Dénombrement d'arbres plus petit dans l'ordre de Tamari

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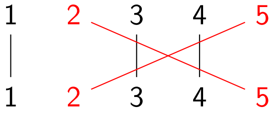
Université Paris-Est Marne-la-Vallée

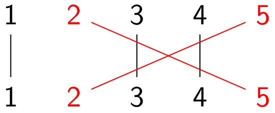
6 Novembre 2013

Transpositions



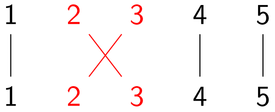
$(2, 5)$

Transpositions



$(2, 5)$

Transpositions simples

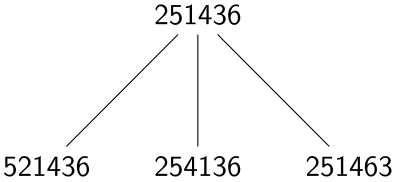


$(2, 3) = s_2$

Ordre faible droit

$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

Ordre faible droit



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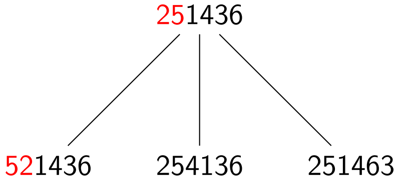

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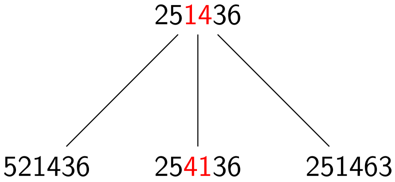

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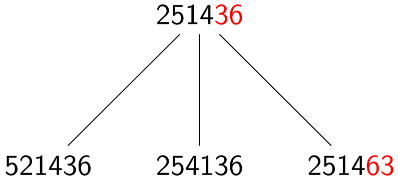
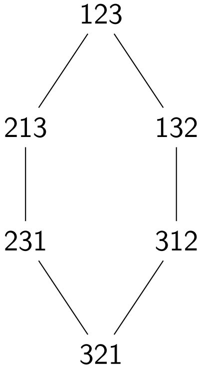
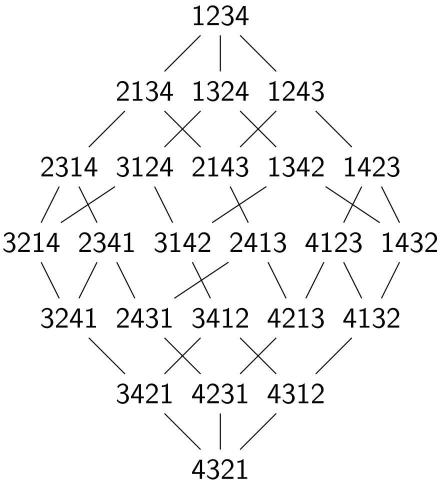
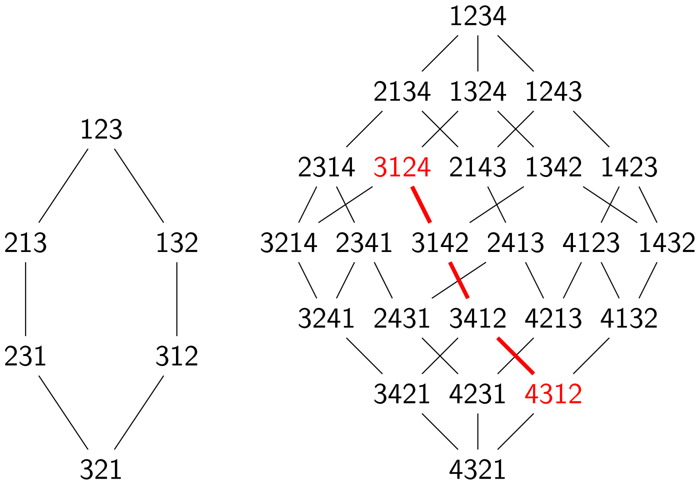


Ordre faible droit

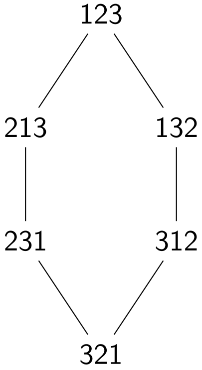


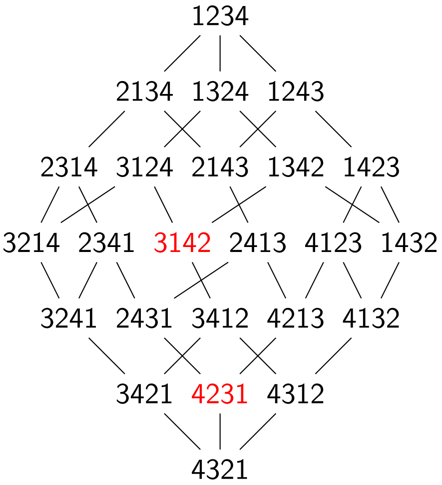


Ordre faible droit

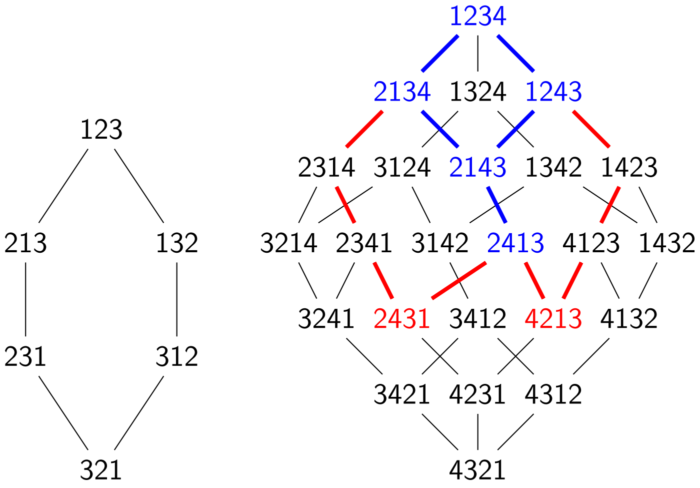


Ordre faible droit





Ordre faible droit



Arbres binaires

Définition récursive :

- l'arbre vide ou
- une racine possédant 2 sous-arbres (gauche et droit)

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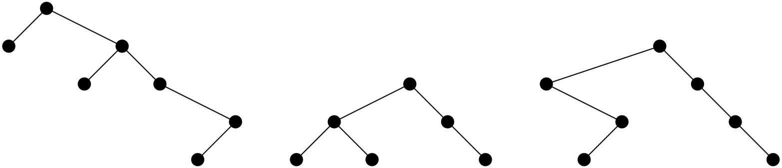
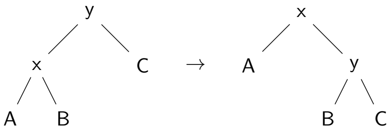
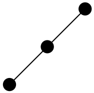
Exemples

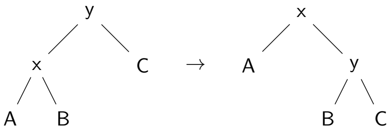


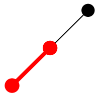
Rotation droite



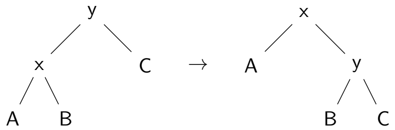


Rotation droite

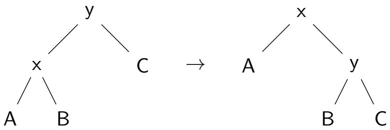


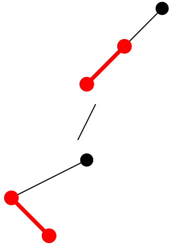


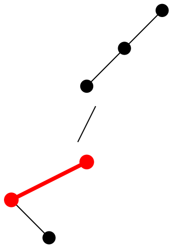
Rotation droite



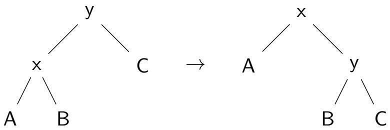
Rotation droite







Rotation droite



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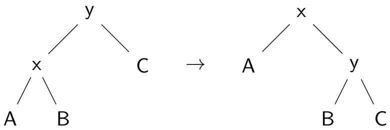

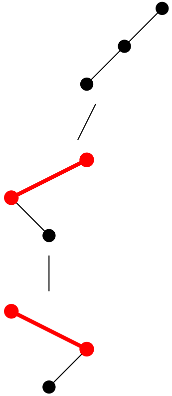

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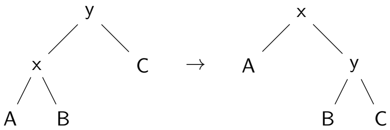

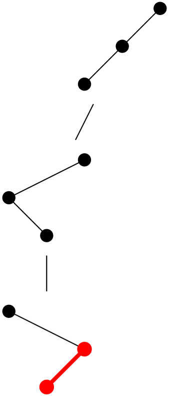

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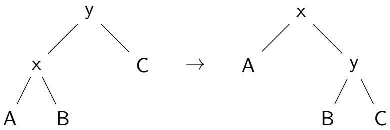
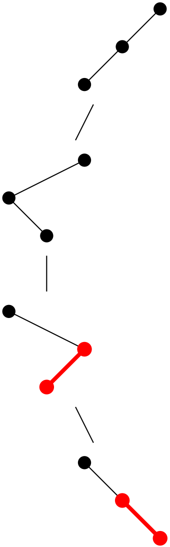
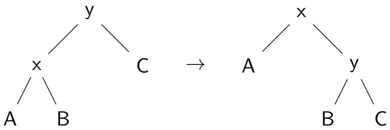
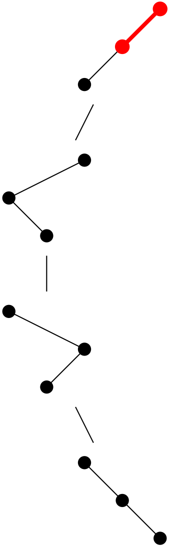


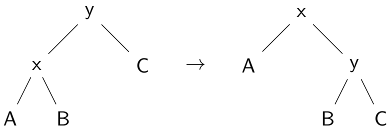


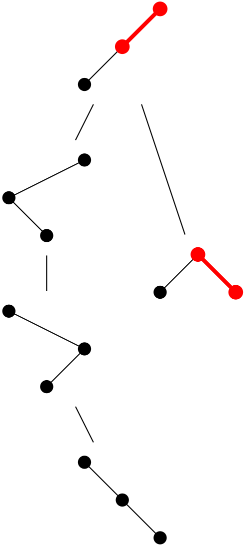
Rotation droite





Rotation droite





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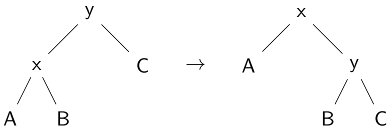

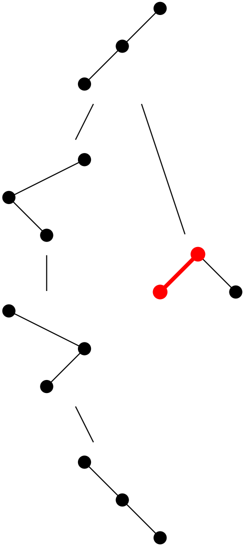

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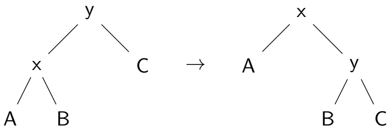

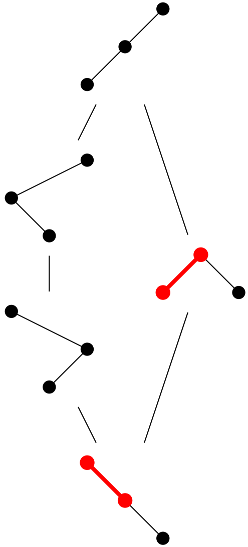

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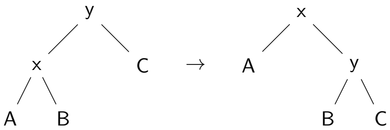

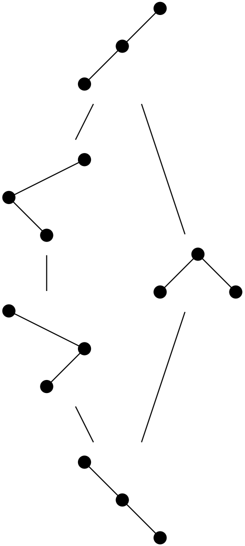

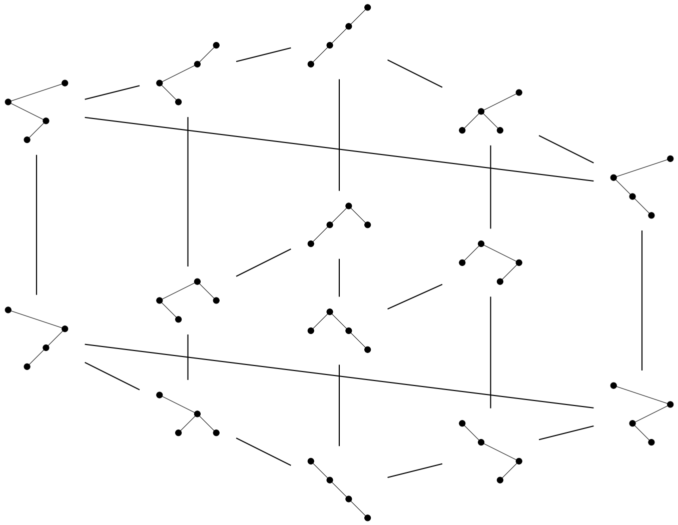

Quelques résultats sur l'ordre de Tamari

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- 1962, Tamari : ordre sur les parenthésages formels

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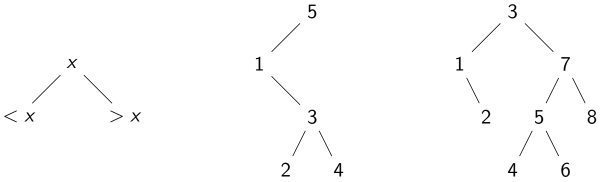
- 1962, Tamari : ordre sur les parenthésages formels
- 1972, Huang, Tamari : structure de treillis
- 2007, Chapoton : nombre d'intervalles

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

Lien avec l'ordre faible

Étiquetage canonique

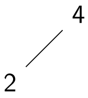


Insertion dans un arbre binaire de recherche

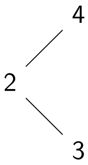
4

15324 $\rightarrow$

Insertion dans un arbre binaire de recherche

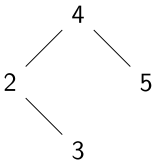
15324 → 

Insertion dans un arbre binaire de recherche

15324 $\rightarrow$ 

Insertion dans un arbre binaire de recherche

15324 $\rightarrow$



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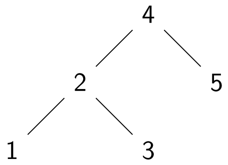

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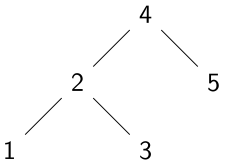
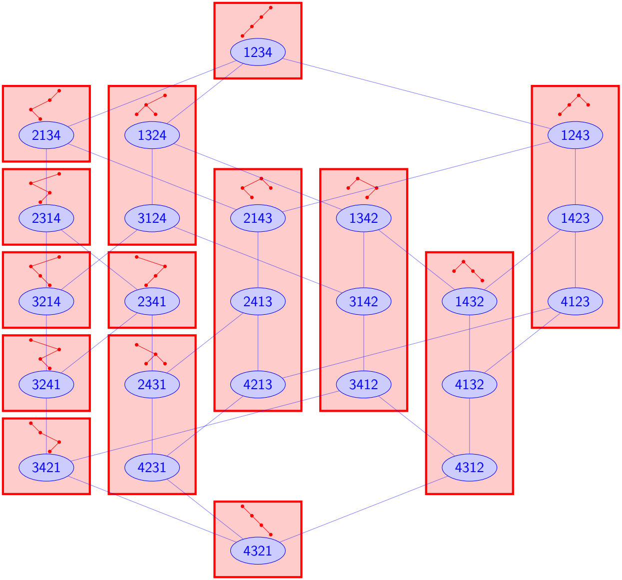


Caractérisation : les permutations qui correspondent à un arbre donné sont ses extensions linéaires

$15324, 31254, 35124, 51324, \ldots$

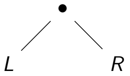

1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

Polynômes de Tamari

On définit récursivement $\mathcal{B}_T$ par

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

avec T = 

Théorème (C., Pons)

$\mathcal{B}_T(x)$ compte le nombre d'arbres inférieurs ou égaux à T dans le treillis de Tamari en fonction du nombre de nœuds sur leur branche gauche.

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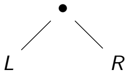

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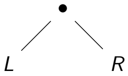

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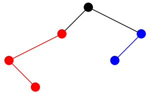

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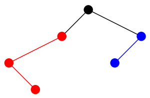

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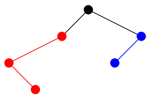

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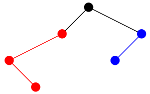
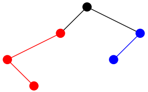
$$\mathcal{B}_R(x) = x^2$$



$$\mathcal{B}_{\emptyset} := 1$$

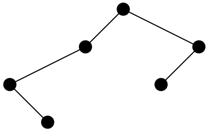
$$\mathcal{B}_T(x) := x(x^3 + x^2)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

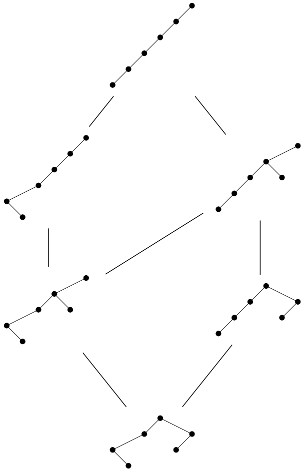
$$\mathcal{B}_R(x) = x^2$$



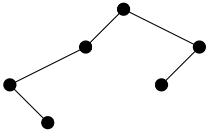
$$\mathcal{B}_\emptyset := 1$$

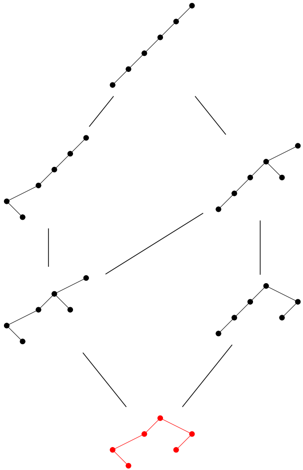
$$\mathcal{B}_T(x) := x(x^3 + x^2)(1 + x + x^2)$$





$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$





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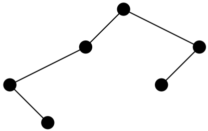
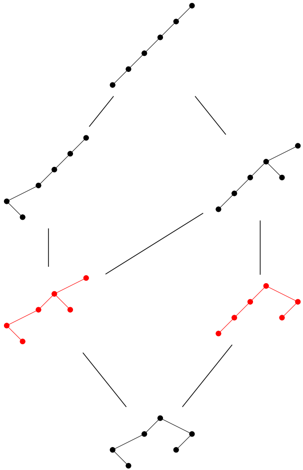
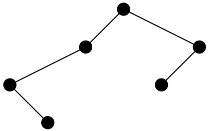
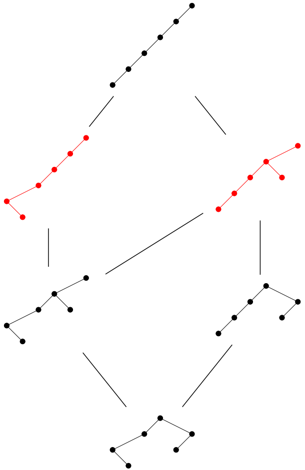


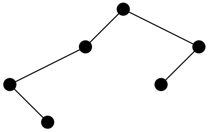


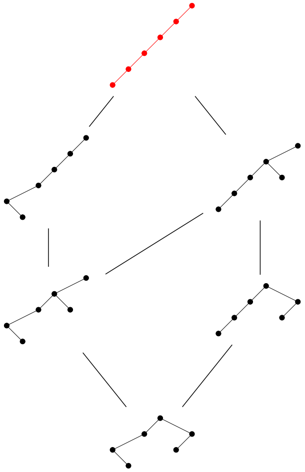
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



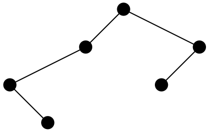


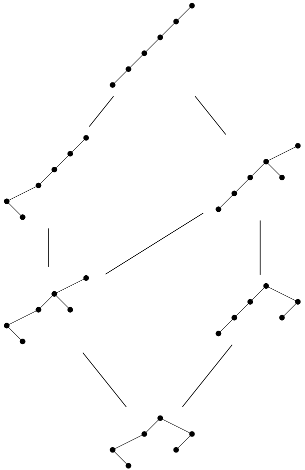
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



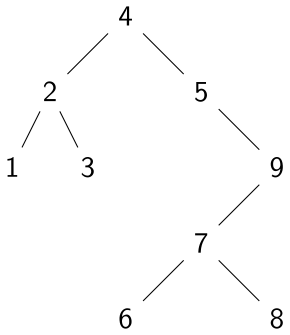


$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$





$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$
$$\mathcal{B}_T(1) = 6$$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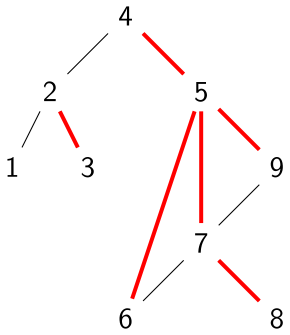
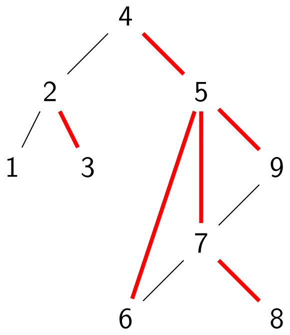
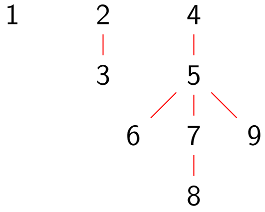
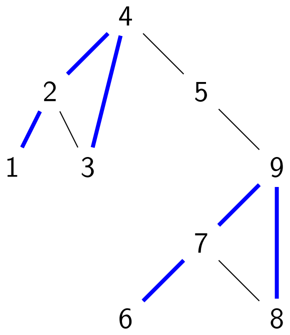
Forêt finale $F_{\geq}(T)$



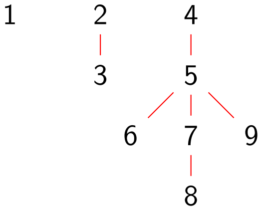


Forêt finale $F_{\geq}(T)$

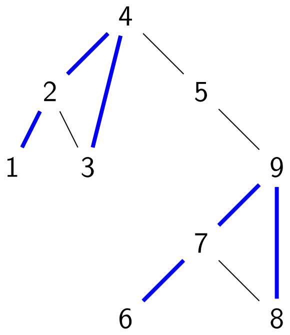




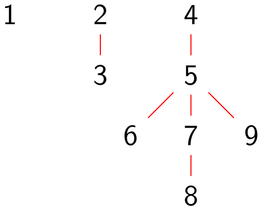
Forêt finale $F_{\geq}(T)$



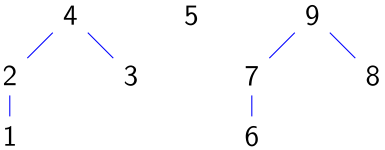
Forêt initiale $F_{\leq}(T)$

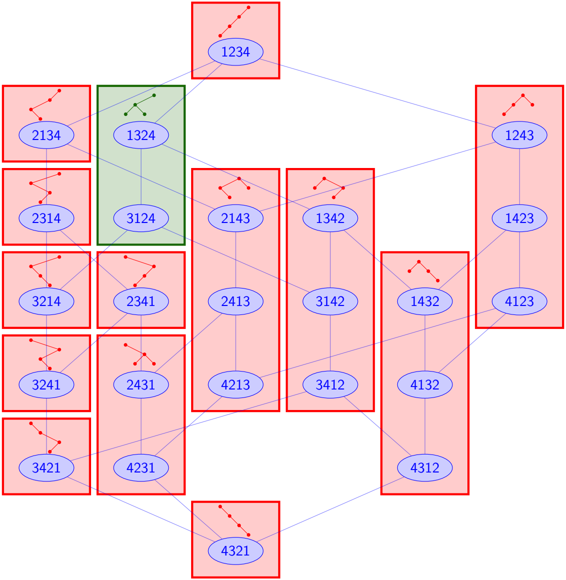


Forêt finale $F_{\geq}(T)$



Forêt initiale $F_{\leq}(T)$

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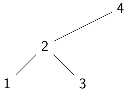
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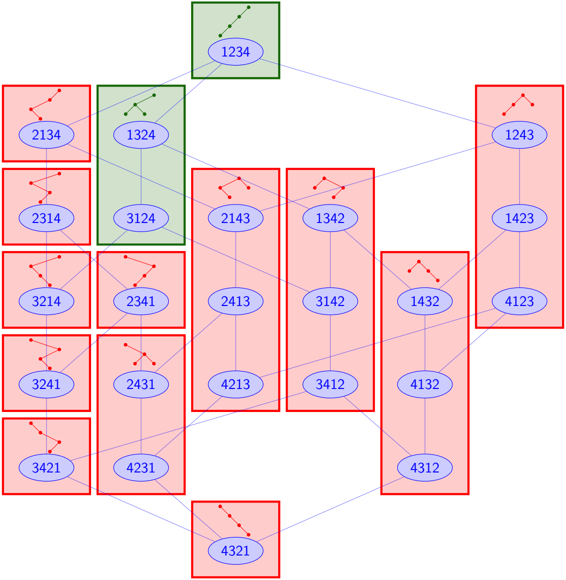
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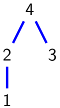
$F_{\leq}(T)$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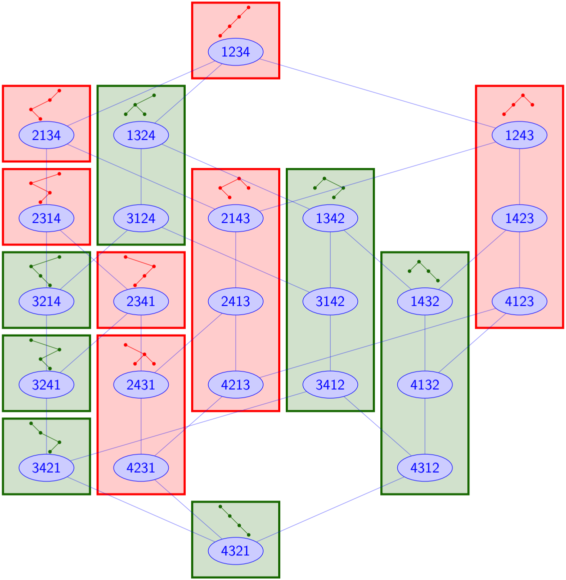
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$F_{\geq}(T)$


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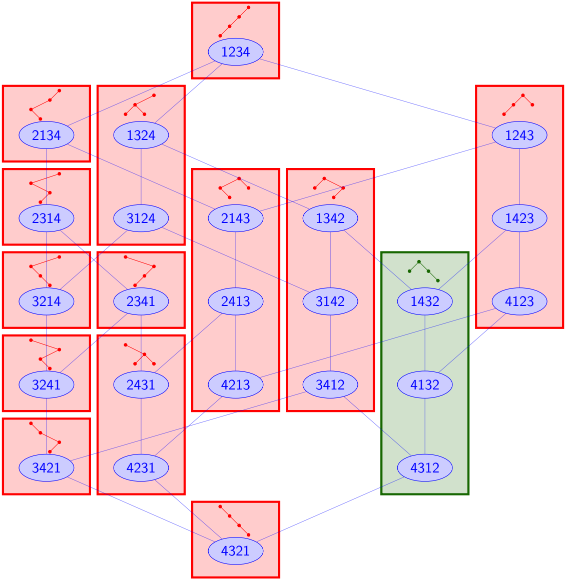
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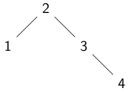
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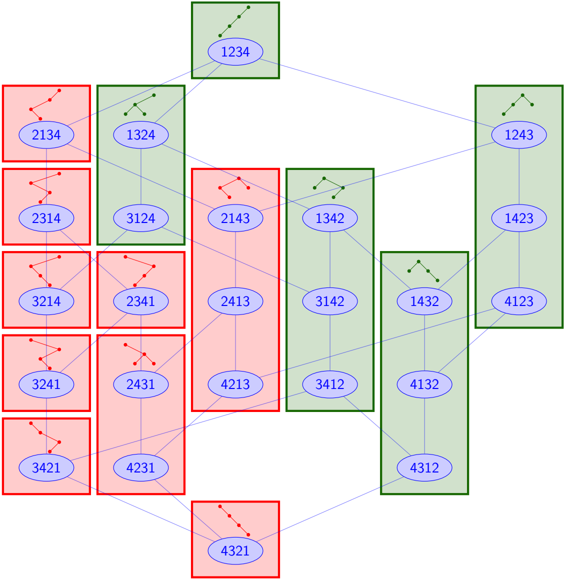
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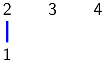
$F_{\leq}(T')$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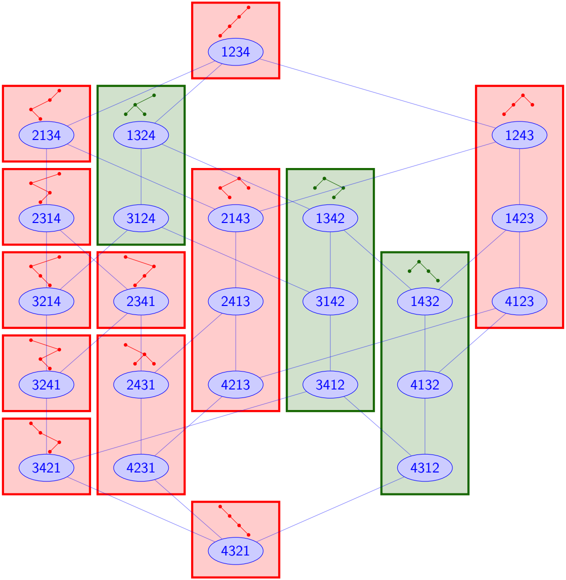
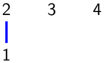
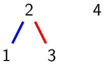


$F_{\geq}(T)$

$F_{\leq}(T')$



Intervalle-poset
$[T, T']$



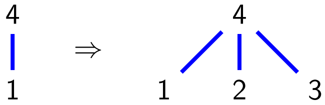
Théorème (C., Pons)

Les intervalles de Tamari sont en bijection avec les posets étiquetés sur $1, \ldots, n$ *de taille* n *vérifiant :*

Théorème (C., Pons)

Les intervalles de Tamari sont en bijection avec les posets étiquetés sur $1, \ldots, n$ *de taille* n *vérifiant :*

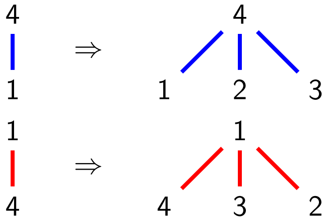
- *Si* $a < c$ *et* $a \lhd c$ *alors* $b \lhd c$ *pour tout* $a < b < c$.

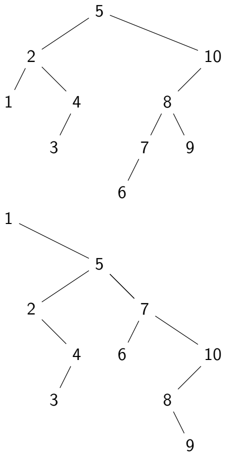


Théorème (C., Pons)

Les intervalles de Tamari sont en bijection avec les posets étiquetés sur $1, \dots, n$ *de taille* n *vérifiant :*

- *Si* $a < c$ *et* $a \lhd c$ *alors* $b \lhd c$ *pour tout* $a < b < c$.
- *Si* $a < c$ *et* $c \lhd a$ *alors* $b \lhd a$ *pour tout* $a < b < c$.

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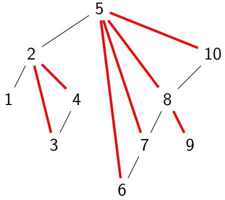
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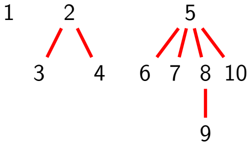
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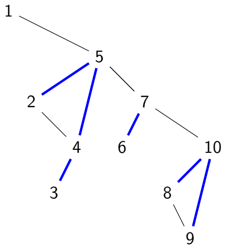
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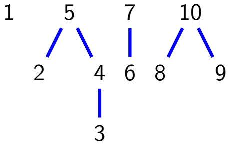
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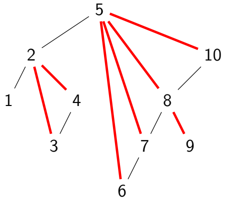
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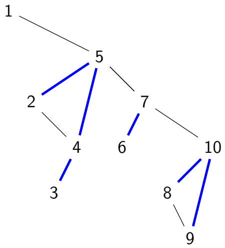
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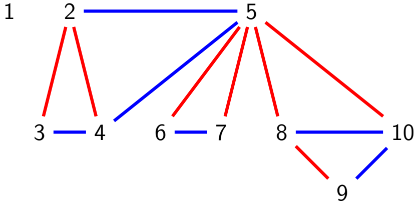
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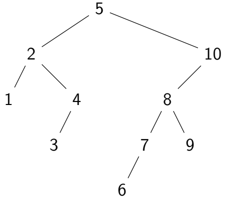
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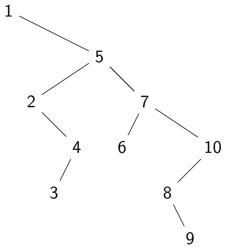
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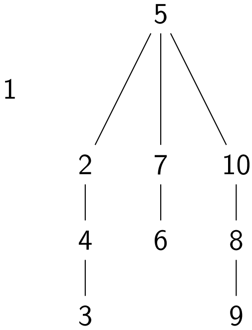
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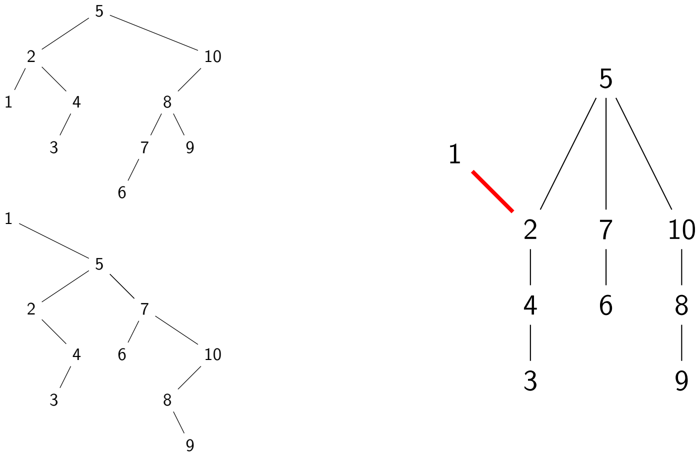
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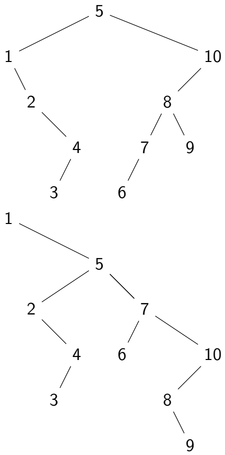
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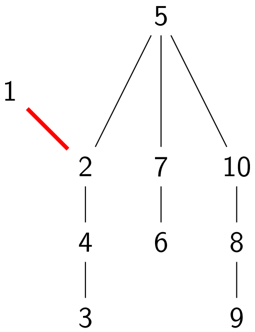
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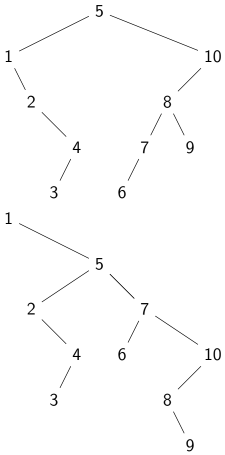
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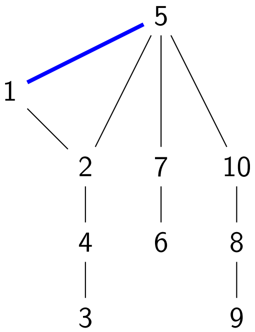
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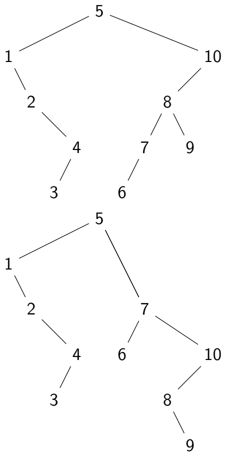
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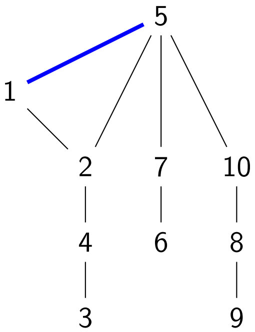
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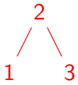
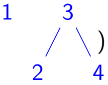
$\mathbb{B}($ 

, 

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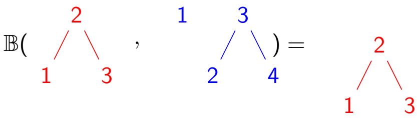
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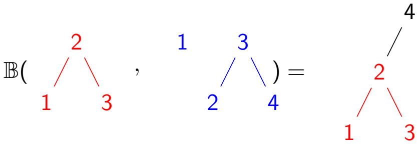
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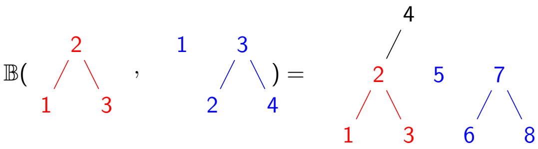

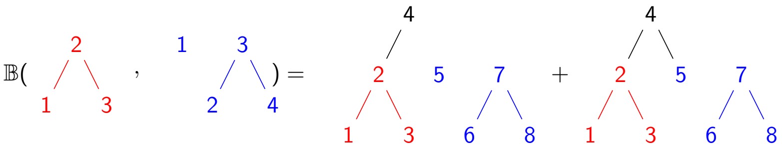
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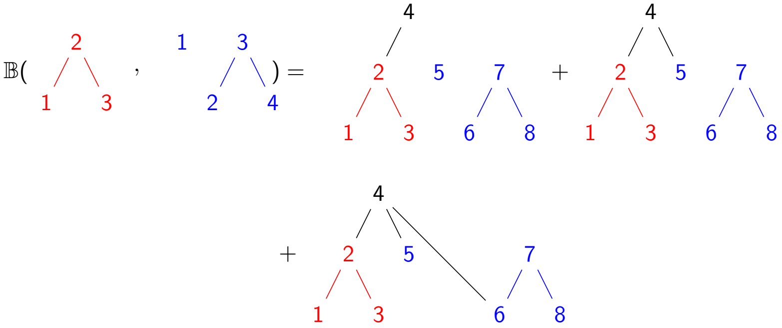

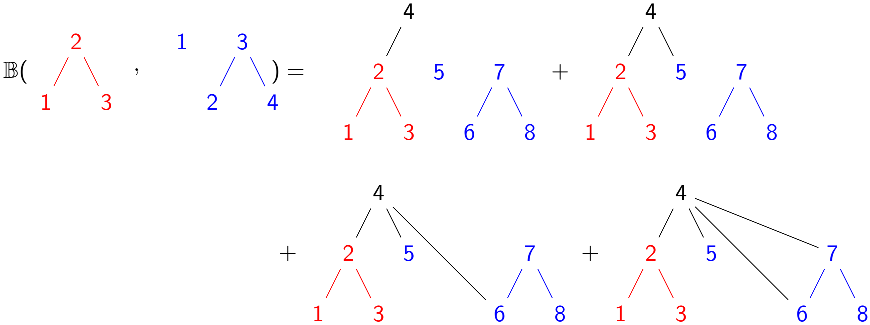
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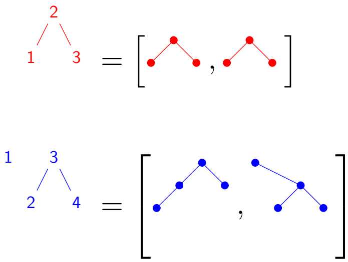
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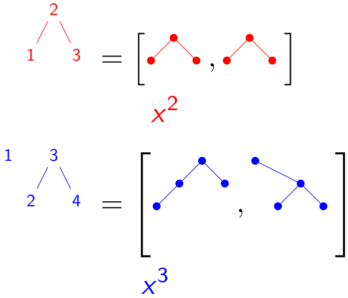
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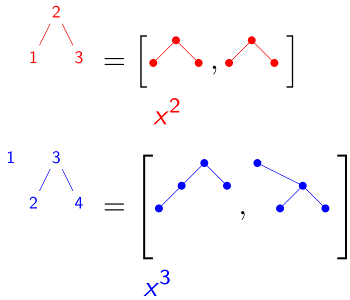
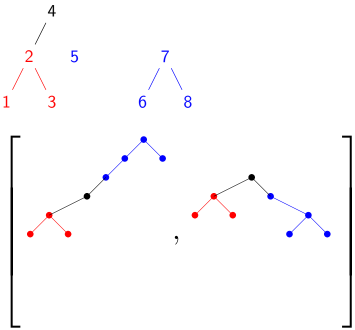
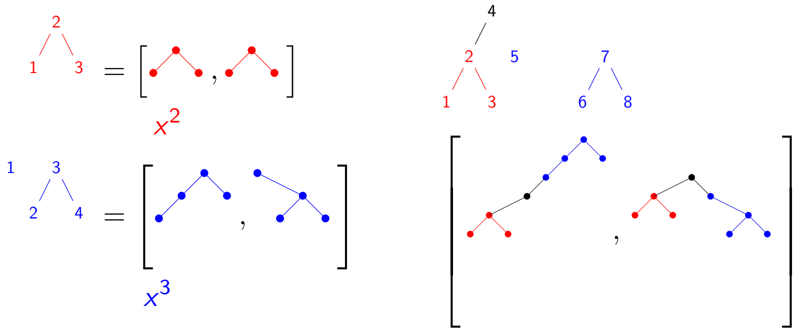
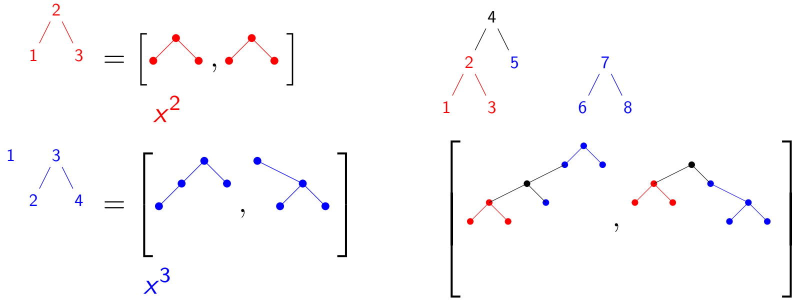

2
1
3
=
x^2
1
3
2
4
=
x^3

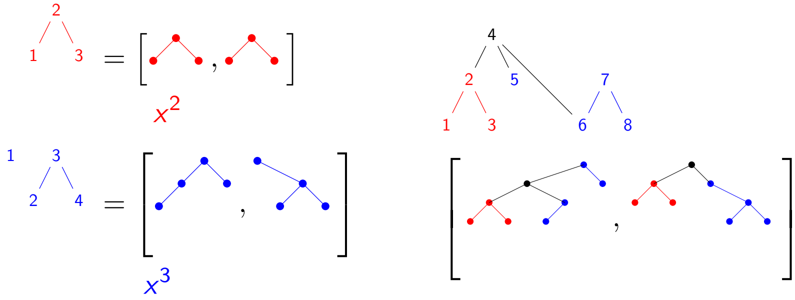

4
2
5
7
1
3
6
8



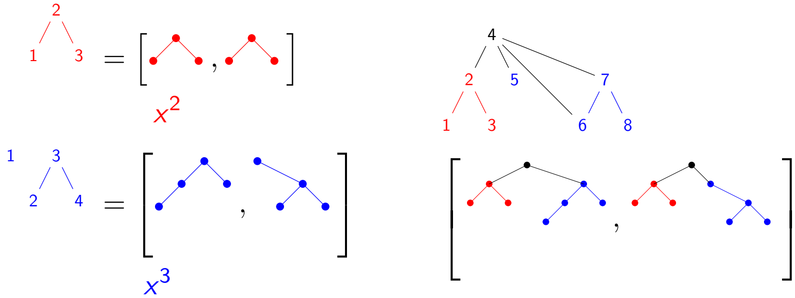
$x^2.x.x^3$



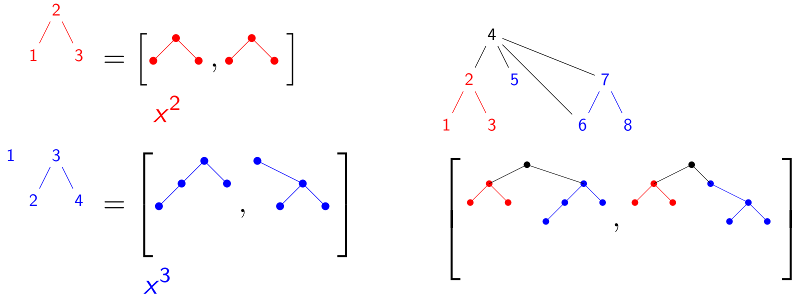
$$x^2.x.x^3 + x^2.x.x^2$$



$$x^2.x.x^3 + x^2.x.x^2 + x^2.x.x$$



$$x^2.x.x^3 + x^2.x.x^2 + x^2.x.x + x^2.x$$



$$x^2.x.(x^3 + x^2 + x + 1)$$

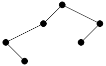
Théorème (Chapoton)

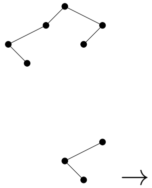
La série génératrice des intervalles de Tamari vérifie l'équation fonctionnelle

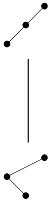
$$\Phi(x,y) = \mathrm{B}(\Phi,\Phi) + 1$$

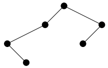
où

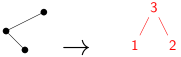
$$\mathrm{B}(f,g) = xyf(x,y)\frac{xg(x,y) - g(1,y)}{x-1}$$



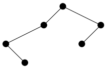


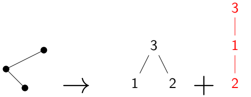




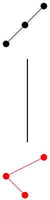


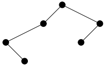
x^3

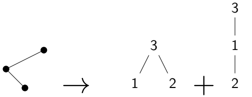




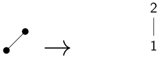
$$x^3 + x^2$$



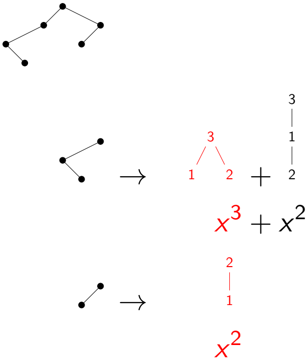


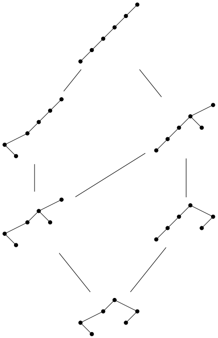


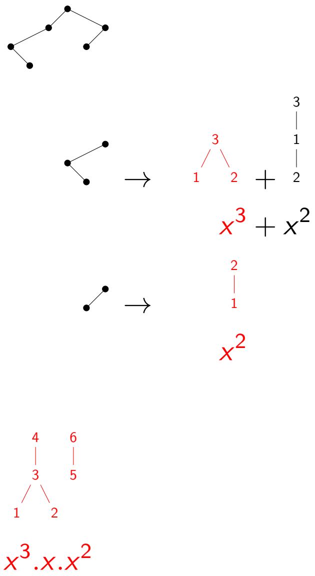
$$x^3 + x^2$$

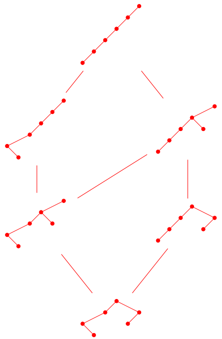


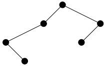
$$x^2$$

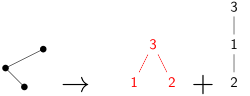

3
1
2
+
3
1
2
$x^3 + x^2$
2
1
x^2



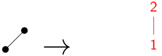

3
1
2
3
1
2
$x^3 + x^2$
2
1
x^2
4
6
3
5
1
2
$x^3.x.x^2$



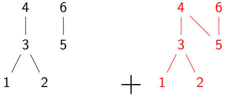




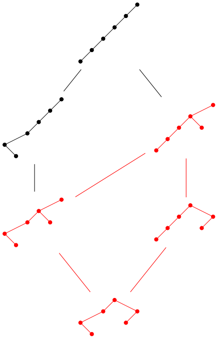
$x^3 + x^2$

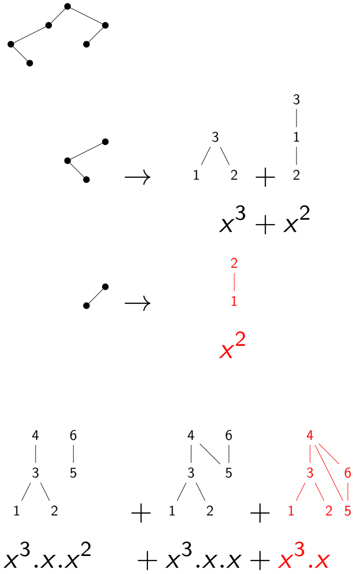


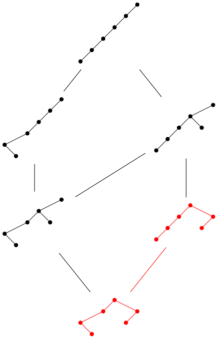
x^2

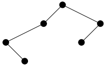


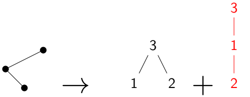
$x^3.x.x^2 \quad + x^3.x.x$



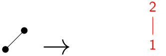

3
3
1
1
2
2
2
$x^3 + x^2$
2
1
x^2
4
6
3
5
1
2
4
6
3
5
1
2
4
3
6
1
2
5
$x^3.x.x^2 \quad + x^3.x.x + x^3.x$



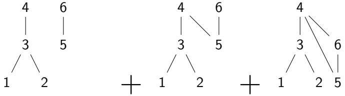




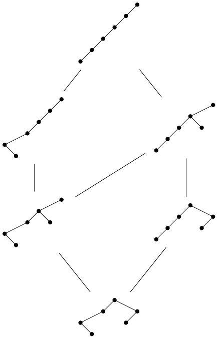
$$x^3 + x^2$$

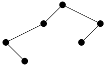


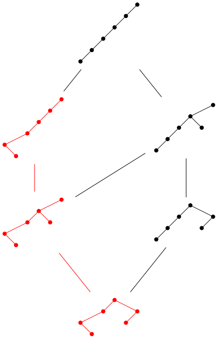
$$x^2$$

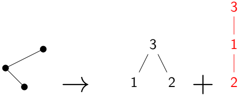


$$x^3.x.x^2 + x^3.x.x + x^3.x$$

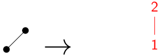




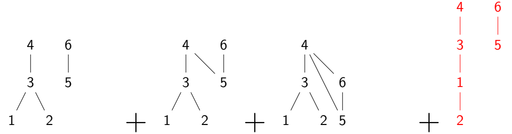




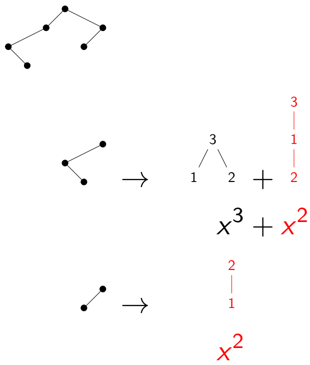
$$x^3 + x^2$$

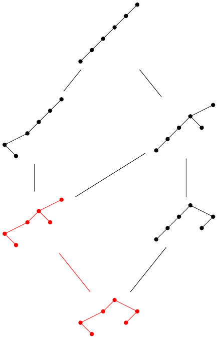


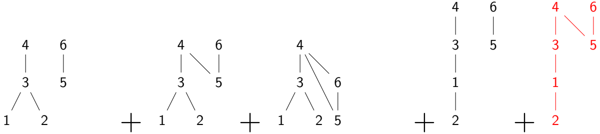
$$x^2$$



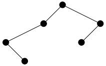
$$x^3.x.x^2 + x^3.x.x + x^3.x + x^2.x.x^2$$

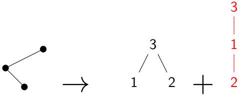




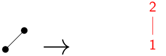


$$x^3.x.x^2 + x^3.x.x + x^3.x + x^2.x.x^2 + x^2.x.x$$

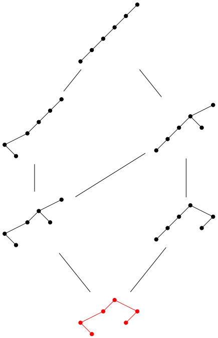


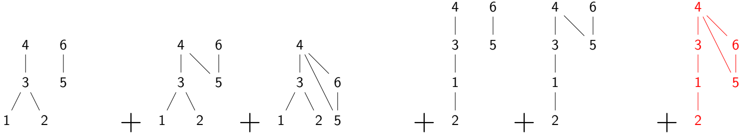


$$x^3 + x^2$$

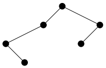


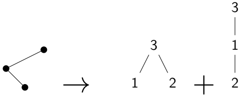
$$x^2$$



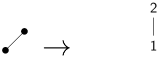


$$x^3.x.x^2 + x^3.x.x + x^3.x + x^2.x.x^2 + x^2.x.x + x^2.x$$

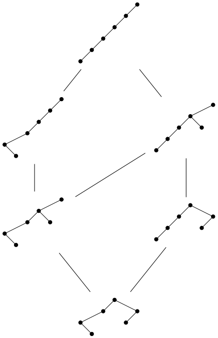




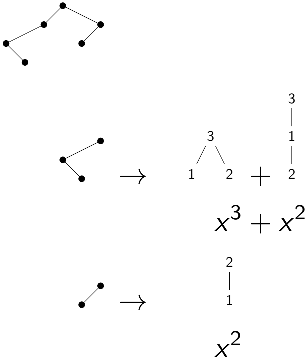
$$x^3 + x^2$$

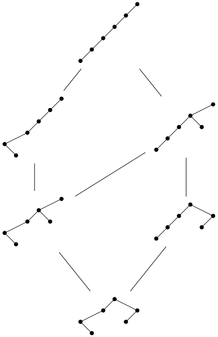


$$x^2$$



$$(x^3 + x^2).x.(x^2 + x + 1)$$





$$x\,\mathcal{B}_L(x)\frac{\mathcal{B}_R(x)-\mathcal{B}_R(1)}{x-1}$$

Posets de m-Tamari

(2011-2012) Bergeron, Préville-Ratelle, *Higher trivariate diagonal harmonics via generalized Tamari posets.*

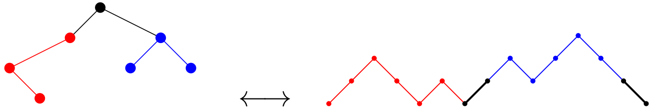
Posets de m-Tamari

(2011-2012) Bergeron, Préville-Ratelle, *Higher trivariate diagonal harmonics via generalized Tamari posets.*

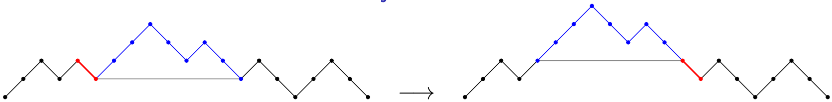
Structure de treillis, intervalles

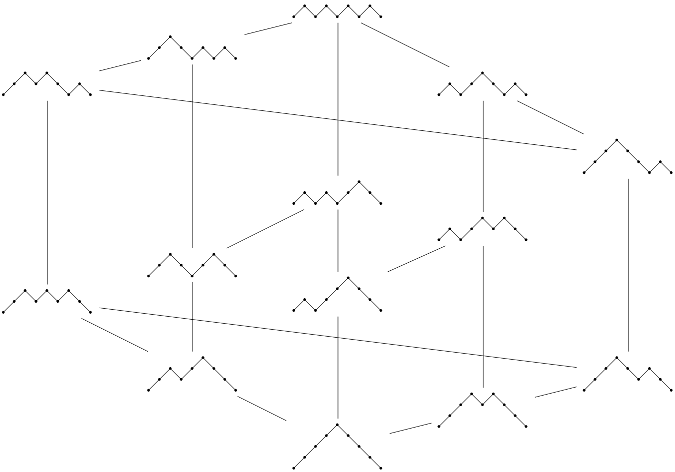
(2011) Bousquet-Mélou, Fusy, Préville-Ratelle, *The number of intervals in the m-Tamari lattices.*

Bijection arbres - chemins



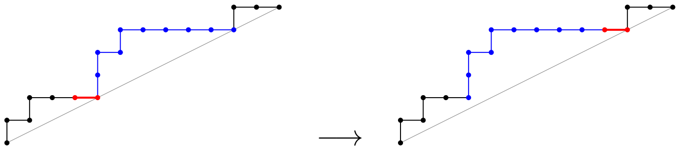
Rotation sur les chemins de Dyck

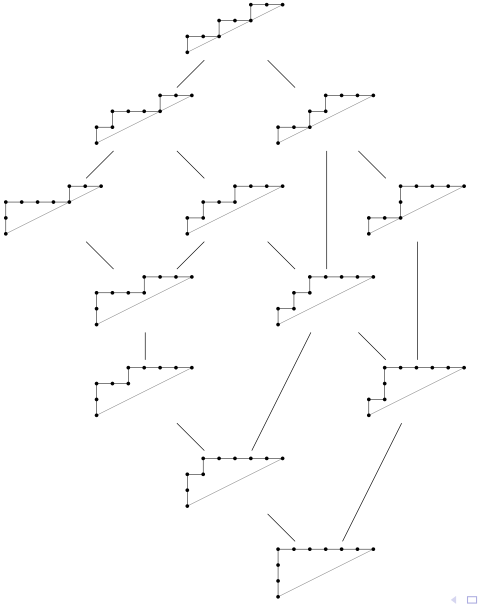




Chemins m-ballots

Exemple $m = 2$.

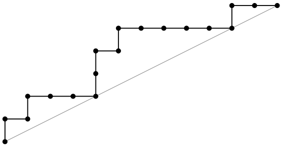




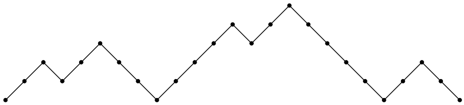
Les posets de m-Tamari sont des treillis

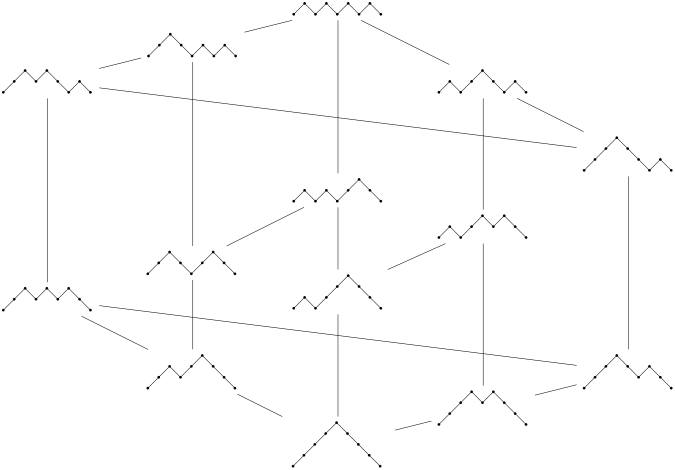
Bousquet-Mélou, Fusy, Préville-Ratelle : plongement des posets m-Tamari dans le treillis de Tamari $n \times m$.

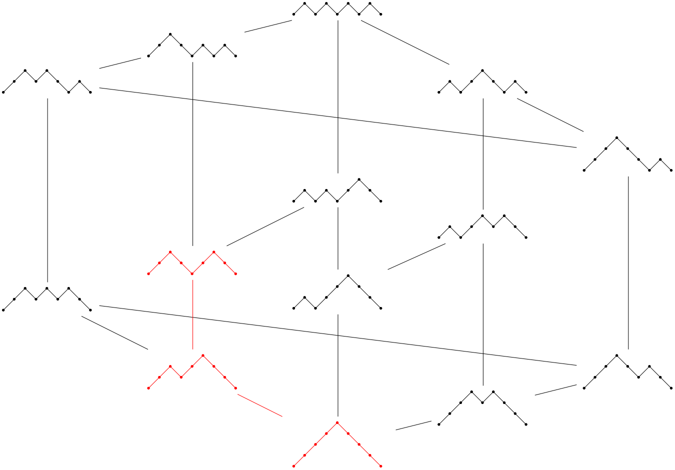
Chemin m-ballot

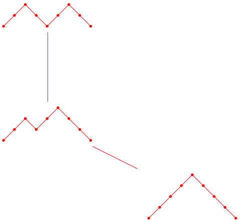


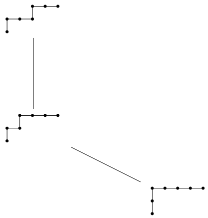
chemin de Dyck



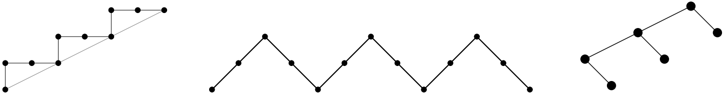


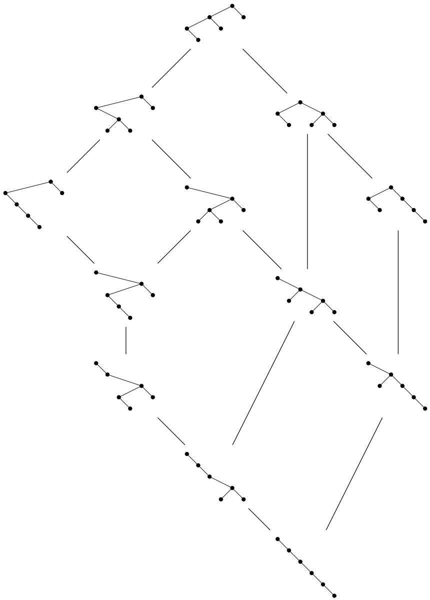




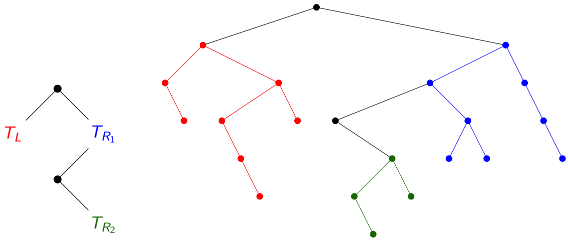


Arbres m-binaires





Structure ternaire



Intervalles-posets

Intervalles-posets

- polynômes de Tamari multivariés et flots

Intervalles-posets

- polynômes de Tamari multivariés et flots
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Structures "m"

- structures algébriques

Intervalles-posets

- polynômes de Tamari multivariés et flots
- bijection avec les triangulations planes, statistiques symétriques, etc.
- treillis cambriens et généralisations

Structures "m"

- structures algébriques
- treillis des chaînes de permutations