(\alpha)}\right)_\mu$ - linear basis of Jack symmetric functions	• $\left(\mathrm{Ch}_\mu^{(\alpha)}\right)_\mu$ - linear basis of Jack characters

Reduction for graded algebras

- Proving some properties of the elements $(u_k)_k$, which form a basis of the algebra A - HARD;
- Proving same properties of the elements $(M_k)_k$, which form a basis of the algebra A - EASY;
- Define gradation on algebra A such that

$$u_k = M_k + \text{ terms of lower degree};$$

- Deducing required proporties of the elements $(u_k)_k$.

Reduction for graded algebras

Example

- $\Lambda_\star^{(\alpha)} \subset \left(\Lambda_\star^{(\alpha)}\right)^{\text{ext}}$ - localisation over the $(\sqrt{\mathrm{Ch}_{(1)}^{(\alpha)}})$;
- $\mathrm{Ch}_{(1)}^{m/2}\,\widetilde{\mathrm{Ch}}_\mu^{(\alpha)} := \mathrm{Ch}_{(1)}^{m/2}\prod_{i=1}^{\ell}\mathrm{Ch}_{(\mu_i)}^{(\alpha)}$ - linear basis of $\left(\Lambda_\star^{(\alpha)}\right)^{\text{ext}}$, where $m_1(\mu) = 0$, $m \in \mathbb{Z}$;
- $\deg\left(Ch_{(1)}^{m/2}\widetilde{\mathrm{Ch}}_\mu^{(\alpha)}\right) = m + |\mu|$;
- $\left(\Lambda_\star^{(\alpha)}\right)^{\text{ext}} \ni u_k^{(\alpha)} = \frac{\mathrm{Ch}_{(k+1)}^{(\alpha)}}{(k+1)\,\mathrm{Ch}_{(1)}^{(k+1)/2}} - \frac{\gamma}{k+1}[k \text{ is odd}]$

$+$ terms of negative degree;

Reduction for graded algebras

Theorem (D., Féray)

Choose a sequence $(\Xi_k)_{k=2,3,\dots}$ of independent standard Gaussian random variables. As $n \to \infty$, we have:

$$\left(\frac{\mathrm{Ch}^{(\alpha)}_{(k)}(\lambda_{(n)})}{\sqrt{k}n^{k/2}} \right)_{k=2,3,\dots} \xrightarrow{d} (\Xi_k)_{k=2,3,\dots},$$

where the distribution of $\lambda_{(n)}$ is Jack measure of size n and where $\xrightarrow{d}$ means convergence in distribution of the finite-dimensional law.

$$\mathbb{E}_{\mathbb{P}^{(\alpha)}_n}(\mathrm{Ch}^{(\alpha)}_\mu) = \begin{cases} n(n-1)\cdots(n-k+1) & \text{if } \mu = 1^k \text{ for some } k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Trick with polynomial interpolation

Theorem (D., Féray)

Let

$$\mathrm{Ch}^{(\alpha)}_{\mu}\,\mathrm{Ch}^{(\alpha)}_{\nu} = \sum_{\rho} g^{(\alpha)}_{\mu,\nu;\pi}\,\mathrm{Ch}^{(\alpha)}_{\pi}.$$

Then, structure constants $g^{(\alpha)}_{\mu,\nu;\pi}$ are polynomials in $\gamma := \alpha^{1/2} - \alpha^{-1/2}$ of degree less than

$$\min_{i=1,2,3} \left(n_i(\mu) + n_i(\nu) - n_i(\pi)\right),$$

with rational coefficients, where $n_i(\lambda)$ - natural valued function of λ.

Trick with polynomial interpolation

Let $\mu, \nu, \pi \in \mathbb{Y}_n$.

$$c^{(\alpha)}_{\mu,\nu;\pi} = \frac{\alpha^{d(\mu,\nu;\pi)/2}}{z_{\tilde{\mu}} z_{\tilde{\nu}}} \sum_{0 \le i \le m_1(\pi)} g^{(\alpha)}_{\tilde{\mu},\tilde{\nu};\tilde{\pi}1^i} \cdot z_{\tilde{\pi}} \cdot i! \cdot \binom{n - |\tilde{\pi}|}{i},$$

where

- $\tilde{\mu}$ is created from μ by removing all parts equal to 1,
- $z_\mu = \mu_1 \mu_2 \cdots m_1(\mu)! m_2(\mu)! \cdots,$
- $m_i(\mu)$ - the number of parts equal to i in μ,
- $d(\mu, \nu; \pi) = |\mu| - \ell(\mu) + |\nu| - \ell(\nu) - (|\pi| - \ell(\pi)).$

Trick with polynomial interpolation

Let $\mu, \nu, \pi \in \mathbb{Y}_n$.

$$c^{(\alpha)}_{\mu,\nu;\pi} = \frac{\alpha^{d(\mu,\nu;\pi)/2}}{z_{\tilde{\mu}} z_{\tilde{\nu}}} \sum_{0 \leq i \leq m_1(\pi)} g^{(\alpha)}_{\tilde{\mu},\tilde{\nu};\tilde{\pi}1^i} \cdot z_{\tilde{\pi}} \cdot i! \cdot \binom{n - |\tilde{\pi}|}{i},$$

- LHS and RHS of the equation above are polynomials in n;
- knowing $c^{(\alpha)}_{\mu,\nu;\pi}$ one can calculate $g^{(\alpha)}_{\tilde{\mu},\tilde{\nu};\tilde{\pi}1^i}$;
- $c^{(\alpha)}_{\mu,\nu;\pi}$ have combinatorial interpretation for $\alpha = 1, 2, 1/2$.

$\alpha = 1$ - Structure contants of the $Z(\mathbb{C}[\mathfrak{S}_n])$

Let $\mathbb{C}[\mathfrak{S}_n] := \{f : f : \mathfrak{S}_n \to \mathbb{C}\}$ be a group algebra of the symmetric group. This is algebra with the multiplication defined by:

$$f \cdot g(\sigma) := \sum_{\sigma_1\sigma_2=\sigma} f(\sigma_1)g(\sigma_2).$$

Let

$$Z(\mathbb{C}[\mathfrak{S}_n]) := \{f \in \mathbb{C}[\mathfrak{S}_n] : \forall g \in \mathbb{C}[\mathfrak{S}_n], fg = gf\}$$

be the center of that algebra. It has a basis $(f_\mu)_{|\mu|=n}$, where

$$f_\mu(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ has cycle type } \mu, \\ 0 & \text{otherwise.} \end{cases}$$

$\alpha = 1$ - Structure contants of the $Z(\mathbb{C}[\mathfrak{S}_n])$

Let

$$f_\mu f_\nu = \sum_{|\rho|=n} c_{\mu,\nu;\rho} f_\rho .$$

Lemma

The structure constant $c_{\mu,\nu;\rho}$ is equal to the number of pairs of permutation (σ_1, σ_2) such that

- σ_1 *has cycle type* μ,
- σ_2 *has cycle type* ν,
- $\sigma_1\sigma_2 = \sigma$, *where* σ *is a fixed permutation of the cycle-type* ρ.

$\alpha = 1$ - Structure contants of the $Z(\mathbb{C}[\mathfrak{S}_n])$

One has a following relation:

$$c^{(1)}_{\mu,\nu;\rho} = c_{\mu,\nu;\rho}.$$

Remark

From the previous theorem and a relation above one can deduce a classical result of Farahat and Higman: $c_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\rho 1^{n-|\rho|}}$ is a polynomial in n.

$\alpha = 2$ - Structure contants of the Hecke algebra of $(\mathfrak{S}_{2n}, H_n)$

Let $\mathfrak{S}_{2n}$ acts on the set $X_n := \{1, \bar{1}, \dots, n, \bar{n}\}$ by permutations and let

$$\mathfrak{S}_{2n} > H_n := \{\sigma \in \mathfrak{S}_{2n} : \forall i \in X_n \sigma(\bar{i}) = \overline{\sigma(i)}$$

be a hyperoctahedral subgroup.
Hecke algebra $\mathbb{C}[H_n \backslash \mathfrak{S}_{2n} / H_n] < \mathbb{C}[\mathfrak{S}_{2n}]$ of the pair $(\mathfrak{S}_{2n}, H_n)$ is defined by:

$$\mathbb{C}[H_n \backslash \mathfrak{S}_{2n} / H_n] := \{x \in \mathbb{C}[\mathfrak{S}_{2n}] : hxh' = x \forall h, h' \in H_n\}.$$

Double-cosets: equivalence classes for the relation $x \sim hxh'$ (for $x \in \mathfrak{S}_{2n}$ and $h, h' \in H_n$)

- naturally indexed by partitions of size n;
- $F_\mu = \sum_{x \text{ of type } \mu} \delta_x$ - linear basis of $\mathbb{C}[H_n \backslash \mathfrak{S}_{2n} / H_n]$.

$\alpha = 2$ - Structure contants of the Hecke algebra of $(\mathfrak{S}_{2n}, H_n)$

Let

$$F_\mu F_\nu = \sum_{|\rho|=n} h_{\mu,\nu;\rho} F_\rho.$$

Then

$$c^{(2)}_{\mu,\nu;\rho} = \frac{h_{\mu,\nu;\rho}}{2^n n!}.$$

Remark

From the previous theorem and a relation above one can deduce a result of Tout (2013):

$$\frac{h_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\pi 1^{n-|\pi|}}}{n!\ 2^n}$$

is a polynomial in n.

$\alpha = 2$ - Structure contants of the Hecke algebra of $(\mathfrak{S}_{2n}, H_n)$

- $\mathcal{F}_\mathcal{S}$ - the set of all (perfect) matchings on a set $\mathcal{S}$;
- $G(F_1, \ldots, F_k)$ - the multigraph with vertex-set $\mathcal{S}$ whose edges are formed by the pairs in $F_1, \ldots, F_k \in \mathcal{F}_\mathcal{S}$;
- The components of $G(F_1, F_2)$ are even cycles. Let the list of their lengths in weakly decreasing order be $(2\theta_1, 2\theta_2, \ldots) = 2\theta$, and define Λ by $\Lambda(F_1, F_2) = \theta$;
- $\mathcal{F}_n$ - the set of all matchings on the set $\{1, 2 \ldots, 2n\}$.

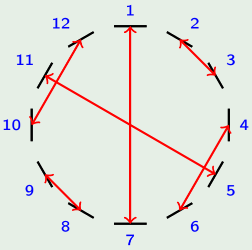
Lemma (Goulden, Jackson 1996)

Let F_1, F_2 be two fixed matchings in $\mathcal{F}_n$ such that $\Lambda(F_1, F_2) = \pi$, where $|\pi| = n$. Then, for any μ, ν of size n we have

$$h_{\mu,\nu;\pi} = 2^n n! |\{F_3 \in \mathcal{F}_n : \Lambda(F_1, F_3) = \mu, \Lambda(F_2, F_3) = \nu\}|.$$

$\alpha = 2$ - Structure contants of the Hecke algebra of $(\mathfrak{S}_{2n}, H_n)$

Example



- $F_1 = \{\{1,7\},\{2,3\},$
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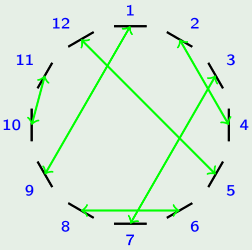

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  $\{3,7\},\{5,12\},\{6,8\},\{10,11\}\}$

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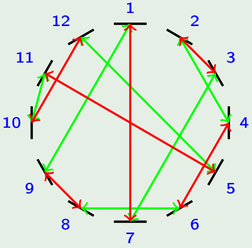

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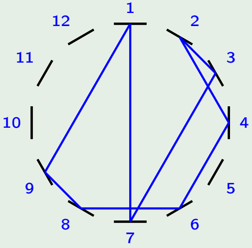
- $F_2 = \{\{1,9\},\{2,4\},$
  $\{3,7\},\{5,12\},\{6,8\},\{10,11\}\}$
- $\Lambda(F_1, F_2) = (,)$

$\alpha = 2$ - Structure contants of the Hecke algebra of $(\mathfrak{S}_{2n}, H_n)$

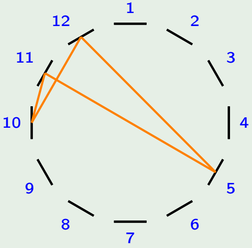
Example



- $F_1 = \{\{1,7\},\{2,3\},\{4,6\},\{5,11\},\{8,9\},\{10,12\}\}$
- $F_2 = \{\{1,9\},\{2,4\},\{3,7\},\{5,12\},\{6,8\},\{10,11\}\}$
- $\Lambda(F_1, F_2) = (4,)$

$\alpha = 2$ - Structure contants of the Hecke algebra of $(\mathfrak{S}_{2n}, H_n)$

Example



- $F_1 = \{\{1,7\},\{2,3\},$ $\{4,6\},\{5,11\},\{8,9\},\{10,12\}\}$
- $F_2 = \{\{1,9\},\{2,4\},$ $\{3,7\},\{5,12\},\{6,8\},\{10,11\}\}$
- $\Lambda(F_1, F_2) = (4, 2)$

Main steps in the proof of the main Theorem

- We want to estimate some mixed moments of $\mathrm{Ch}^{(\alpha)}_{(k)}$;
- $\mathbb{E}_{\mathbb{P}^{(\alpha)}_n}\left(\mathrm{Ch}^{(\alpha)}_{(k_1)} \cdots \mathrm{Ch}^{(\alpha)}_{(k_l)}\right)$ is a polynomial in n;
- the coefficients of the polynomial above are polynomials in γ;
- the coefficients of the dominant terms are polynomials in γ of small degree;
- the only interesting coefficients have degree bounded by 2;
- it is enough to calculate $g^{(\alpha)}_{\mu,\nu;\rho}$ for some special partitions and $\alpha = 1, 2, 1/2$;
- it is possible because of the combinatorial interpretation.

The end

Thank you for your attention.

Any questions?