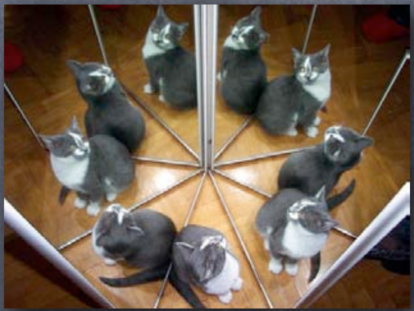
Mots et racines dans les groupes de Coxeter

– 2, 9 et 16 octobre 2013 au LIX –

Christophe Hohlweg, LaCIM, UQAM
(et LIX pour un mois)



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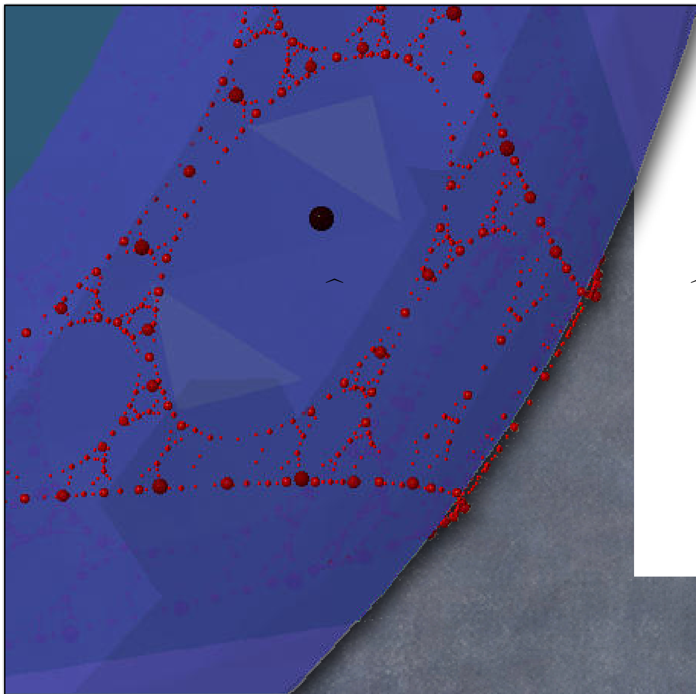
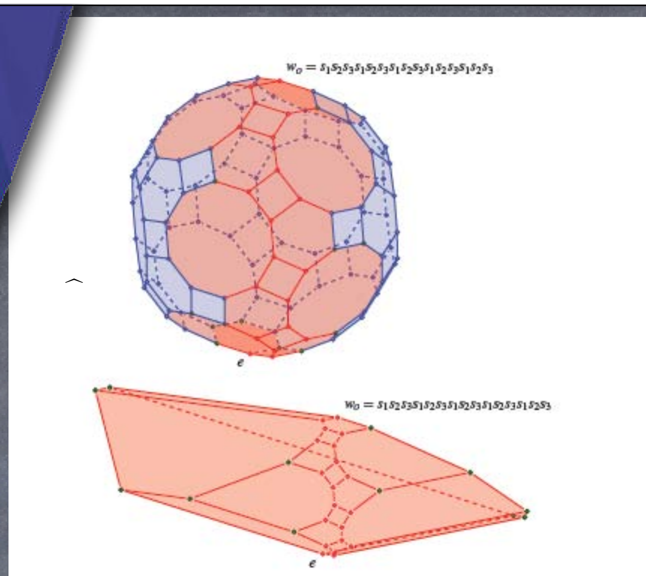
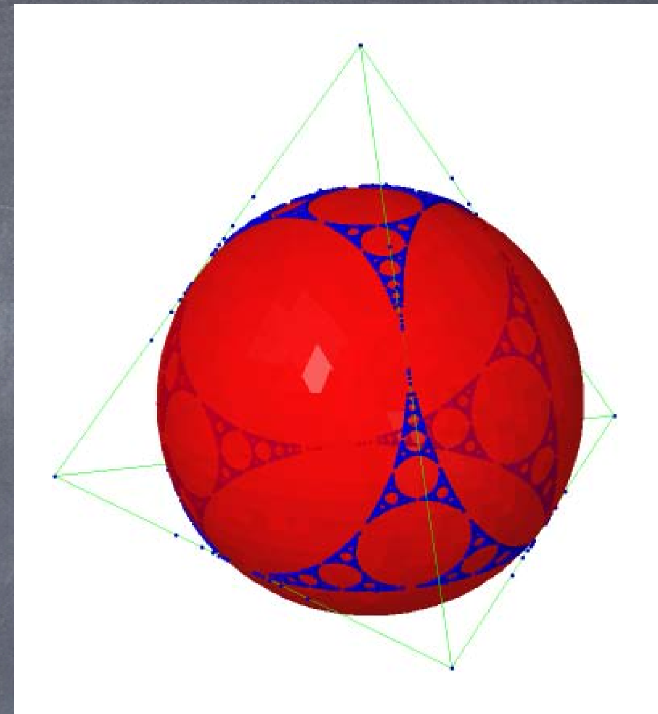
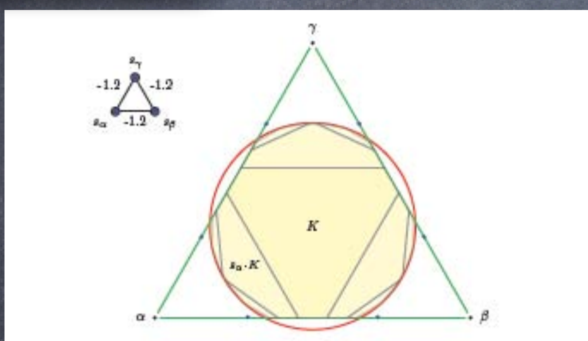
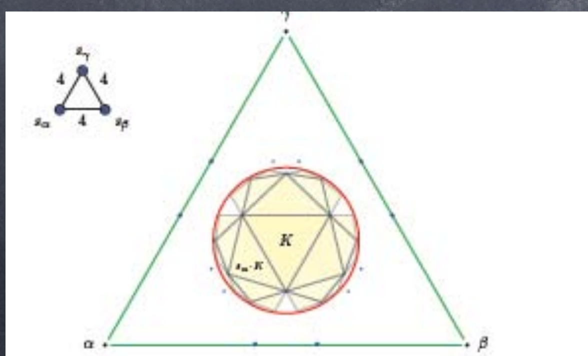


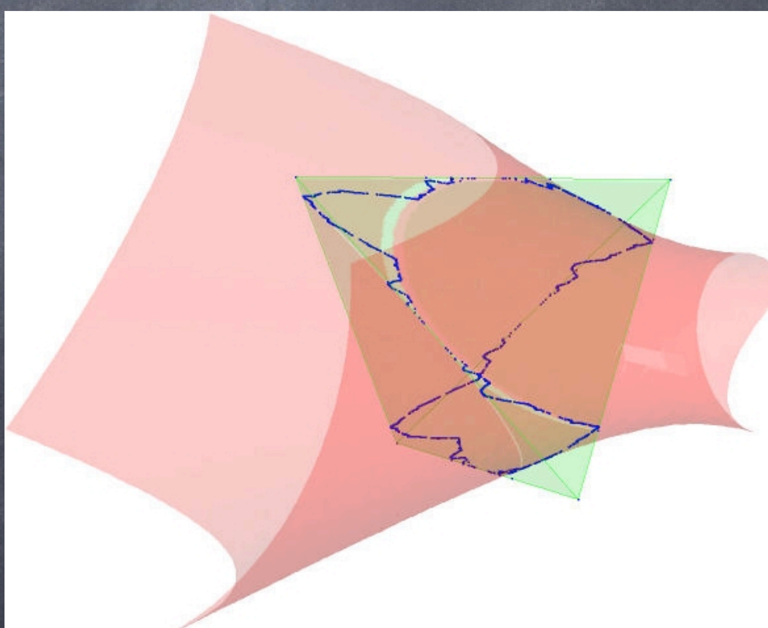




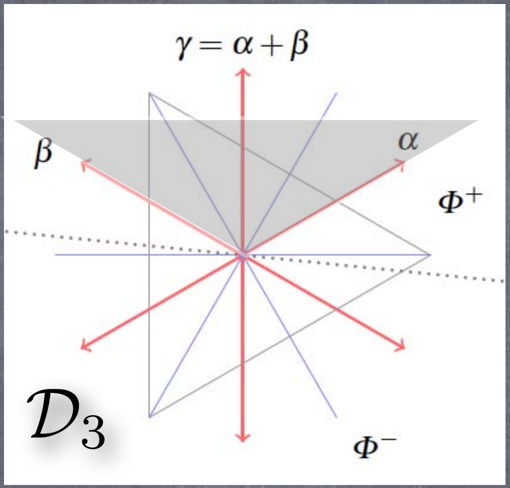

γ
-1.2
-1.2
-1.2
K
α
β

©?


4
4
4
K
α
β



In the last episode



- $W \leq \mathrm{O}(V)$ FRG $\longleftrightarrow$ Φ root system in V
- Separating Φ by a (linear) hyperplane we have:

reflections T $\overset{1:1}{\longleftrightarrow}$ Φ^+ positive roots

s_β $\longleftarrow\!\!|$ β

simple reflections $S \subseteq T$ $\overset{1:1}{\longleftrightarrow}$ Δ basis of $\mathrm{cone}(\Phi^+)$

Theorem. W is generated by $S = \{s_\alpha \mid \alpha \in \Delta\}$

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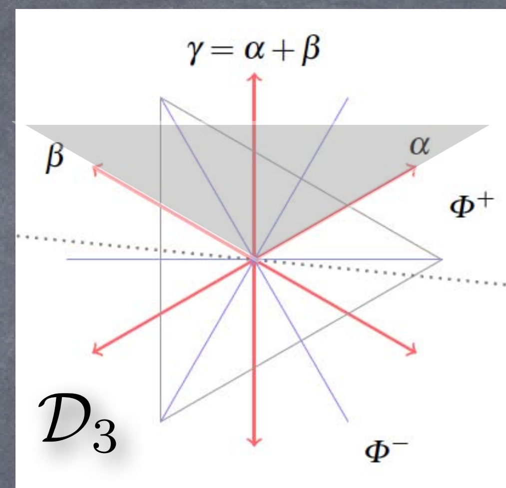

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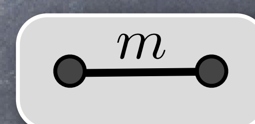
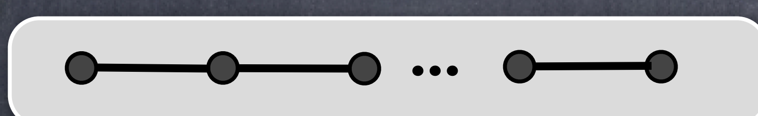
Problem: find the relations for $W = \langle S \rangle$!

Examples: $\mathcal{D}_m = \langle s, t \mid s^2 = t^2 = (st)^m = e \rangle$



$$\mathcal{S}_n = \langle \tau_i \mid \tau_i^2 = (\tau_i \tau_j)^2 = (\tau_i \tau_{i+1})^3 = e,\ 1 \leq i < n,\ |j - i| > 1 \rangle$$



Length, inversions and relations in FRG

Theorem. W is generated by $S = \{s_\alpha \,|\, \alpha \in \Delta\}$

- any $w \in W$ is a word in the alphabet S ;
- Length function $\ell : W \to \mathbb{N}$ with $\ell(e) = 0$ and

$$\ell(w) = \min\{k \,|\, w = s_1 s_2 \dots s_k,\ s_i \in S\}$$

How to relate words on S representing w? Is a word $s_1 s_2 \dots s_k$ a reduced expression for w (i.e. $k = \ell(w)$) ?

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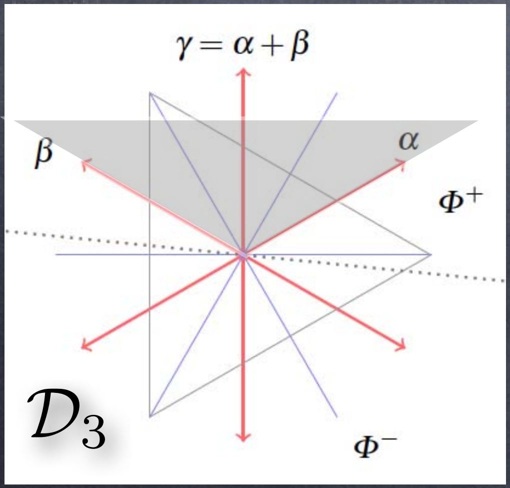

	e	$s = s_\alpha$	$t = s_\beta$	st	ts	$sts = tst$
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γ						

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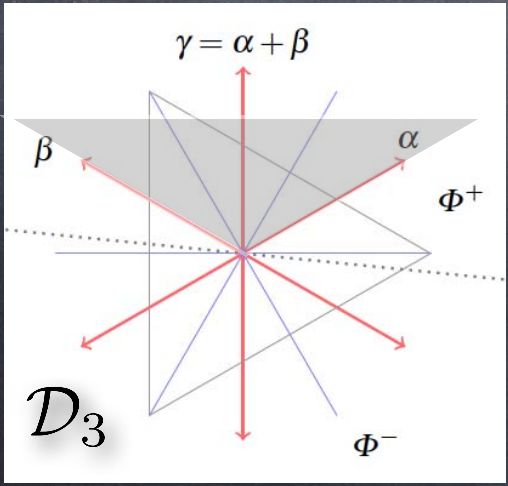

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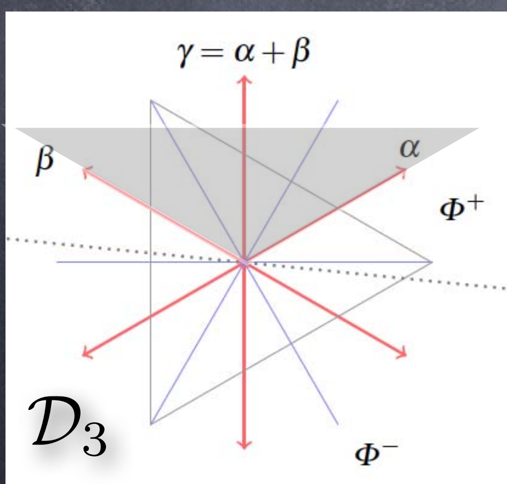

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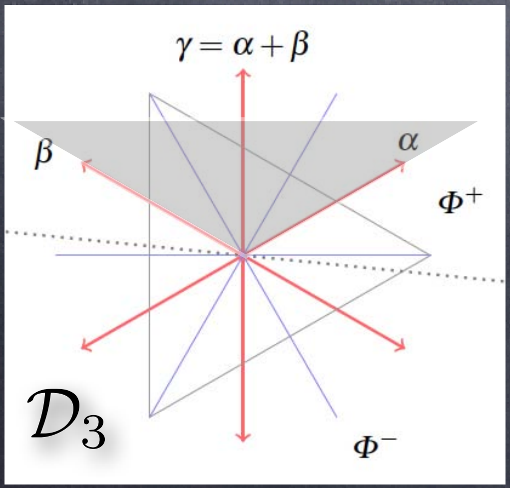

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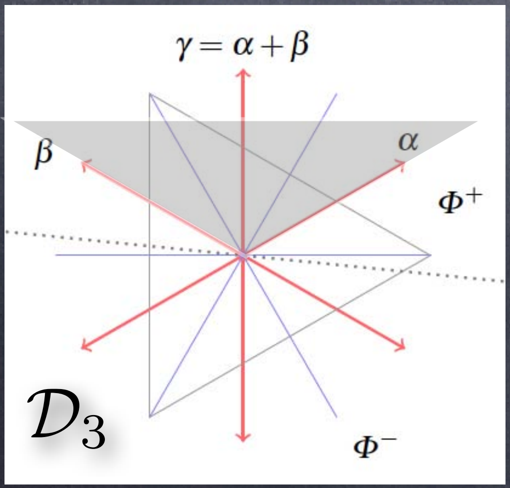

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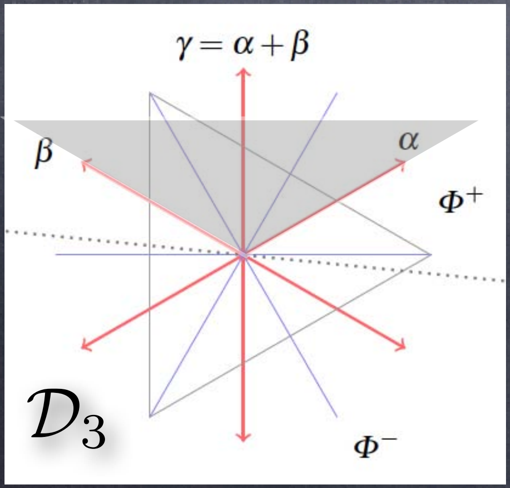

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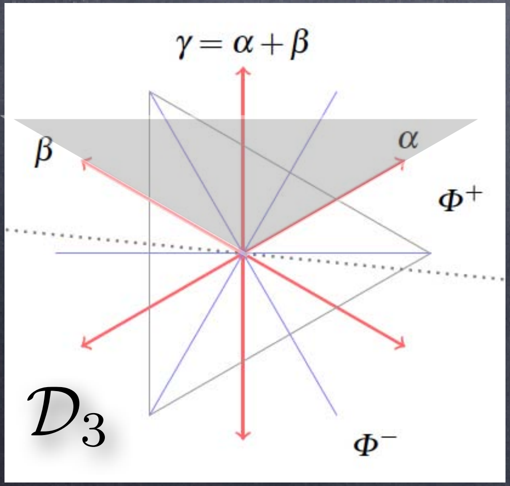

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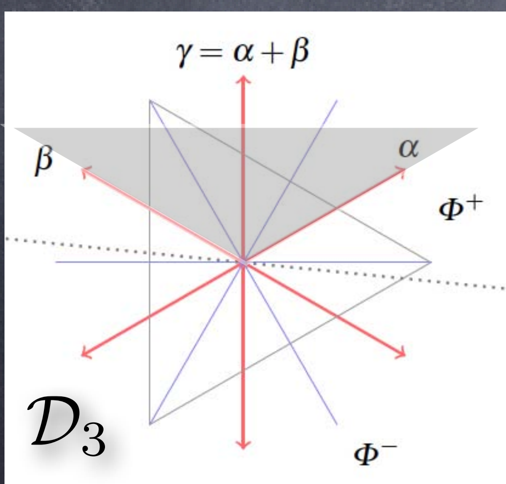

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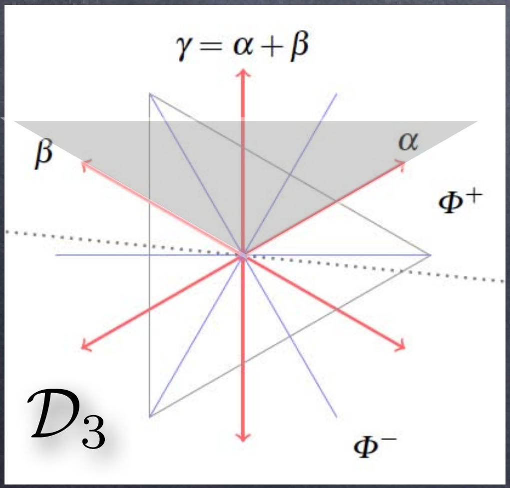

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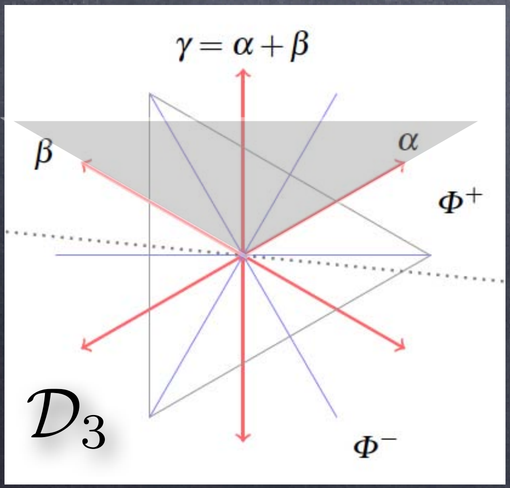

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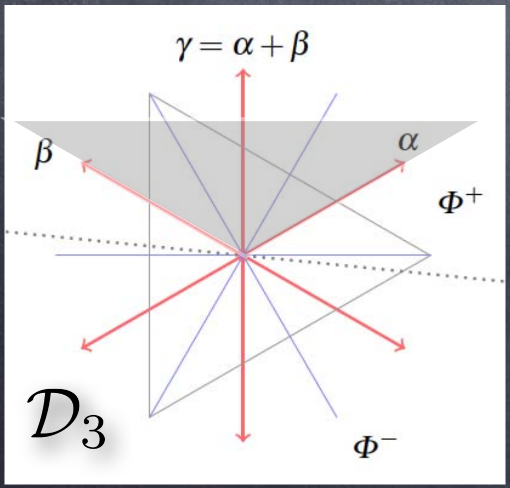

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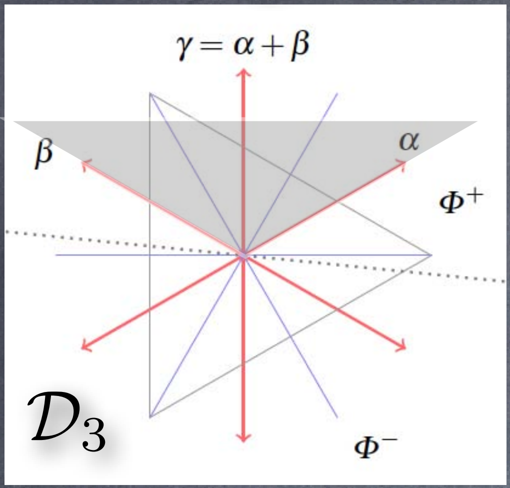
$$\ell(w) = |\{\nu \in \Phi^+ \mid w(\nu) \in \Phi^-\}|$$

Christophe Hohlweg, 2013

Length, inversions and relations in FRG



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Definition. The inversion set of $w \in W$ is

$$\mathrm{inv}(w) = \Phi^+ \cap w^{-1}(\Phi^-) = \{\nu \in \Phi^+ \,|\, w(\nu) \in \Phi^-\}$$

and $\mathrm{des}(w) = \mathrm{inv}(w) \cap \Delta$ is its descent set.

- If $W = S_n$ then those "are" the natural inversion and descent statistics:

$$\mathrm{inv}(\sigma) = \{e_j - e_i \,|\, 1 \leq i < j \leq n,\ e_{\sigma(j)} - e_{\sigma(i)} \in \Phi^-\}$$

$$\mathrm{des}(\sigma) = \{e_{i+1} - e_i \,|\, 1 \leq i < n,\ e_{\sigma(i+1)} - e_{\sigma(i)} \in \Phi^-\}$$

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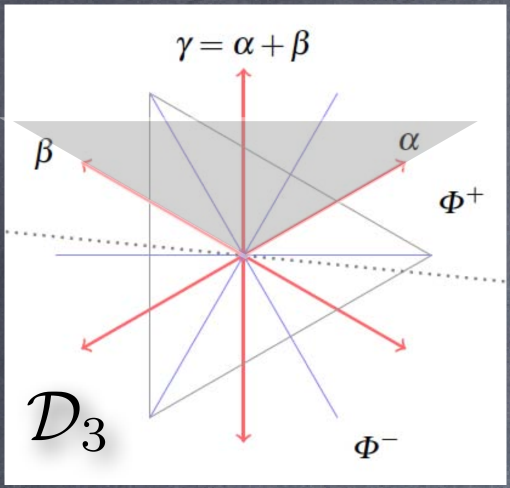

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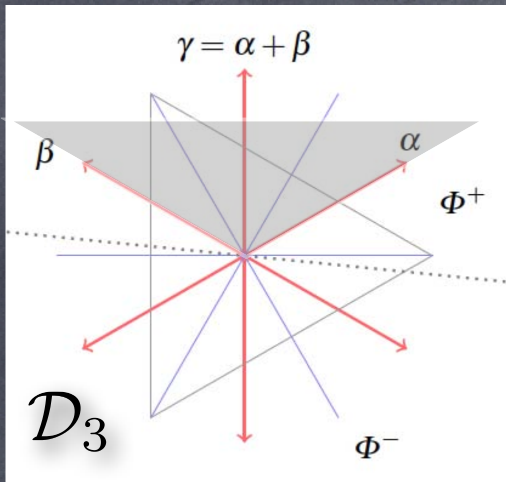

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- If $W = S_n$ then those "are" the natural inversion and descent statistics:

$$\mathrm{inv}(\sigma) \simeq \{(i,j) \,|\, 1 \leq i < j \leq n,\ \sigma(i) > \sigma(j)\}$$

$$\mathrm{des}(\sigma) \simeq \{i \,|\, 1 \leq i < n,\ \sigma(i) > \sigma(i+1)\}$$

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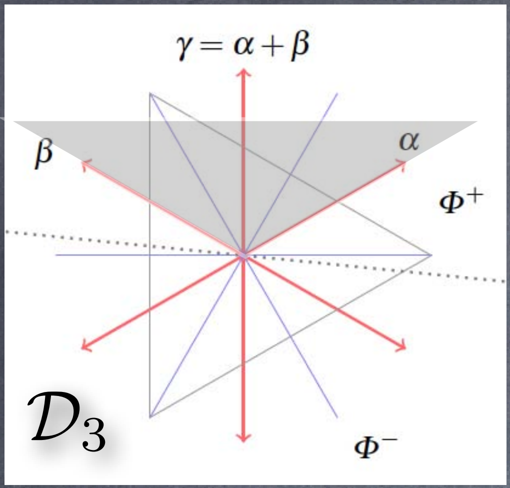

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Proposition. *Let $w \in W$ and $\alpha \in \Phi^+$, then:*

(i) $\ell(ws_\alpha) < \ell(w)$ *if and only if* $\alpha \in \operatorname{inv}(w)$ *(i.e.* $w(\alpha) \in \Phi^-$*).* *Otherwise,* $\ell(ws_\alpha) > \ell(w)$ *if and only if* $w(\alpha) \in \Phi^+$.

(ii) If $\alpha \in \Delta$, *then* s_α *is a bijection on* $\Phi^+ \setminus \{\alpha\}$ *and*

$$\ell(ws_\alpha) = \begin{cases} \ell(w) - 1 & \text{if } \alpha \in \operatorname{des}(w) \text{ i.e. } w(\alpha) \in \Phi^- \\ \ell(w) + 1 & \text{if } \alpha \notin \operatorname{des}(w) \text{ i.e. } w(\alpha) \in \Phi^+ \end{cases}$$

(iii) $\ell(w) = |\operatorname{inv}(w)|$.

N.B.: $(-w) : \Phi^+ \cap w^{-1}(\Phi^-) \to \Phi^+ \cap w(\Phi^-)$ is a bijection, so $\ell(w) = \ell(w^{-1})$.

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(iii) $\ell(w) = |\mathrm{inv}(w)|$.

Critical tools for the proof are the following equivalent statements for a word $w = s_1 \dots s_k \in W$

- Exchange condition. if $\alpha \in \mathrm{des}(w)$ then w may be rewritten with s_α as the last letter: $w = s_1 \dots \widehat{s_i} \dots s_k s_\alpha$
- Deletion condition. the word $w = s_1 \dots s_k$ has a subword that is a reduced expression for w obtained by deleting pairs of letters

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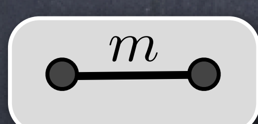
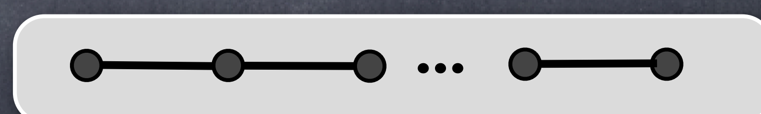
(iii) $\ell(w) = |\mathrm{inv}(w)|$.

And finally

Theorem. $W = \langle S \mid (st)^{m_{st}} = e \rangle$ where $m_{st} = m_{ts}$ is the order of the rotation st (and $m_{ss} = 1$)

- $\mathcal{D}_m$ is 

and S_n is 

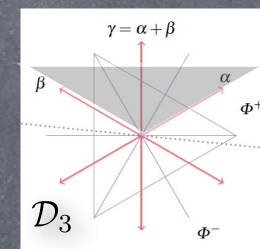
Conclusion: words and roots in FRG

- $W \leq \mathrm{O}(V)$ FRG $\longleftrightarrow$ Φ root system in V

- Separating Φ by a (linear) hyperplane we have:

reflections $T \overset{1:1}{\longleftrightarrow} \Phi^+$ positive roots

$s_\beta \longleftarrow\!\!\mid \beta$



simple reflections $S \subseteq T \overset{1:1}{\longleftrightarrow} \Delta$ basis of $\mathrm{cone}(\Phi^+)$

Theorem. W is generated by $S = \{s_\alpha \mid \alpha \in \Delta\}$

- Length function $\ell(w) \longleftrightarrow \mathrm{inv}(w)$ inversion set

Theorem. $W = \langle S \mid (st)^{m_{st}} = e \rangle$ where $m_{st} = m_{ts}$ is the order of the rotation st (and $m_{ss} = 1$)

On the Cayley graph of FRG

Theorem. W is generated by $S = \{s_\alpha \,|\, \alpha \in \Delta\}$

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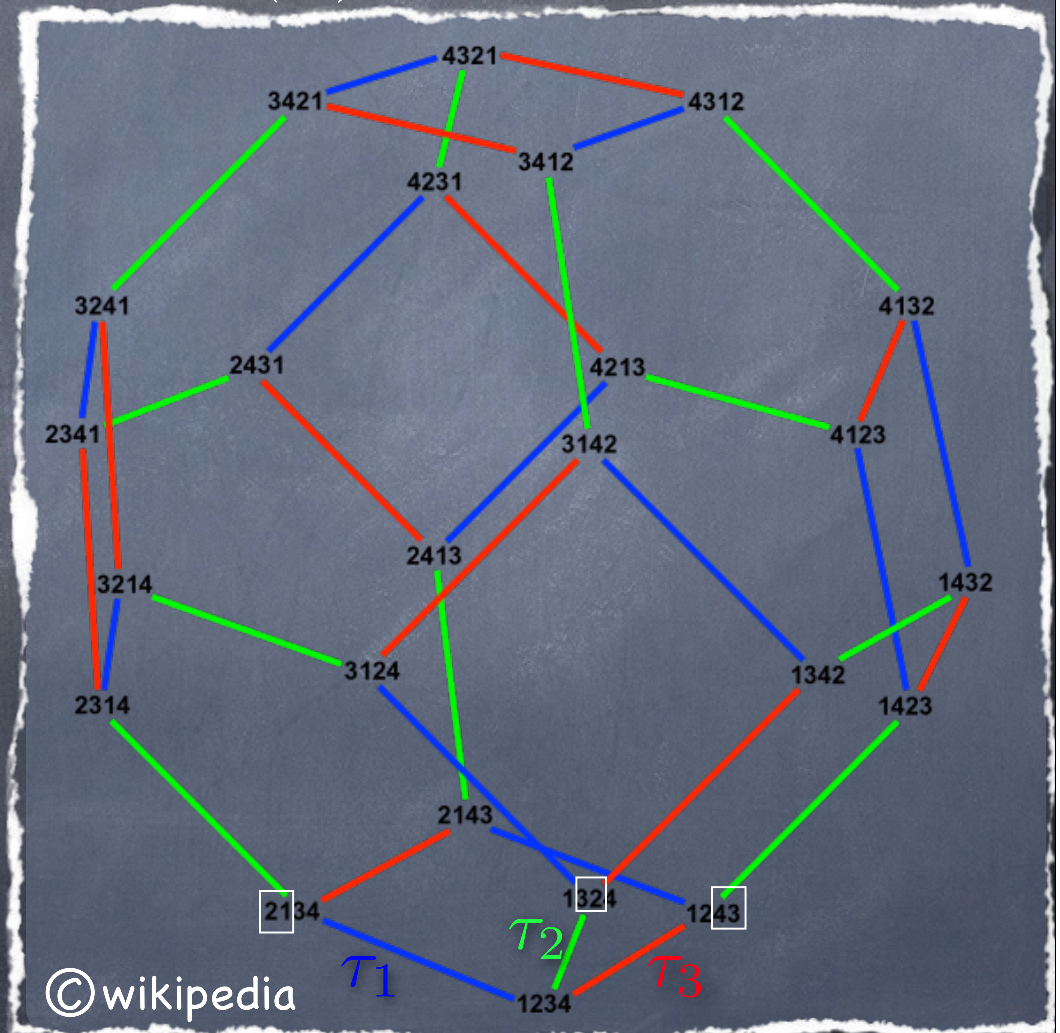
Cayley graph of $W = \langle S \rangle$:

- vertices W
- edges $w \overset{s}{\text{———}} ws$ $(s \in S)$



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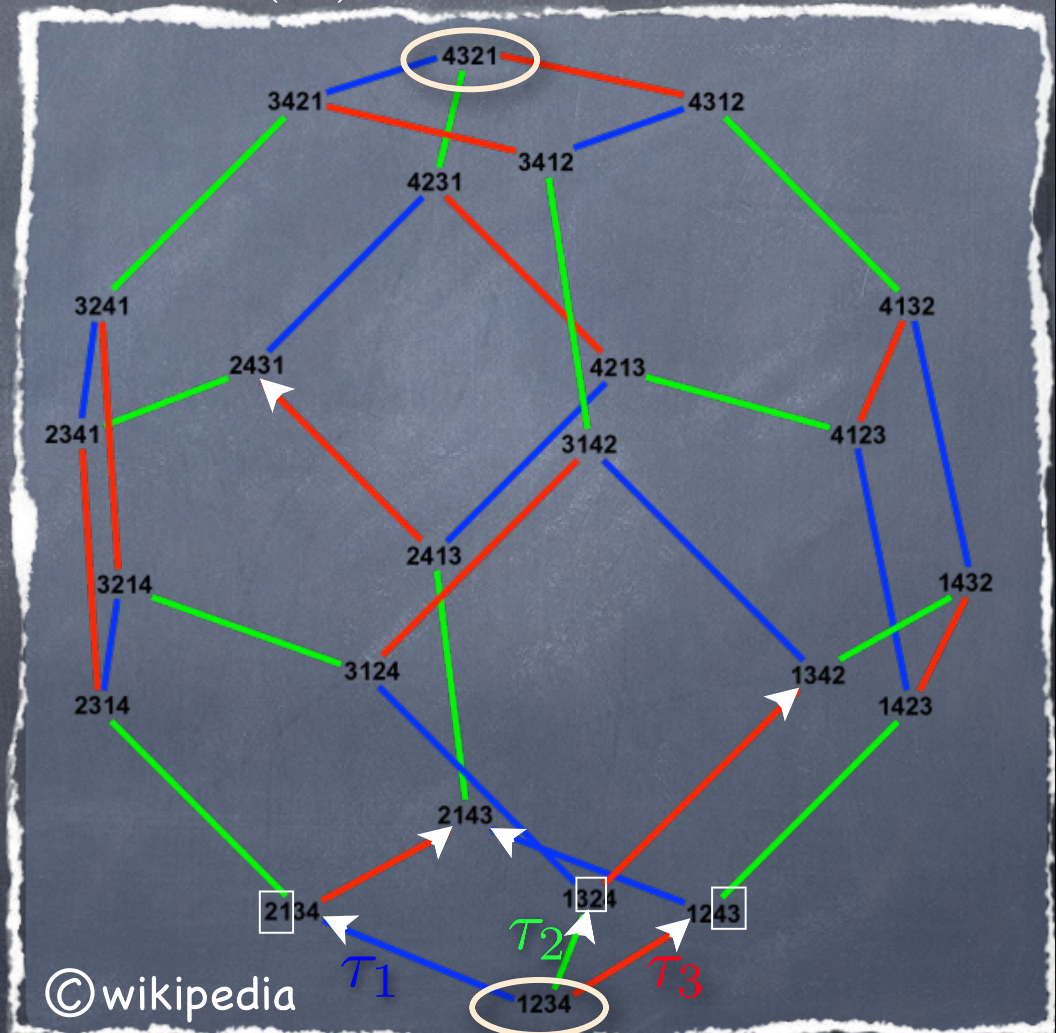
The Cayley graph is naturally oriented by the (right) weak order:

$w < ws$ if $\ell(w) < \ell(ws)$

write: $w \xrightarrow{s} ws$



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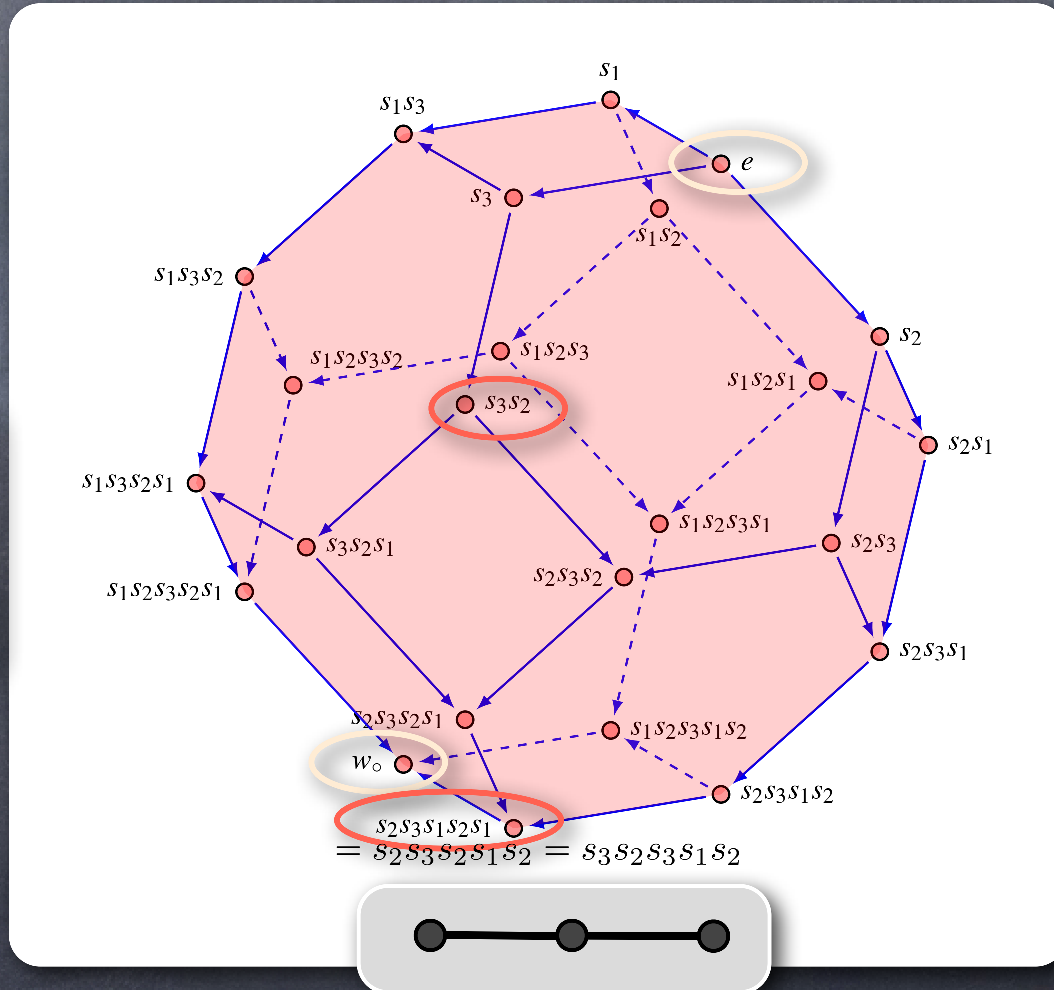

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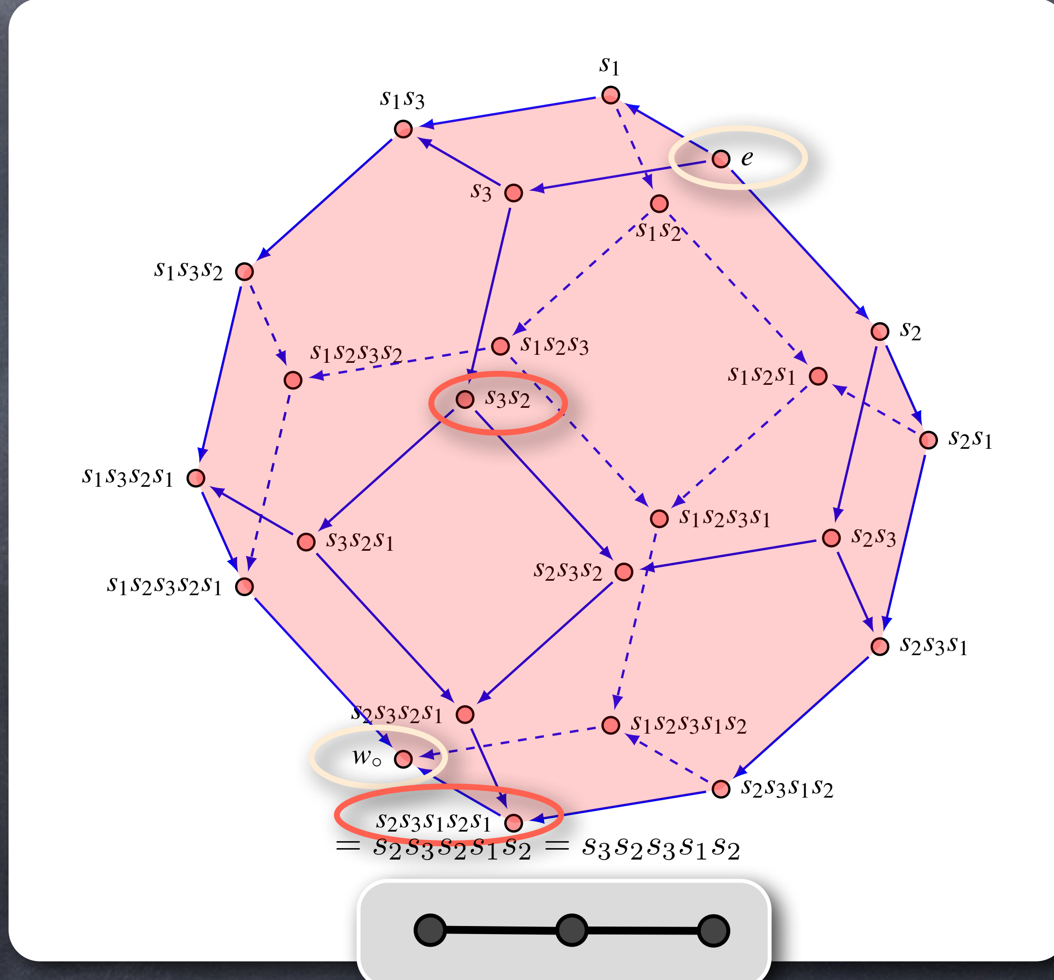
Theorem. The weak order is a lattice. Moreover:

- reduced expression of w are in bijection with maximal chains in the interval $[e, w]$.
- $u \leq w \iff \ell(u^{-1}w) = \ell(w) - \ell(u)$
- $u \leq w$ iff a reduced expression of u is a prefix of a reduced expression of w.
- $w_\circ$ is the unique element of maximal length



Christophe Hohlweg, 2013

On the Cayley graph of FRG

Theorem. W is generated by $S = \{s_\alpha \,|\, \alpha \in \Delta\}$

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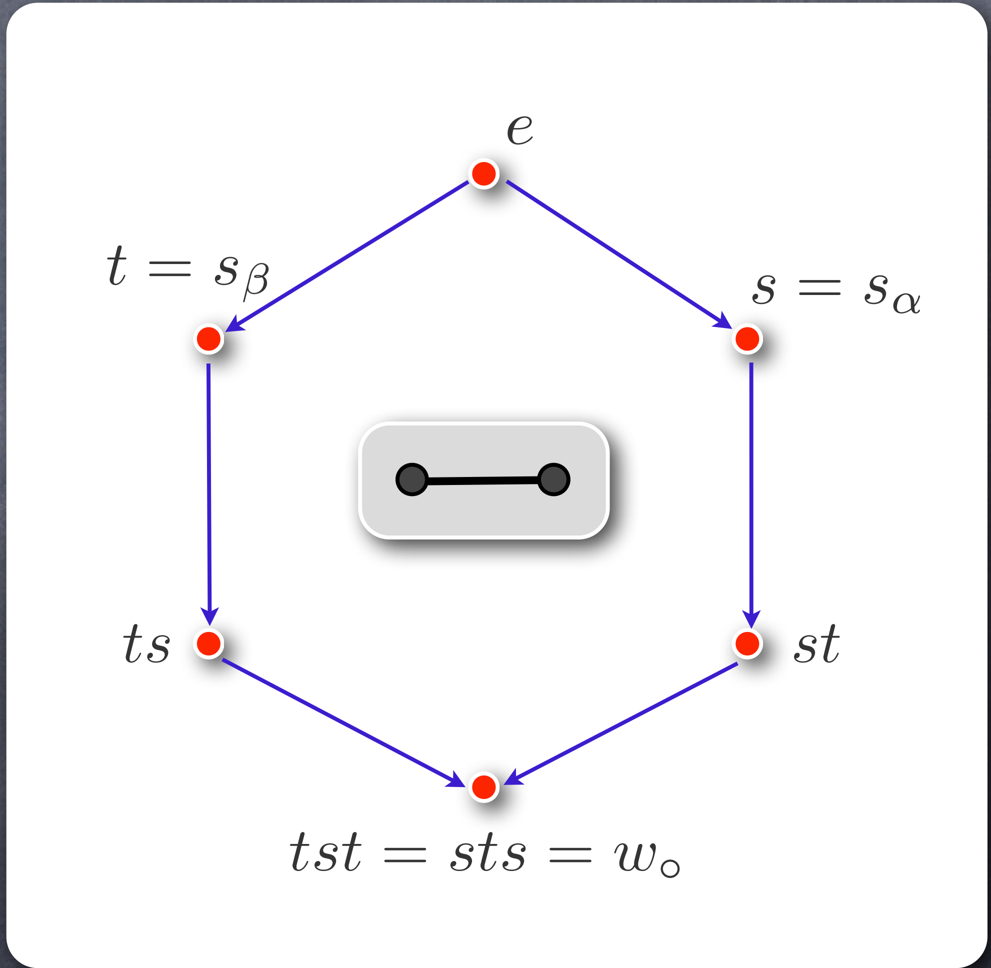

Christophe Hohlweg, 2013

	e	$s=s_\alpha$	$t=s_\beta$	st	ts	$sts=tst$
ℓ	0	1	1	2	2	3
α	α	$-\alpha$	γ	β	$-\gamma$	$-\beta$
β	β	γ	$-\beta$	$-\gamma$	α	$-\alpha$
$\alpha+\beta=\gamma$	γ	β	α	$-\alpha$	$-\beta$	$-\gamma$

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- reduced expression of w are in bijection with maximal chains in the interval $[e,w]$.
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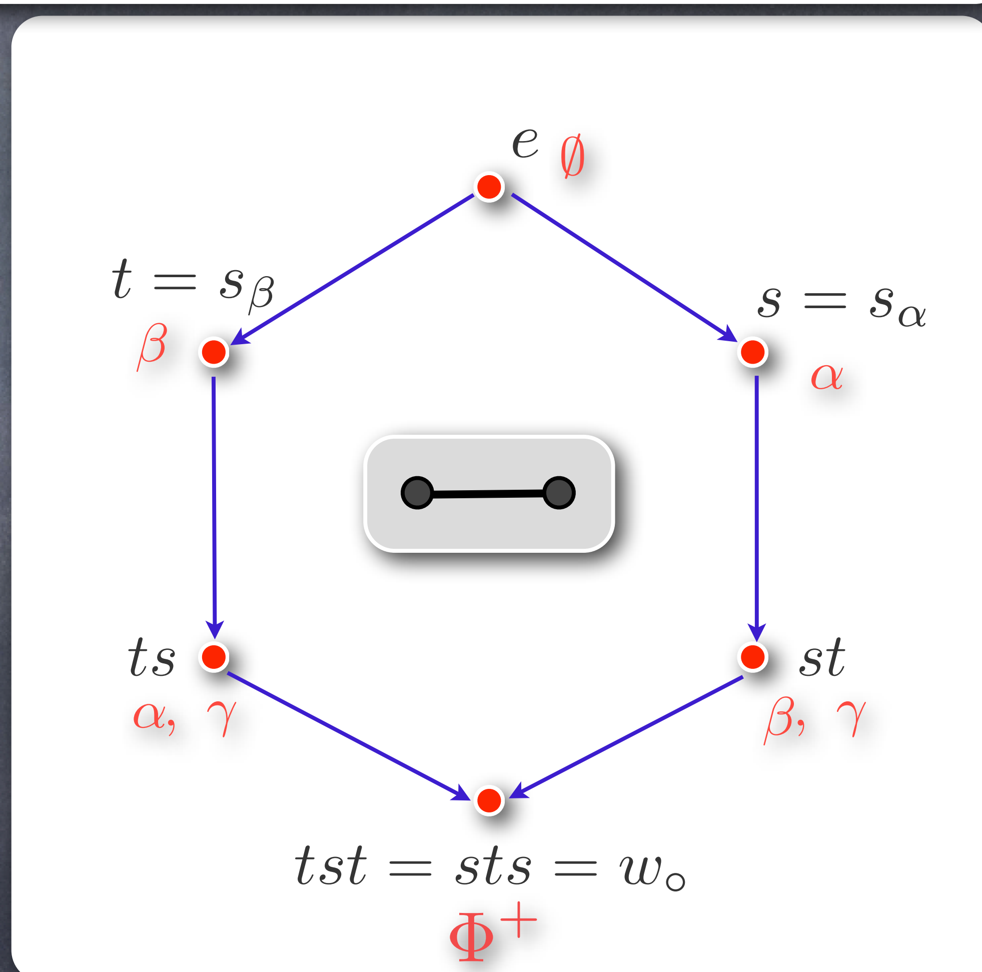

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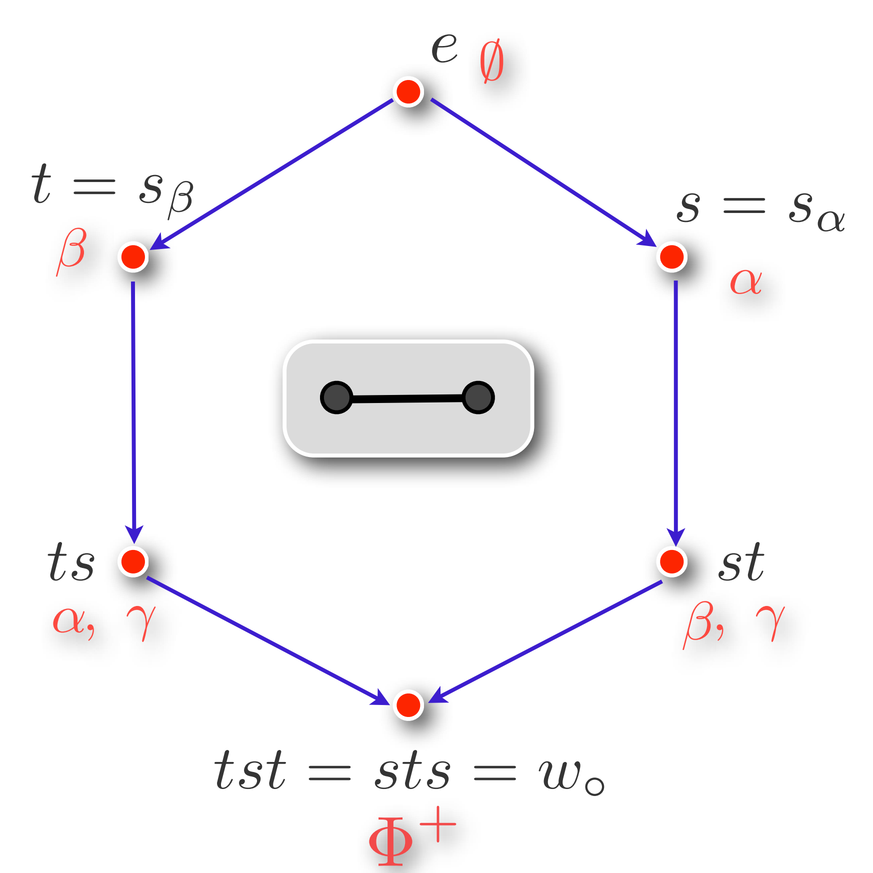


On the Cayley graph of FRG

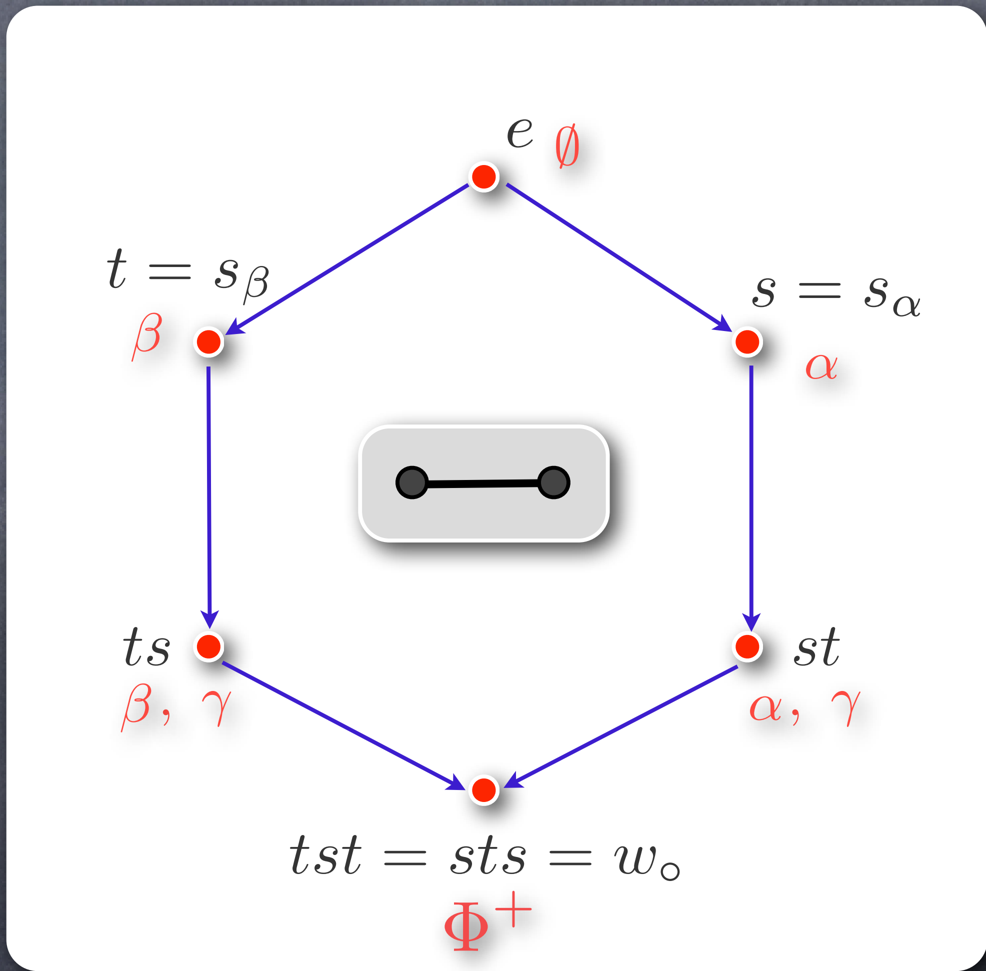
Theorem. W is generated by $S = \{s_\alpha \,|\, \alpha \in \Delta\}$

- Length function $\ell(w) \longleftrightarrow \mathrm{inv}(w^{-1})$ inversion set

Theorem. The weak order is a lattice. Moreover:

- $u \leq w \iff \mathrm{inv}(u^{-1}) \subseteq \mathrm{inv}(w^{-1})$
- $w_\circ$ is the unique element of maximal length: $\ell(w_\circ) = |\Phi^+|$

- $\vee \neq \cup$; $\wedge \neq \cap$ so ... ?
- Problem: to understand reduced expressions as maximal chains in intervals of inversion sets? Count them!

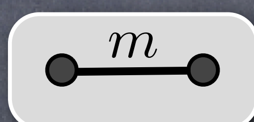


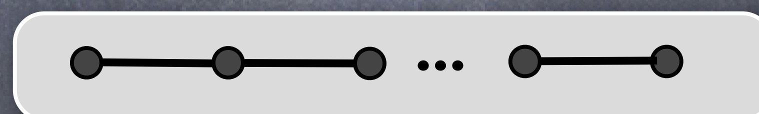
Coxeter groups and Reflection groups

Theorem. $W = \langle S \,|\, (st)^{m_{st}} = e \rangle$ where $m_{st} = m_{ts}$ is the order of the rotation st (and $m_{ss} = 1$)

- So the presentation $W = \langle S \,|\, (st)^{m_{st}} = e \rangle$ is illustrated with a Coxeter graph Γ_S :
  - vertices S (i.e. Δ)
  - edges 

for $m_{st} \geq 3$

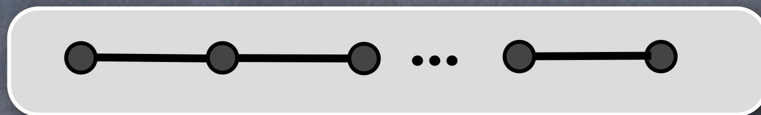
- $\mathcal{D}_m$ is 

and S_n is 

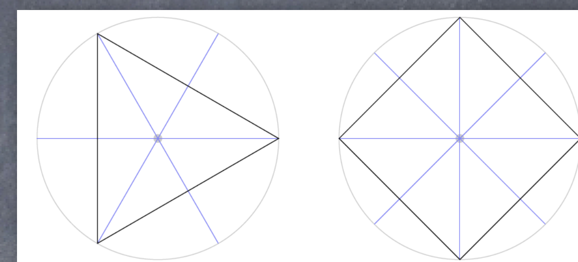
- A Coxeter system (W, S) is a group $W = \langle S \rangle$ obtained from a Coxeter graph (allowing $m_{st} = \infty$ if there is no relation between s and t, and $m_{ss} = 1$)

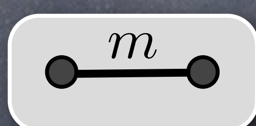
Coxeter groups and Reflection groups

- A Coxeter system (W,S) is a group $W=\langle S\rangle$ obtained from a Coxeter graph (allowing $m_{st}=\infty$ if there is no relation between s and t)

- S_n is  of rank $|S|=n-1$

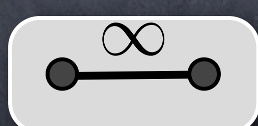
- Rank 2 Coxeter groups

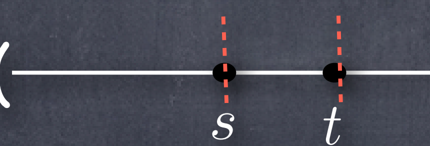


□ $\mathcal{D}_m$ is 

or  $(m=2)$ - FRG

□ A infinite example: the infinite dihedral group

D_∞ is 

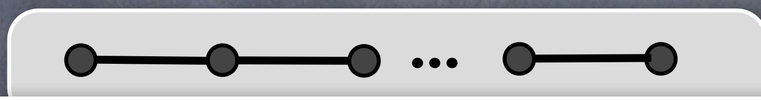
(

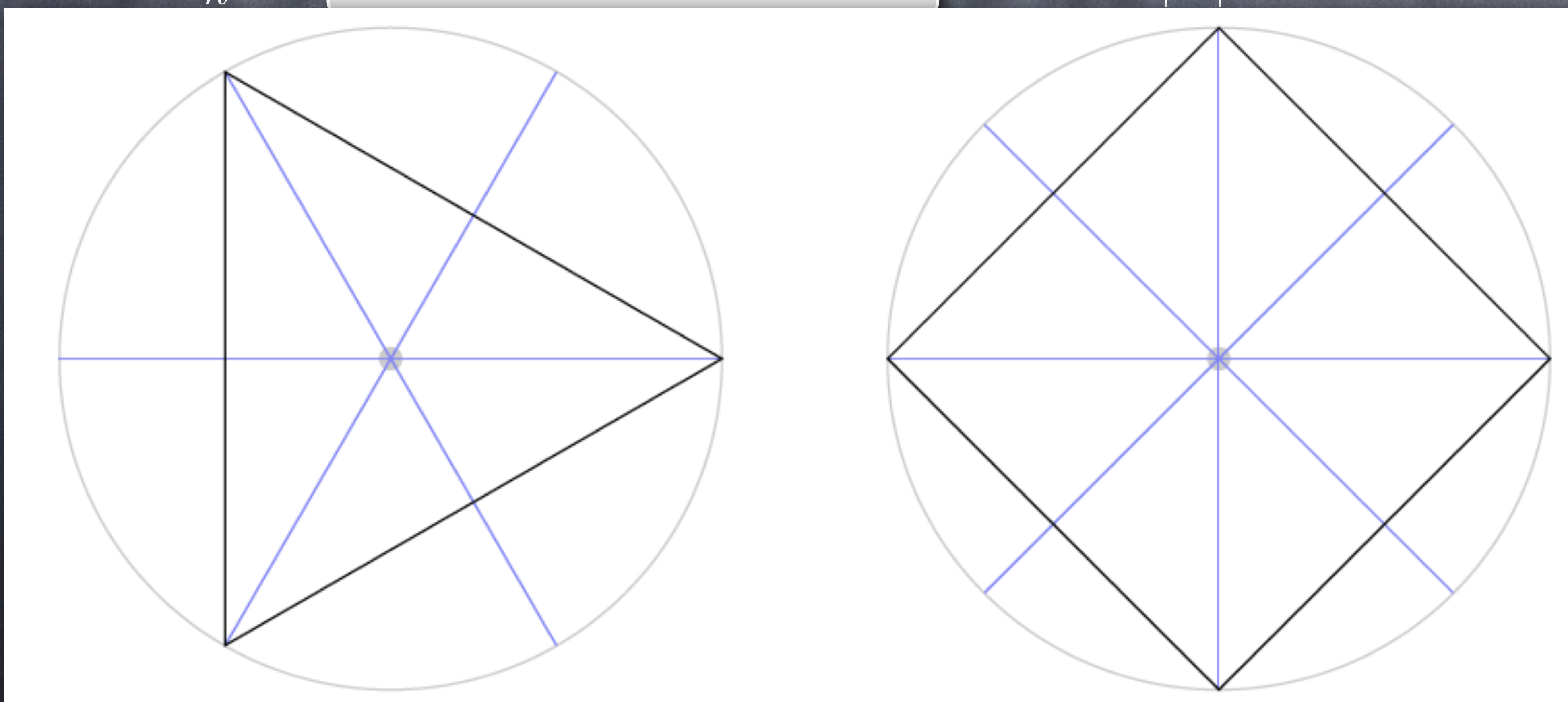
; Infinite number of words:)

$(ts)^p,\ (st)^p,\ s(ts)^p, t(st)^p,\ p\in\mathbb{N}$

Coxeter groups and Reflection groups

- A Coxeter system (W,S) is a group $W=\langle S\rangle$ obtained from a Coxeter graph (allowing $m_{st}=\infty$ if there is no relation between s and t)

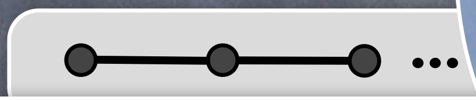
- S_n is  of rank $|S|=n-1$

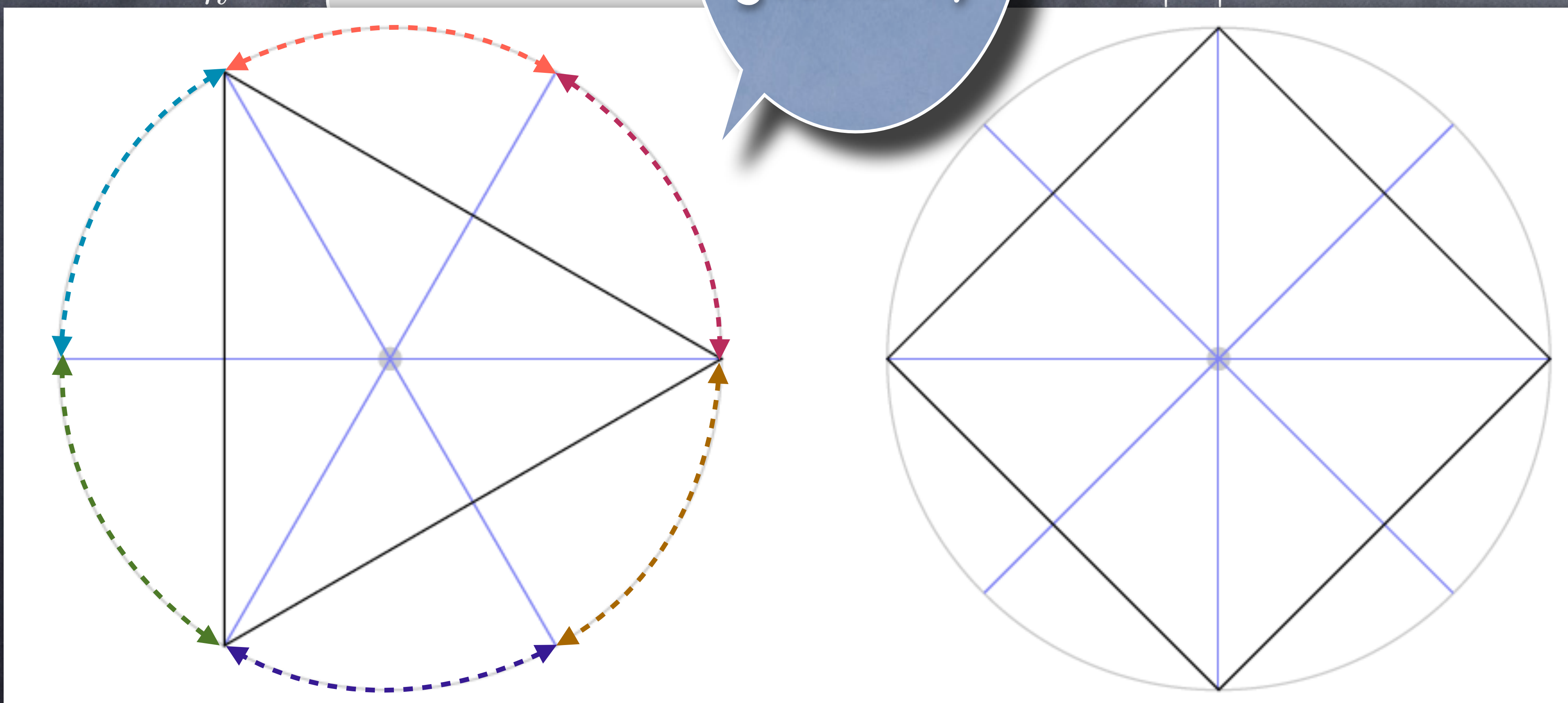


Coxeter groups and Reflection groups

- A Coxeter system $(W, S$ … oup $W = \langle S \rangle$ obtained from a Coxe … allowing $m_{st} = \infty$ if there is no relatio … and t)

Tilings in spherical geometry

- S_n is  … rank $|S| = n - 1$



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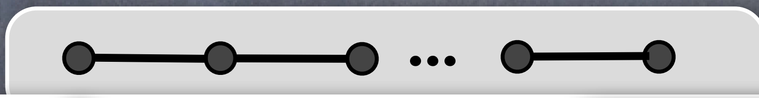
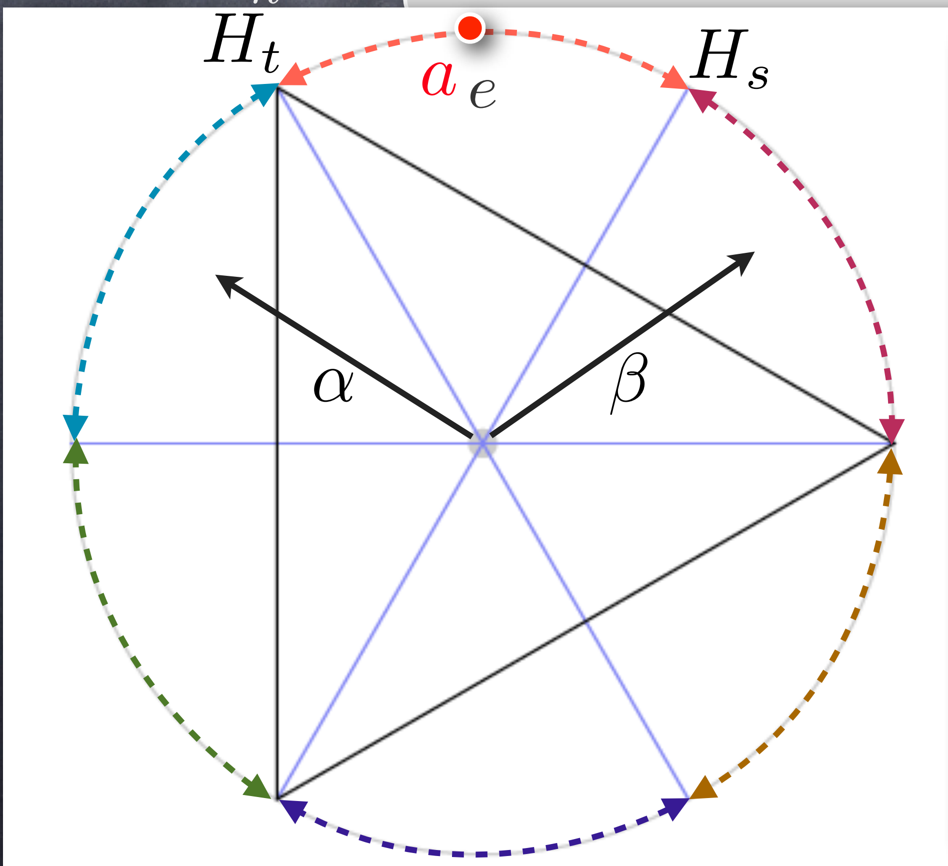
- S_n is  of rank $|S| = n-1$



Tilings in spherical geometry, roots and words

- $\Delta = \{\alpha, \beta\}$ simple system;
- $S = \{s = s_\alpha, t = s_\beta\}$;
- Choose a generic on a tile s.t.

  $$\langle a, \alpha \rangle > 0, \ \langle a, \beta \rangle > 0$$

- Label the corresp. tile by e.
- Then label by acting ...

Coxeter groups and Reflection groups

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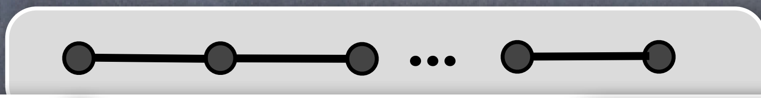
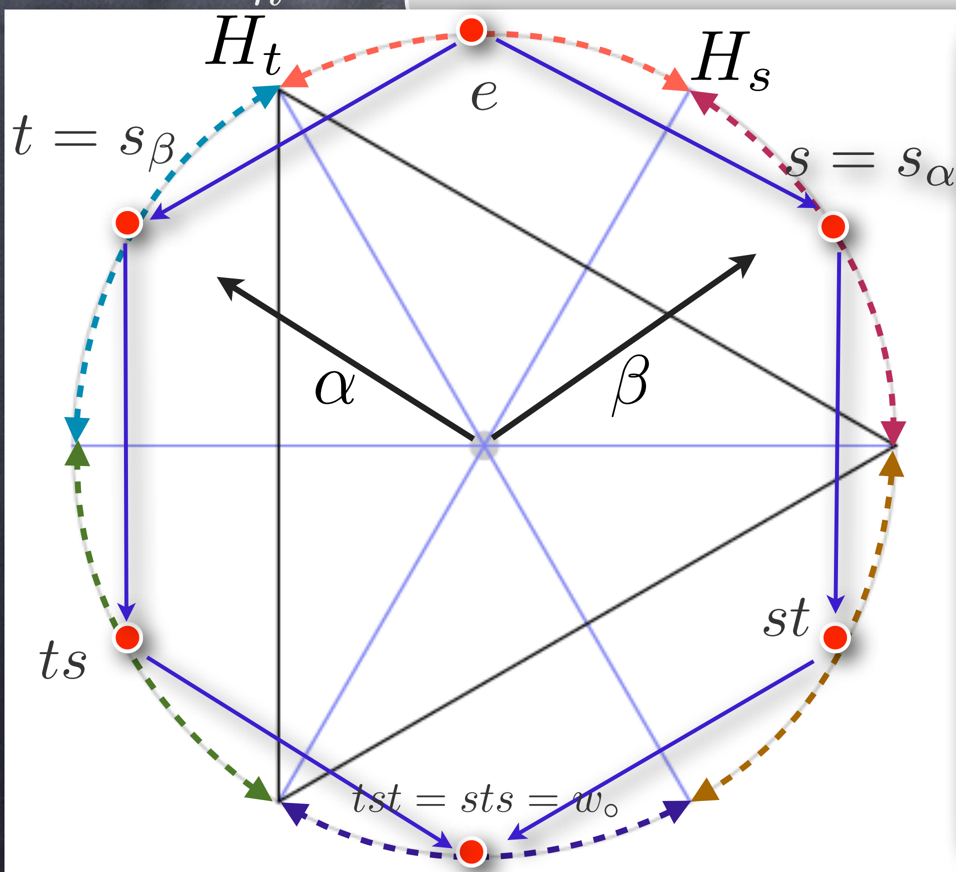
- S_n is  of rank $|S| = n-1$



Tilings in spherical geometry, roots and words

- $\Delta = \{\alpha, \beta\}$ simple system;
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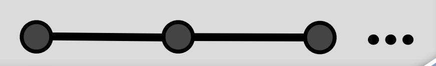
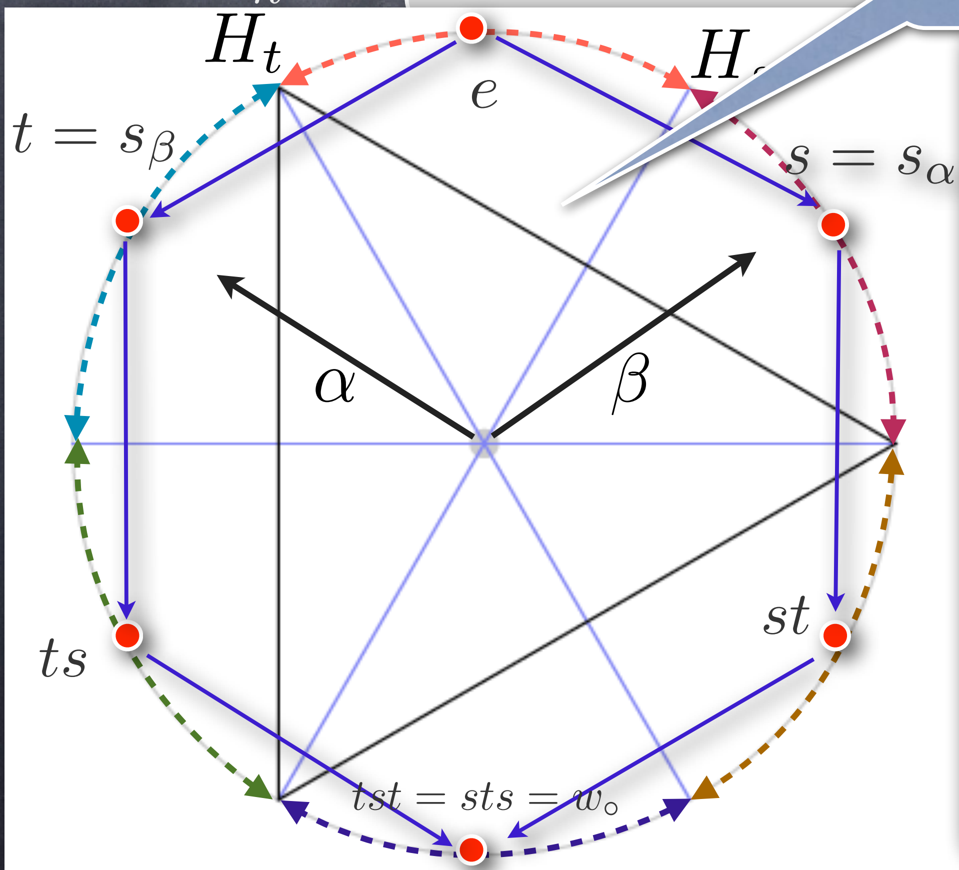
- Label the corresp. tile by e.
- Then label by acting ...

Coxeter groups a[…]

- A Coxeter system $(W,$ […] obtained from a Coxe[…] if there is no relation[…]
- S_n is





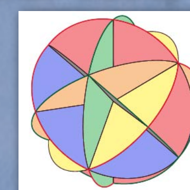
The convex hull of

$$w(a),\ w \in W$$

is a convex polytope called a **permutahedron**

$$\mathrm{Perm}^{\boldsymbol{a}}(W) = \mathrm{conv}\,\{w(\boldsymbol{a}) \mid w \in W\}$$

Tilings in spherical geometry, roots and words (general case)



- Δ simple system;
- $S = \{s_\alpha \mid \alpha \in \Delta\}$;
- Choose a generic on a tile s.t.

$$\langle a, \alpha \rangle > 0,\ \forall \alpha \in \Delta$$

- Label the corresp. tile by e.
- Then label by acting ...

Coxeter groups and Reflection groups

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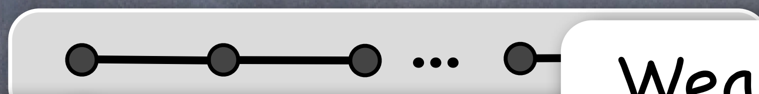
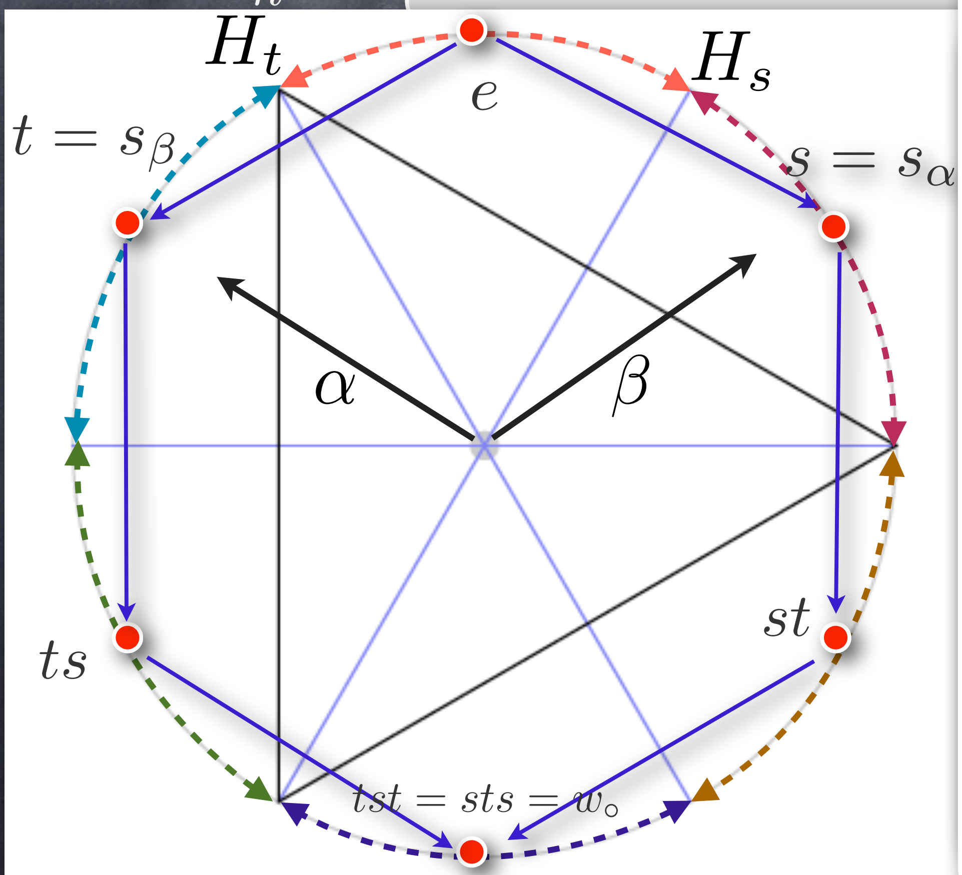
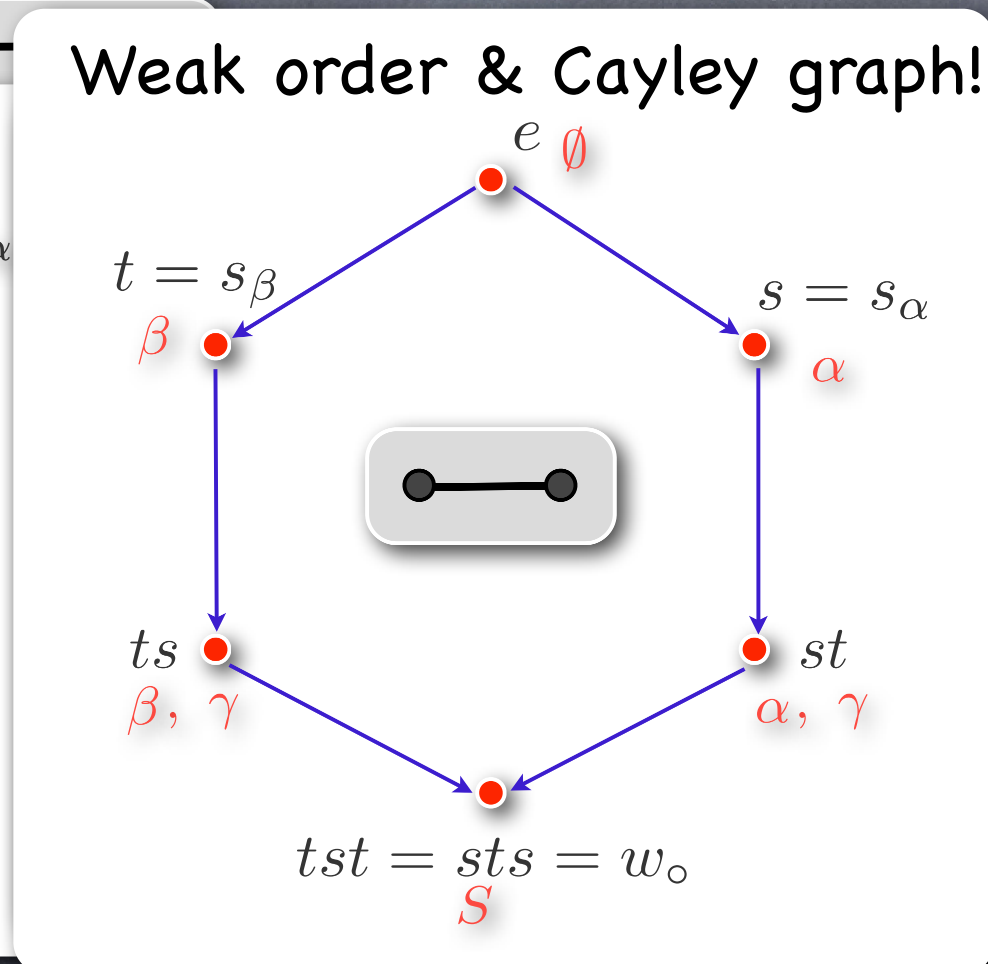
- S_n is 





Coxeter groups and Reflection groups

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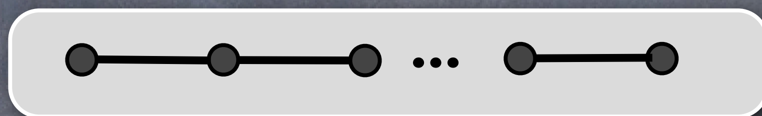
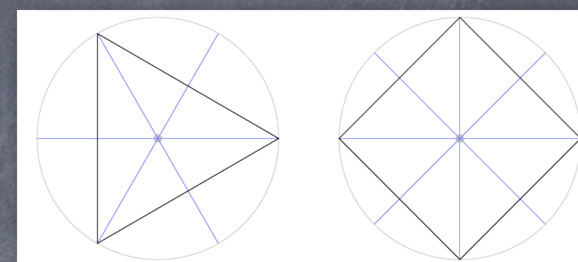
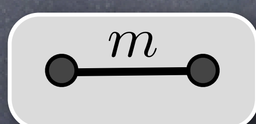
- S_n is  of rank $|S| = n - 1$

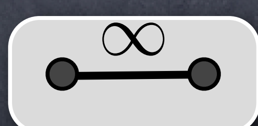
- Rank 2 Coxeter groups

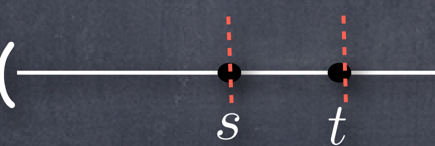


- $\mathcal{D}_m$ is 

or  $(m = 2)$ - FRG

- A infinite example: the infinite dihedral group

D_∞ is 

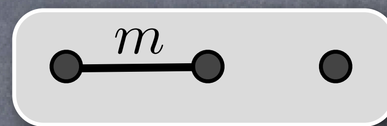
(

; Infinite number of words:)

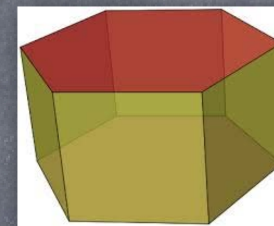
$(ts)^p,\ (st)^p,\ s(ts)^p, t(st)^p,\ p \in \mathbb{N}$

Coxeter groups and Reflection groups

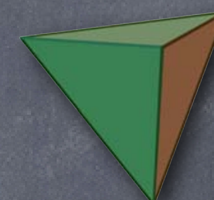
- Rank 3 finite reflection groups: isometry groups of
  - m-gonal regular prisms



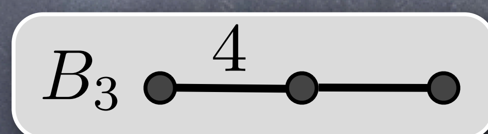
$\mathcal{D}_m \times S_2$

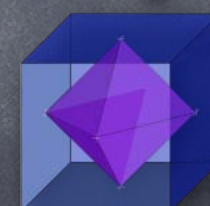


  - Regular polyhedra
    - ✓ tetrahedron

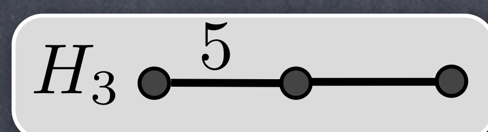


    - ✓ cube/ octahedron



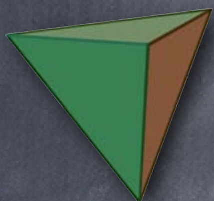


    - ✓ dodecahedron/ icosahedron

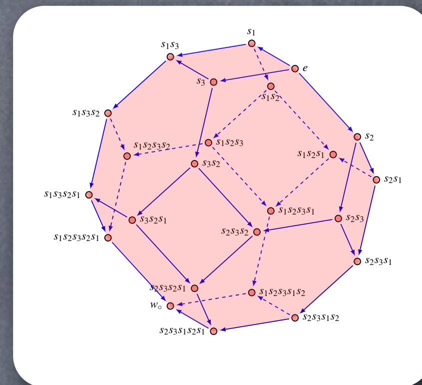


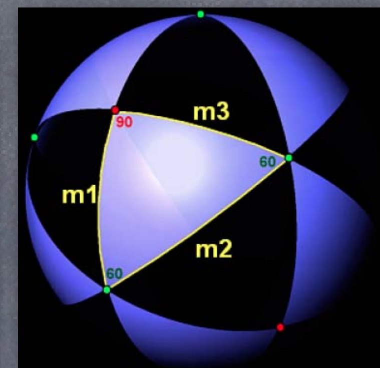
Coxeter groups and Reflection groups

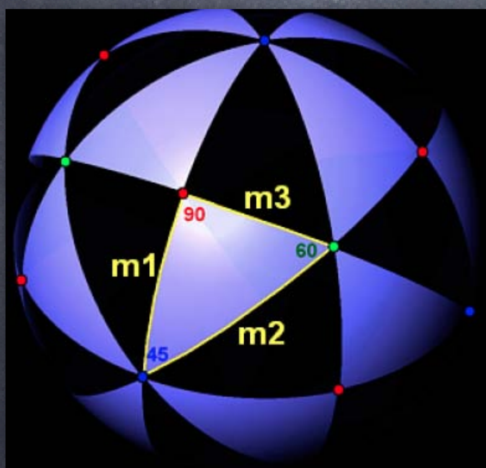
- Regular polyhedra, Permutahedra and tilings in rk 3

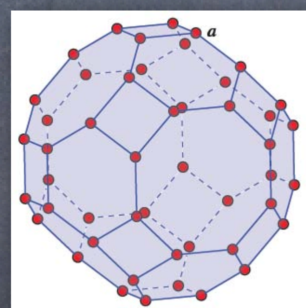


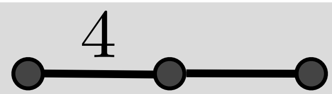
$S_4(A_3)$ 

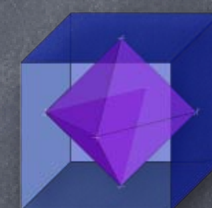


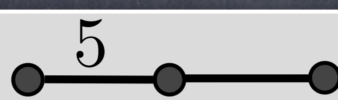


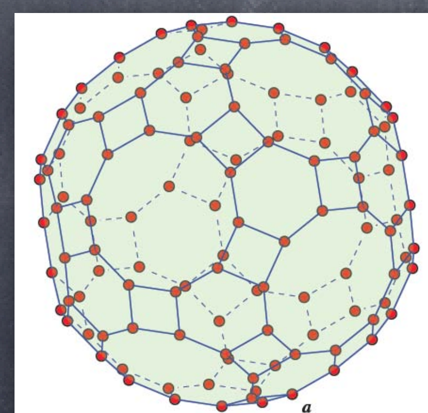


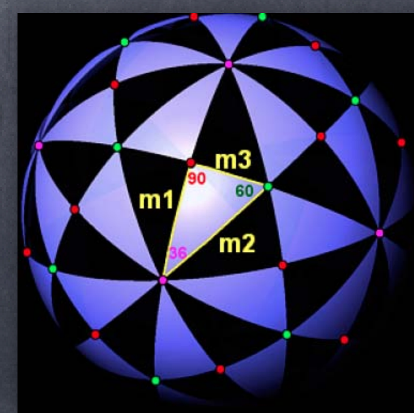


B_3 



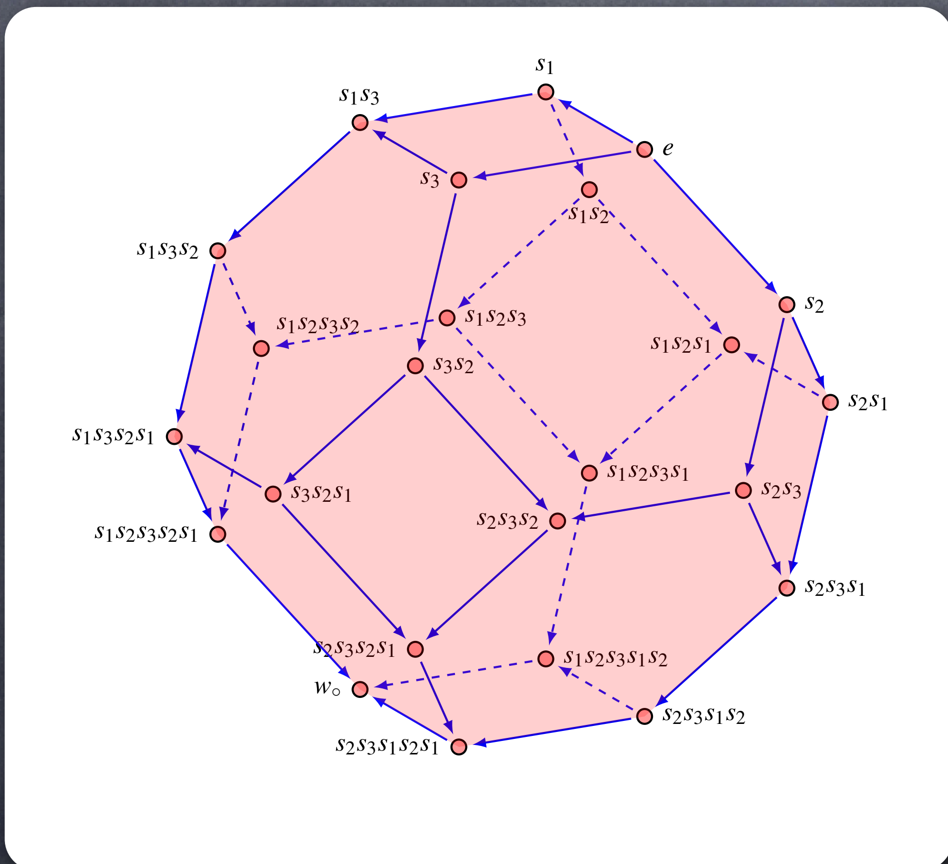
H_3 

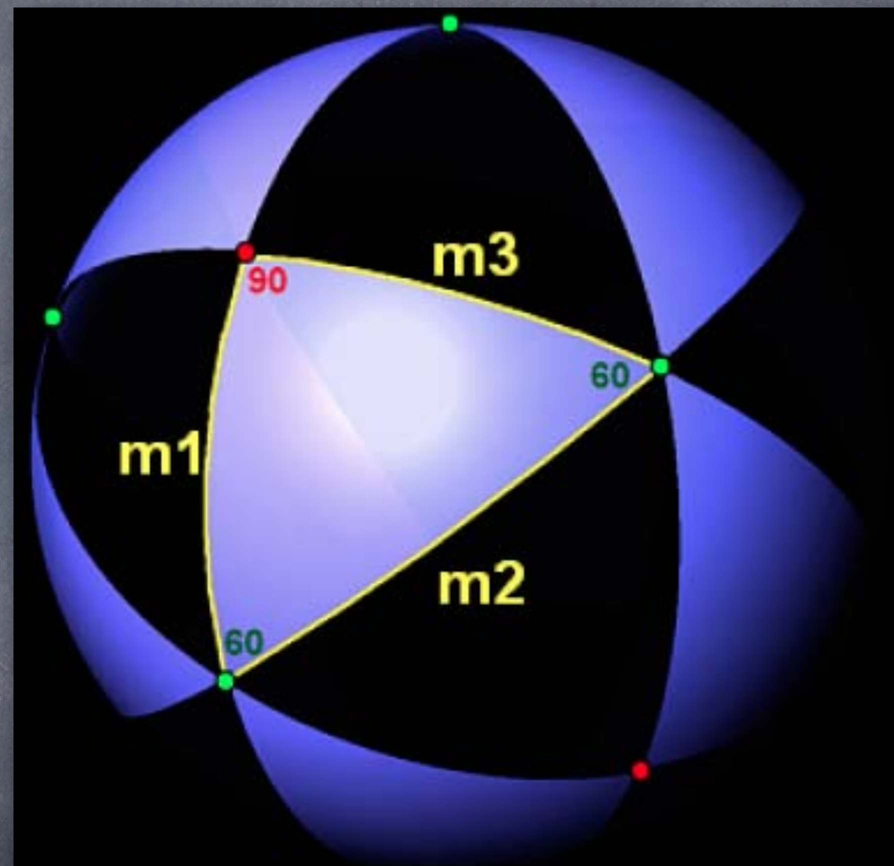


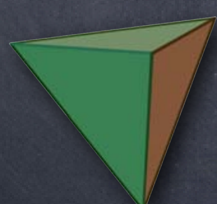


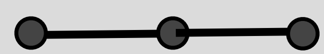
Coxeter groups and Reflection groups

- Regular polyhedra, Permutahedra and tilings in rk 3





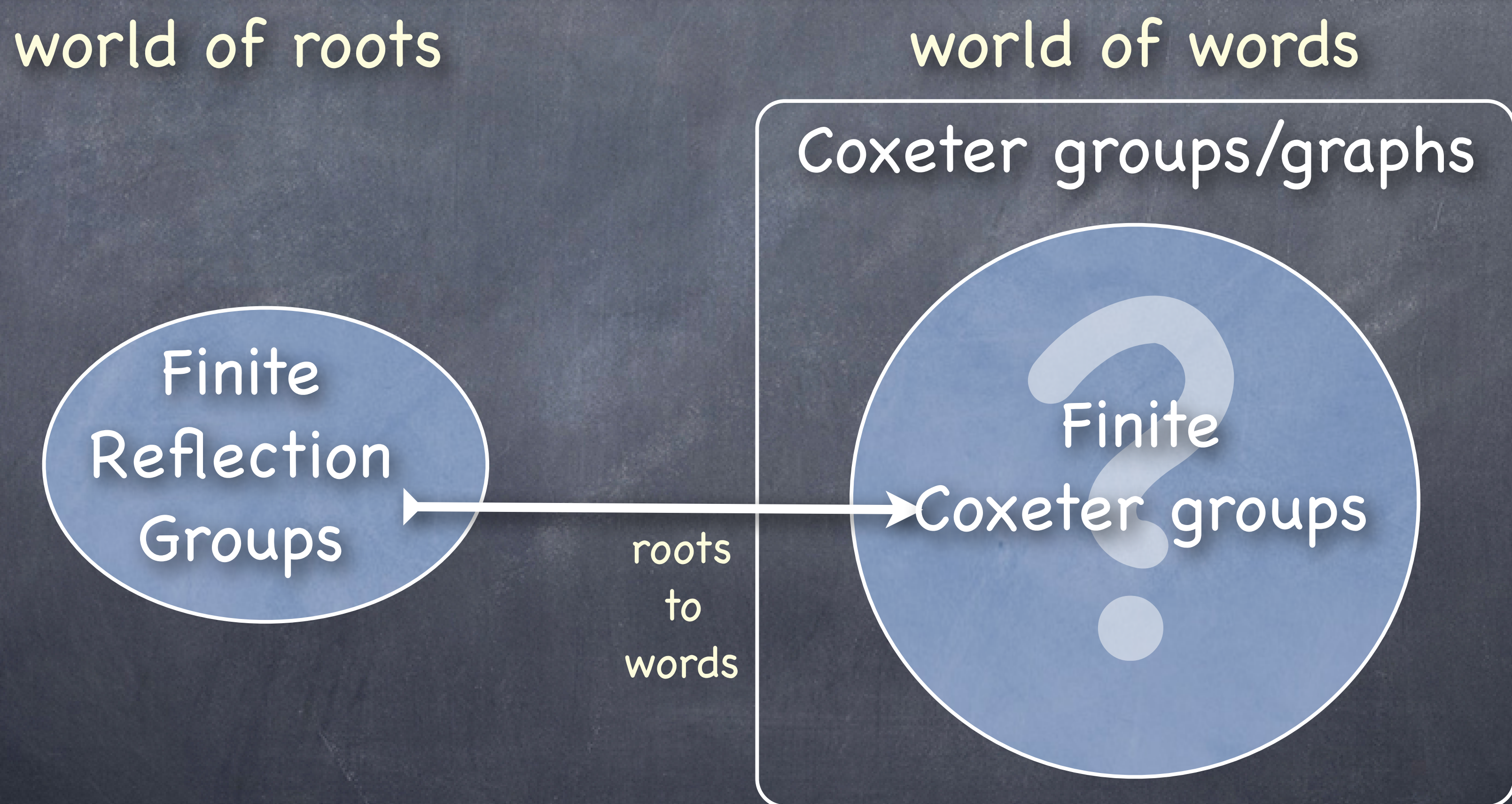


$S_4(A_3)$ 

©wikipedia for tilings and polyhedra

Coxeter groups and Reflection groups

How to find all Coxeter graphs that correspond to Finite Reflection groups (FRG)? to Finite Coxeter groups?



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Coxeter groups and Reflection groups

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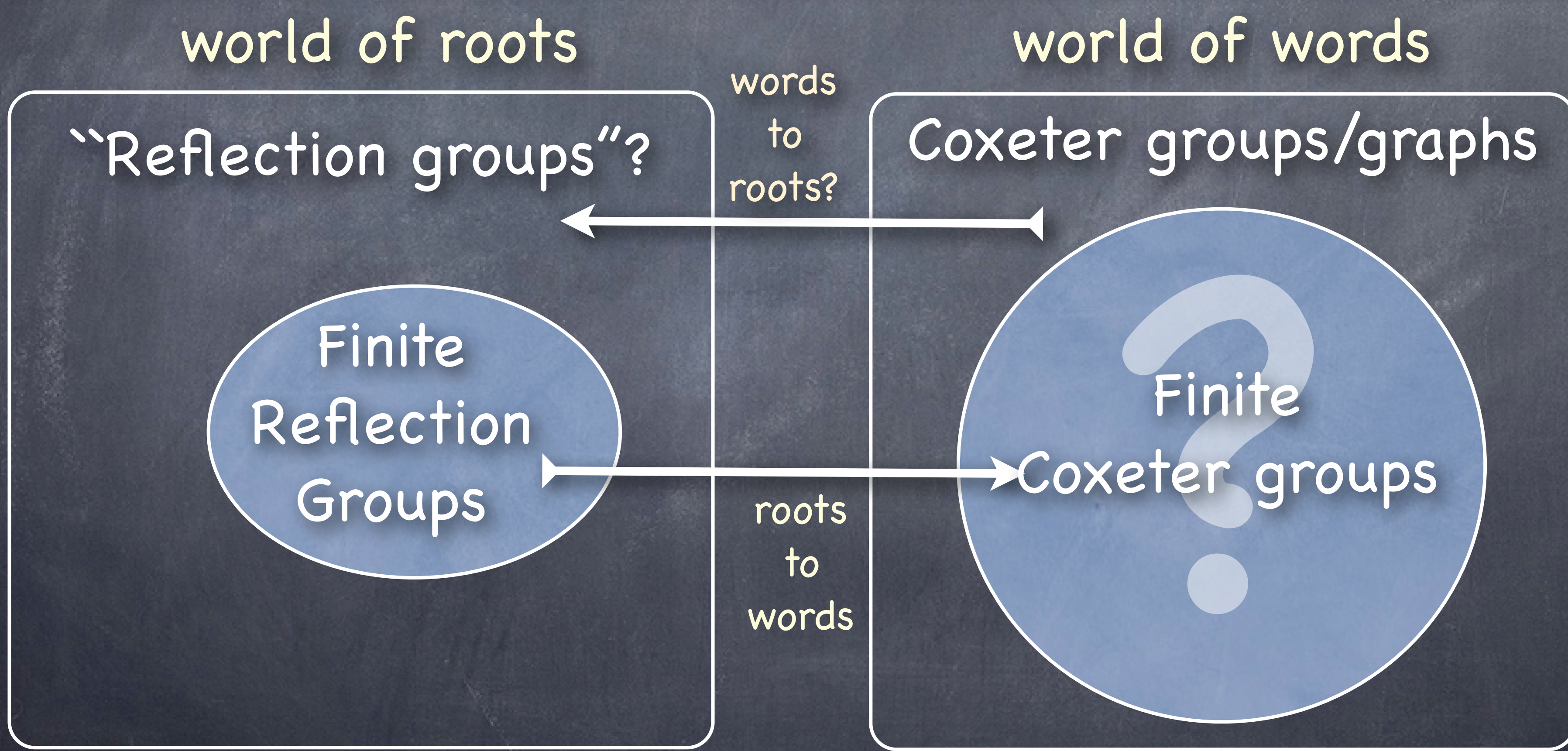


Root systems for Coxeter groups ?

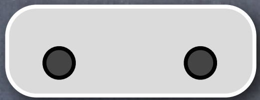
An observation

If (W,S) is a Finite Reflection Group with $\Delta \subseteq \Phi^+ \subseteq \Phi$.

- Dihedral (standard) parabolic subgroups: $I = \{s,t\} \subseteq S$
  - $W_I = \langle I \rangle \leq W$ corresponds to the subgraphs:

$s = s_\alpha \quad t = s_\beta$

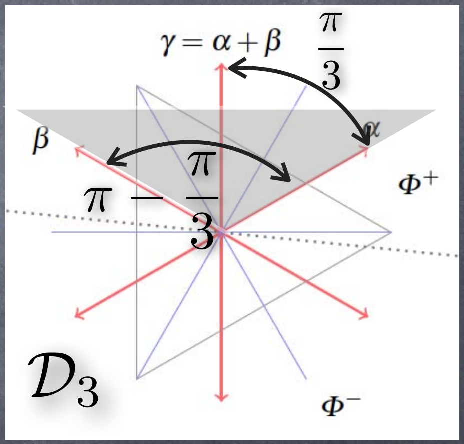
or



$s = s_\alpha \quad t = s_\beta$

  - $W_I = \mathcal{D}_{m_{st}}$ acts on $V_I = \mathrm{span}(\alpha,\beta)$:

$$s_\alpha(\beta) = \beta - 2\langle \alpha,\beta\rangle \alpha$$



  - We have: $\langle \alpha,\beta\rangle = -\cos\left(\frac{\pi}{m_{st}}\right)$

- the scalar product is given on the basis Δ by

$$(\langle \alpha,\beta\rangle)_{\alpha,\beta\in\Delta} = \left(-\cos\left(\frac{\pi}{m_{st}}\right)\right)_{s,t\in S}$$

Root systems for Coxeter groups !

- Geometric representations of a Coxeter system (W,S)
  - V real vector space with basis $\Delta=\{\alpha_s \,|\, s\in S\}$
  - B symmetric bilinear form defined by:

$$B(\alpha_s,\alpha_t)=\begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right) & \text{if } m_{st}<\infty \\ a\leq -1 & \text{if } m_{st}=\infty \end{cases}$$

  - W acts on V: $s(v)=v-2B(v,\alpha)\alpha,\ s\in S$
  - Root system: $\Phi=W(\Delta),\ \Phi^+=\mathrm{cone}(\Delta)\cap\Phi$

Theorem. (i) $W\leq \mathrm{O}_B(V)$ "B-isometries"

(ii) All the properties words/roots for FRG hold in this case (inversion sets/length, weak order etc).

Root systems for Coxeter groups !

$S)$

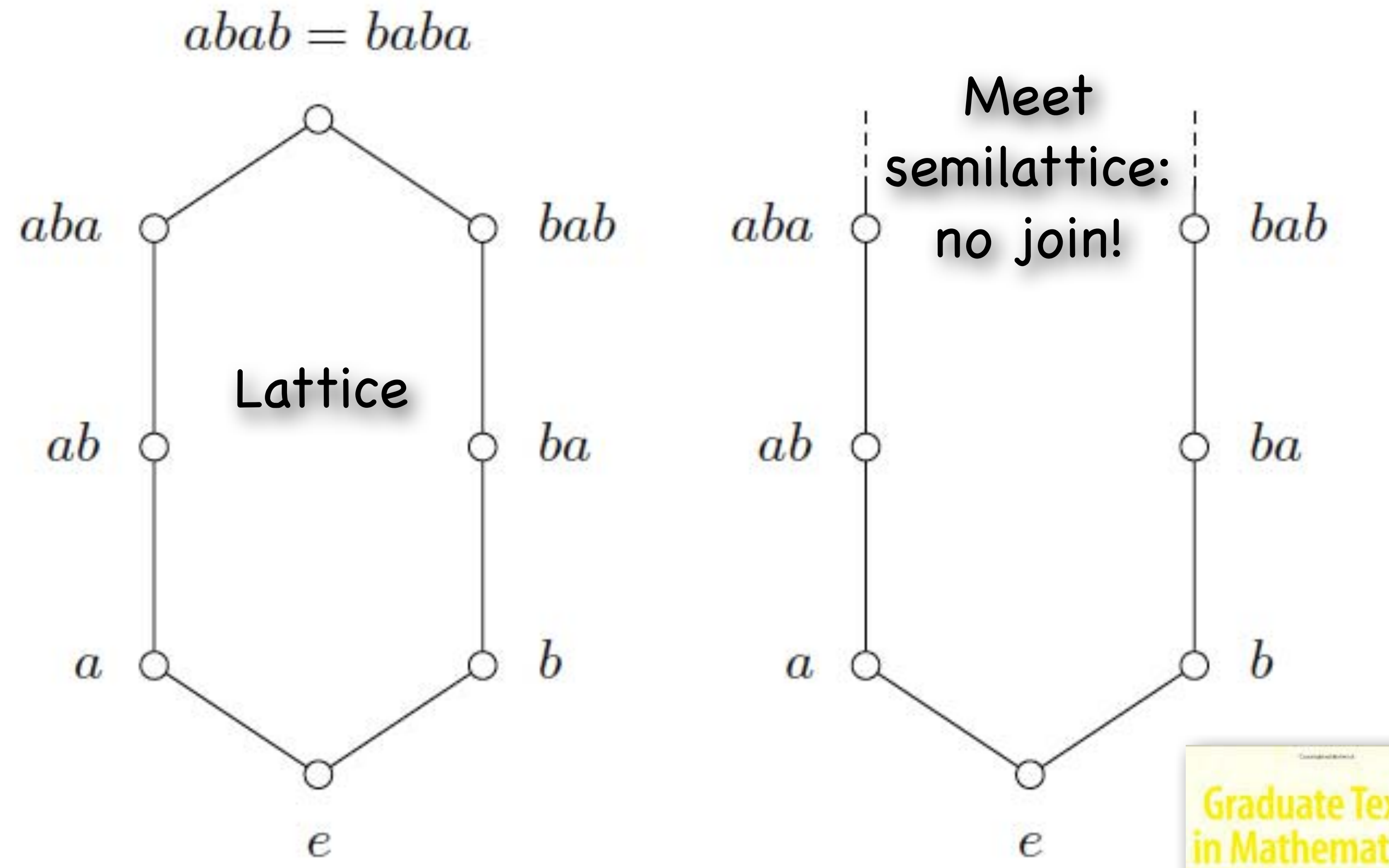
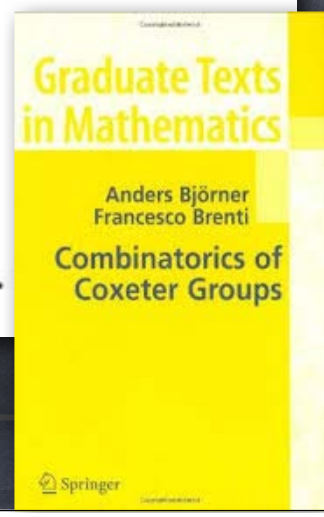


Figure 3.1. Weak order of dihedral groups.

The

this case (inversion sets/length, weak

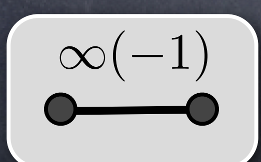


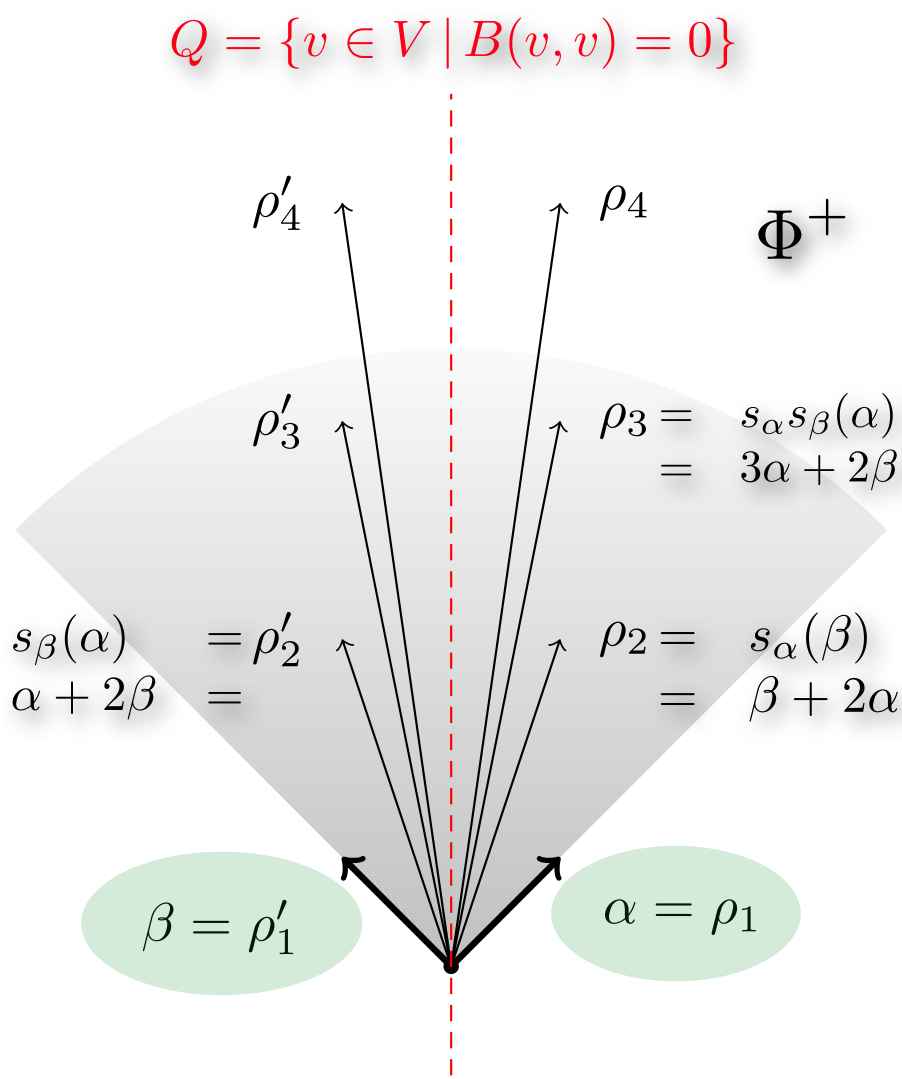
Root systems for Coxeter groups

$\rho'_n = n\alpha + (n+1)\beta$

$\rho_n = (n+1)\alpha + n\beta$

Infinite dihedral group I





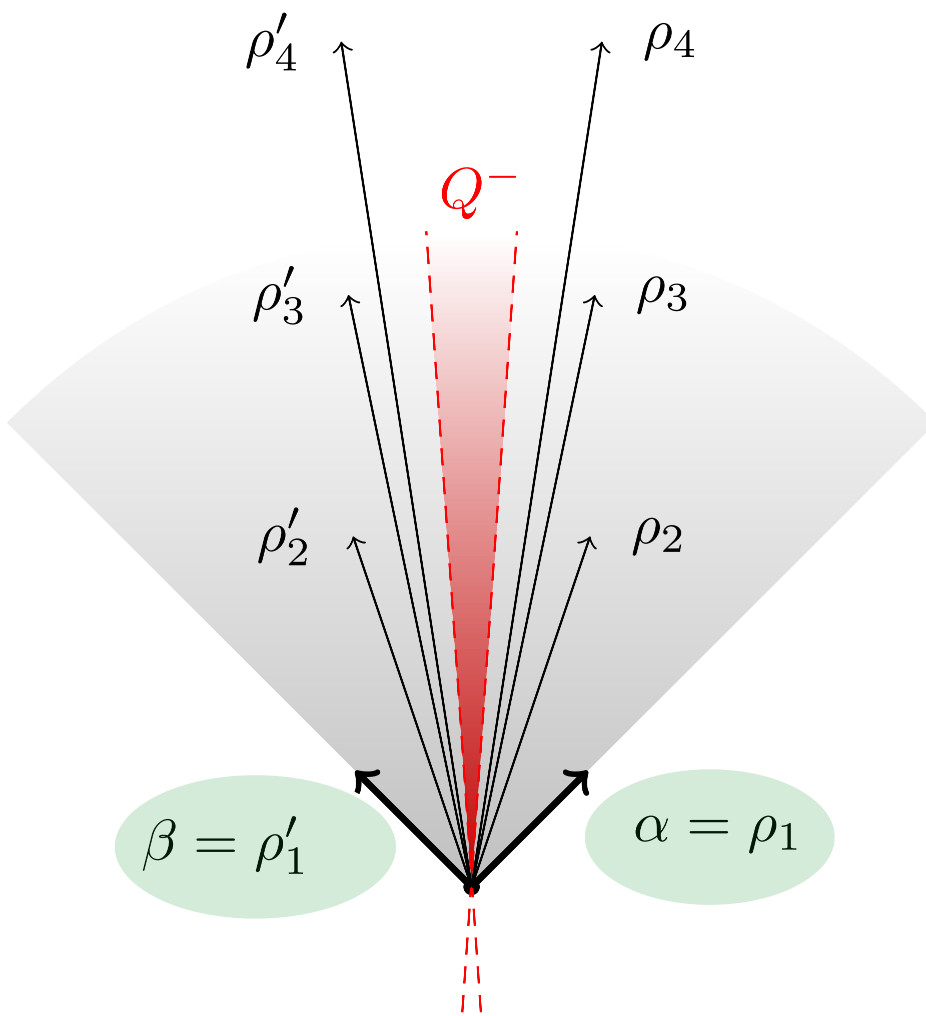
(a) $B(\alpha,\beta) = -1$

$$s_\alpha(v) = v - 2B(v,\alpha)\alpha.$$

Root systems for Coxeter groups

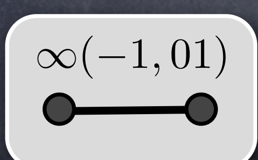
$$Q^- = \{v \in V \mid B(v,v) \leq 0\}$$

Infinite dihedral group II

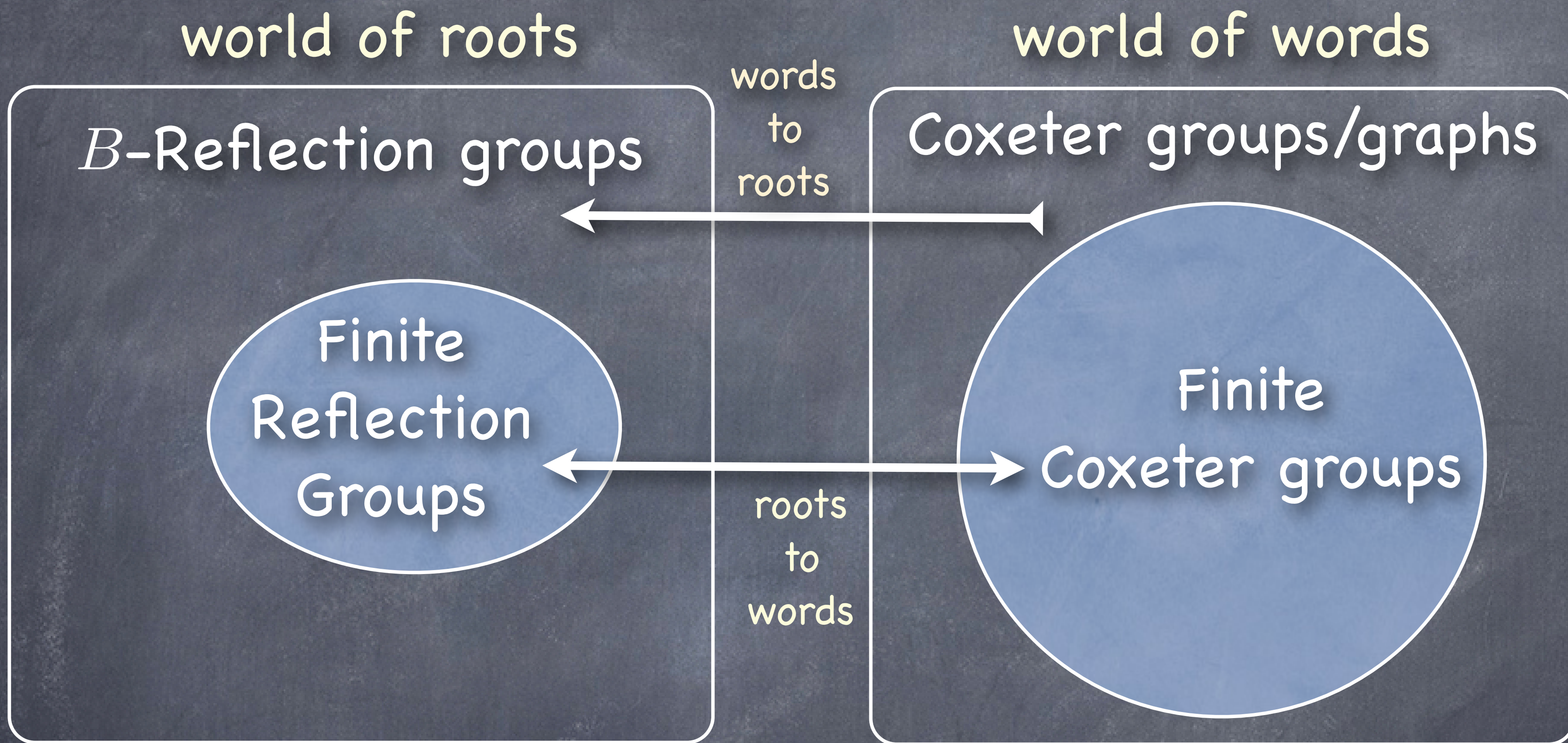


(b) $B(\alpha,\beta) = -1.01 < -1$

$$s_\alpha(v) = v - 2B(v,\alpha)\alpha.$$



Classification of Finite Reflection Groups



Theorem. The following assertions are equivalent:

(i) (W,S) is a finite Coxeter system;

(ii) B is a scalar product and $W \leq \mathrm{O}_B(V)$;

(iii) W is a finite reflection group.

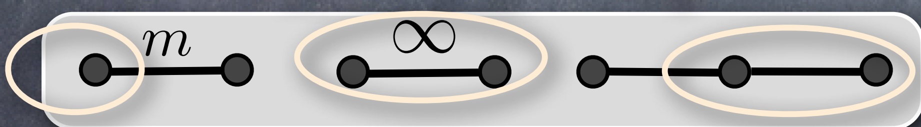
Classification of Finite Reflection Groups

- Subgraphs and standard parabolic subgroups

$I \subseteq S \longleftrightarrow \Gamma_{|I}$; (W_I, I) is a Coxeter system

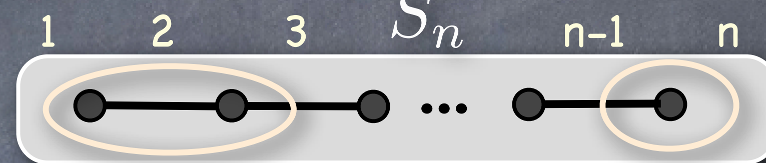
- W is irreducible iff Γ_S is connected

$\mathcal{D}_m \times \mathcal{D}_\infty \times S_4$



$\mathcal{S}_2 \times \mathcal{D}_\infty \times \mathcal{D}_3$

S_n



$W_I \simeq S_3 \times S_2$

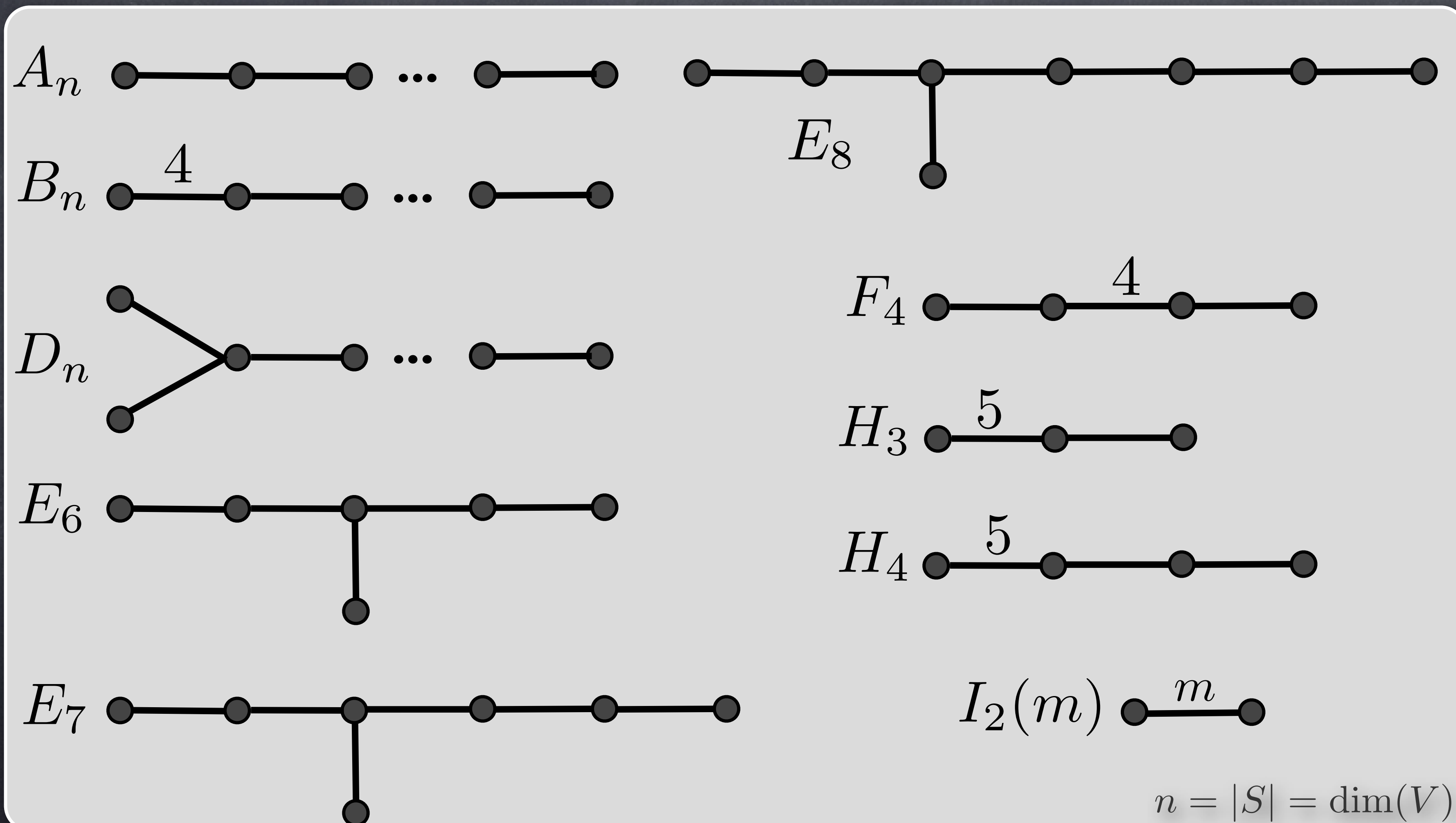
Proposition. *If $I_1, \ldots, I_k$ corresponds to the connected components of Γ_I (I may be S), then*

$$W_I \simeq W_{I_1} \times \cdots \times W_{I_k}$$

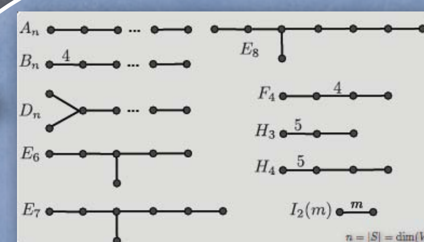
To classify finite reflection groups, i.e., finite Coxeter groups, we just have to find all connected Coxeter graphs that correspond to scalar product!

Classification of Finite Reflection Groups

Theorem. The irreducible FRG are precisely the finite irreducible Coxeter groups. Their graphs are:



Conclusion

world of roots		world of words
B-Reflection groups	words to roots	Coxeter groups/graphs
signature (p,q,r) of B ⟵		Γ_W allowing $\infty (a \leq -1)$
Finite Reflection Groups ⟷	roots to words	

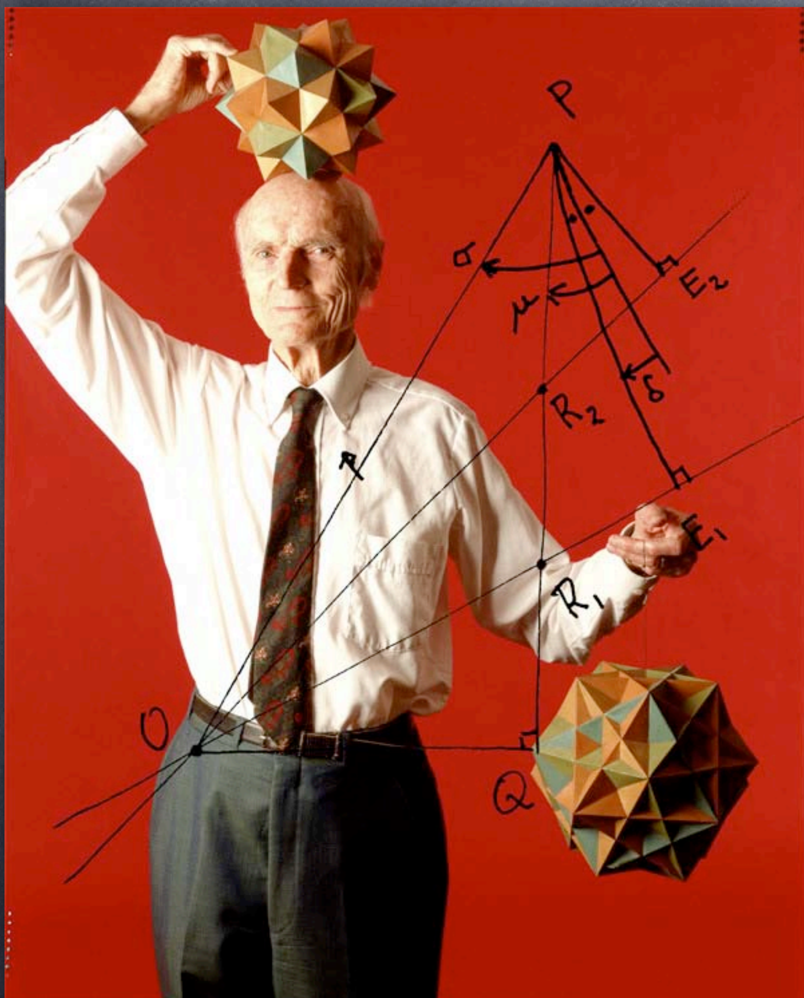
Problem: Let $p, q, r \in \mathbb{N}$, classify all the Coxeter graphs with signature (p,q,r). Count them?

N.B.: Known for $(n,0,0)$ - FRG -; $(n-1,0,1)$ - affine type - and partially for $(n-1,1,0)$ - "weakly hyperbolic" type

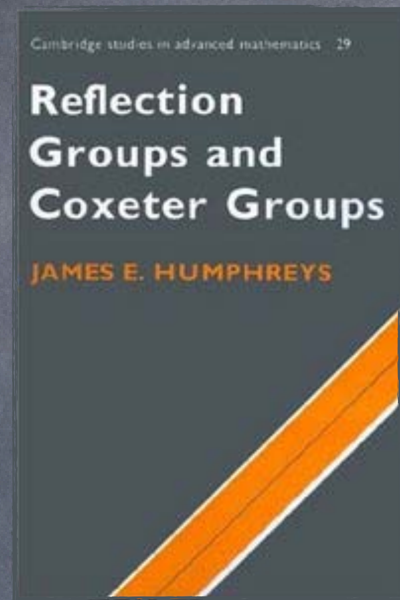
Selected biblio of Part 1 ...

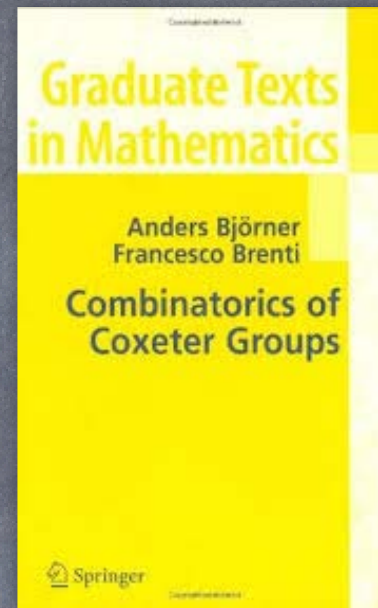
Donald Coxeter

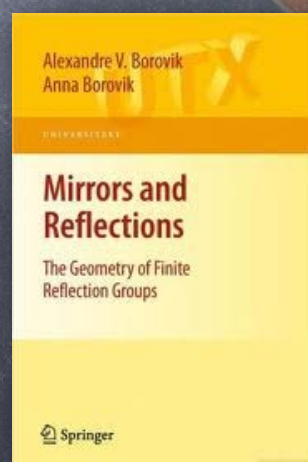
(London 1907, Toronto 2003)
Professor at University of Toronto
(1936-2003)

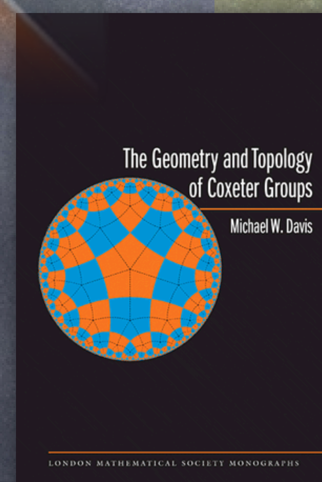


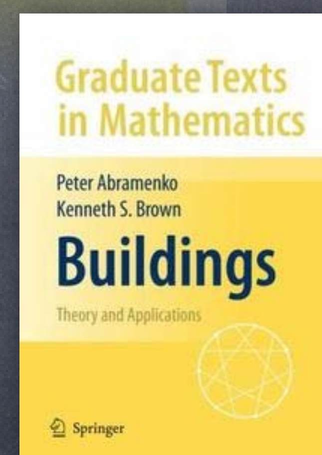
Mandatory photo credit. Mathematics genius Donald Coxeter is the subject of a public talk by journalist Siobhan Roberts, *The Man Who Saved Geometry*, on Sunday, July 31. Photo courtesy of The Banff Centre.



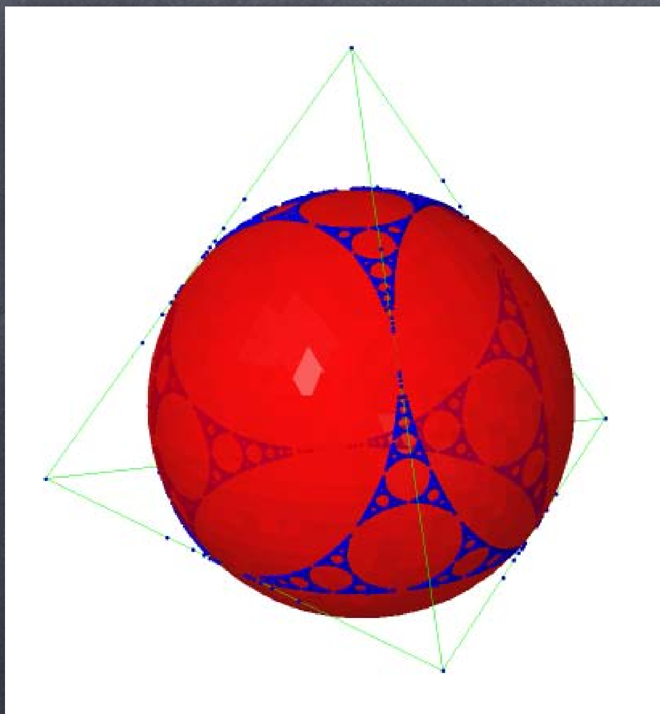


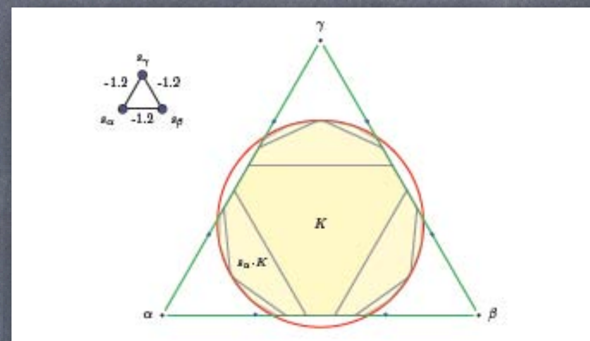


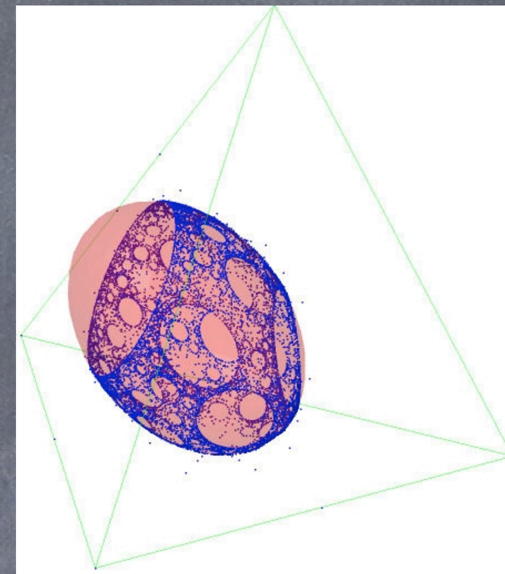


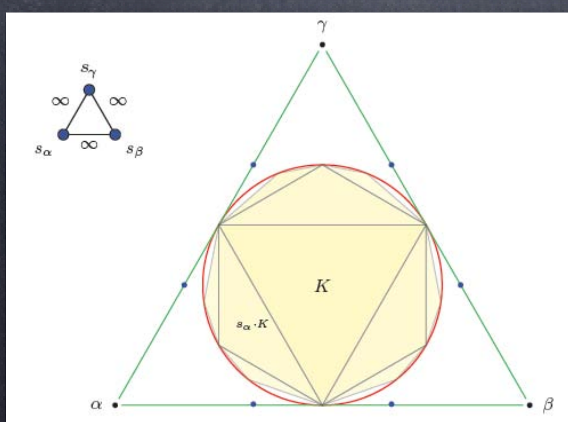


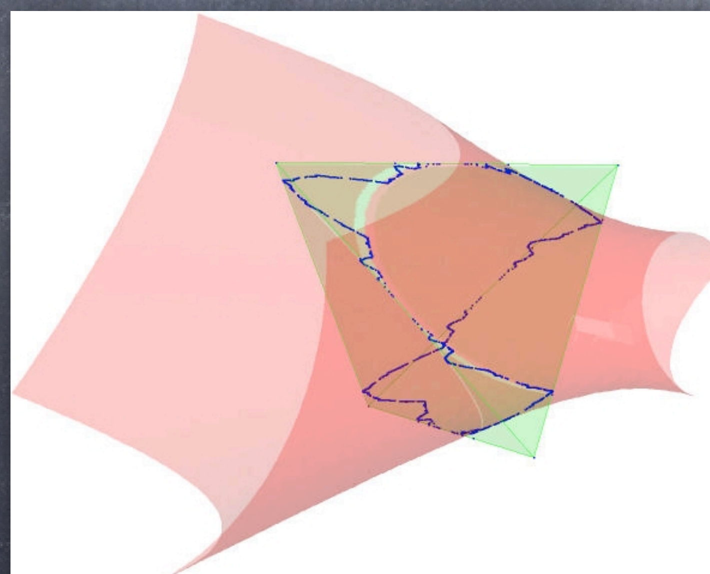
Part 3 - Roots and Words in infinite Coxeter groups

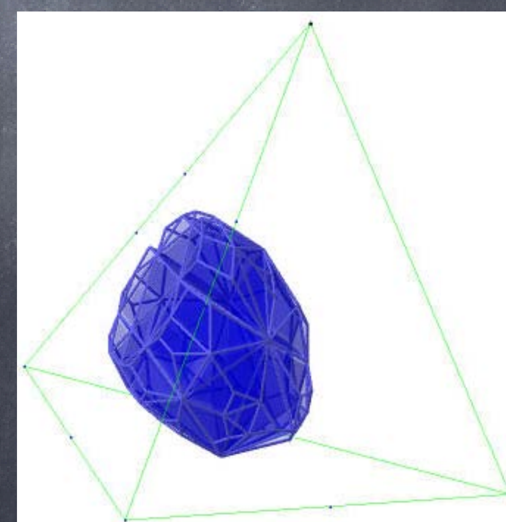












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In the last Episode

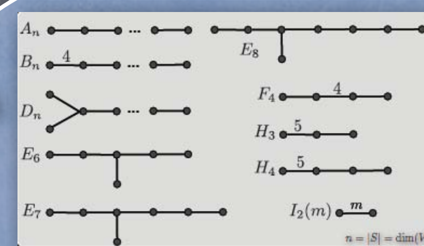
world of roots | words to roots | **world of words**

B-Reflection groups

signature (p,q,r) of B ⟵ Γ_W allowing $\infty(a \leq -1)$

Coxeter groups/graphs

Finite Reflection Groups ⟷ (roots to words)



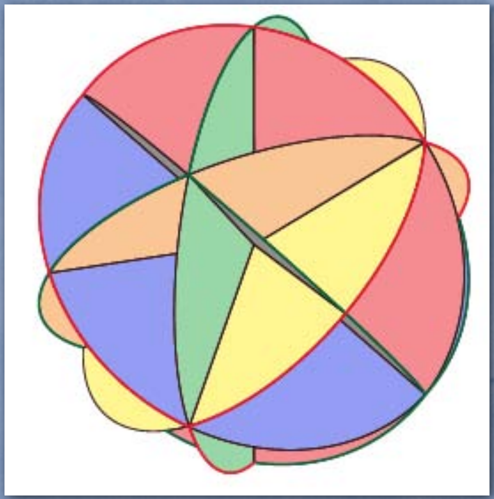
The Cayley graph of (W,S) is naturally oriented by the (right) weak order: $w < ws$ if $\ell(w) < \ell(ws)$.

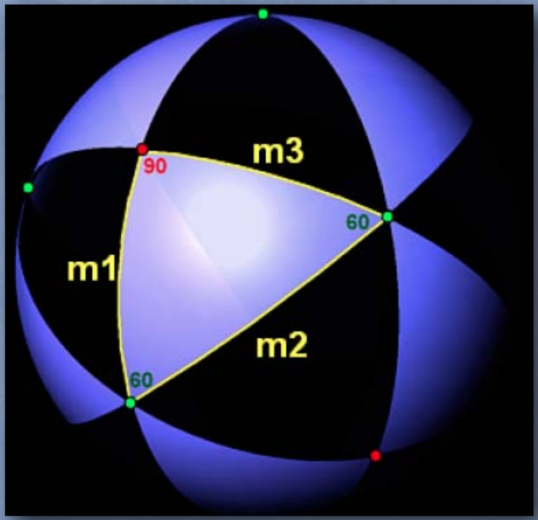
The weak order is a meet-semilattice and

$$u \leq w \iff \mathrm{inv}(u^{-1}) \subseteq \mathrm{inv}(w^{-1}) \qquad (\mathrm{inv}(w^{-1}) = \Phi^+ \cap w(\Phi^-))$$

In the last Episode

In the spherical, euclidean and hyperbolic case, all finitely generated discrete B-reflection groups are Coxeter groups (models for these geometry exist in V or its dual; `cut' these models by the hyperplanes of reflections)



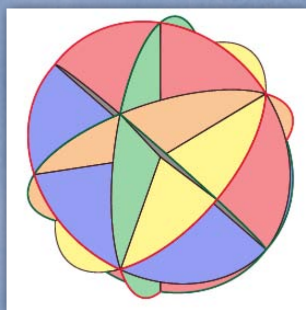


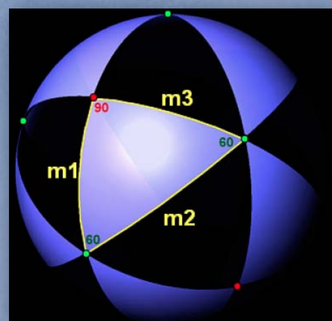
Finite case i.e. B is a scalar product ($V = V^*$): the model is the unit sphere

$$||v||^2 = B(v,v) = 1$$

In the last Episode

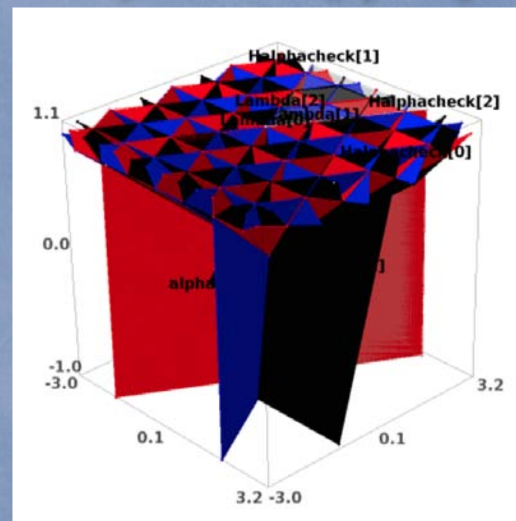
In the spherical, euclidean and hyperbolic case, they are all Coxeter groups (models for these geometry exist in V or its dual; `cut' these models by the hyperplanes of reflections)

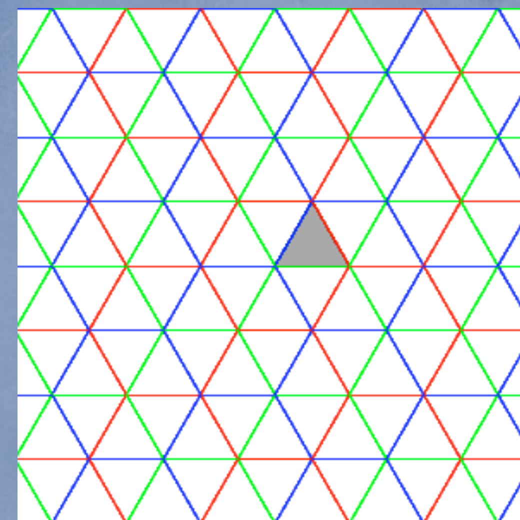




Finite case i.e. B is a scalar product

$sgn(B) = (n, 0, 0)$





Affine case i.e. B is positive degenerate. Its radical is a line: $Rad(B) = \{v \in V \mid B(v, \alpha) = 0, \ \forall \alpha \in \Delta\} = \mathbb{R}x$

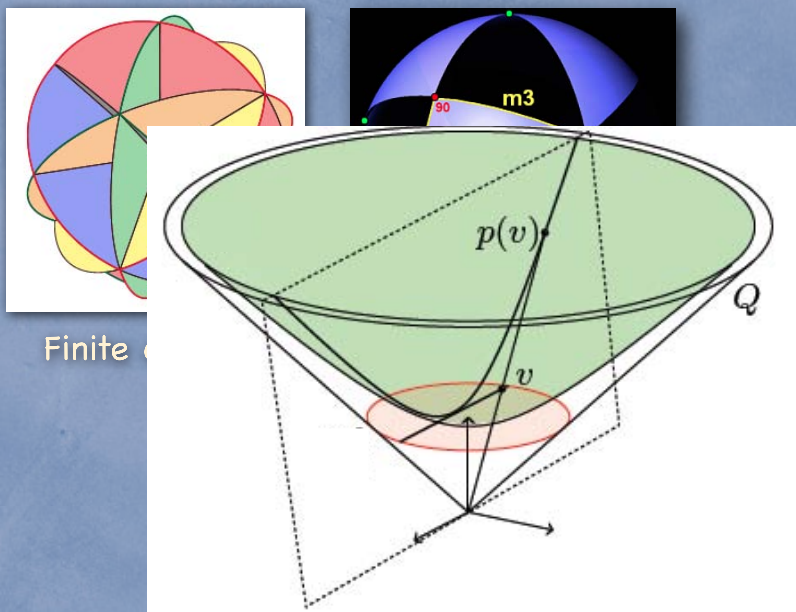
The model is an affine hyperplane in the dual V^*:

$$H = \{\varphi \in V^* \mid \varphi(x) = 1\}$$

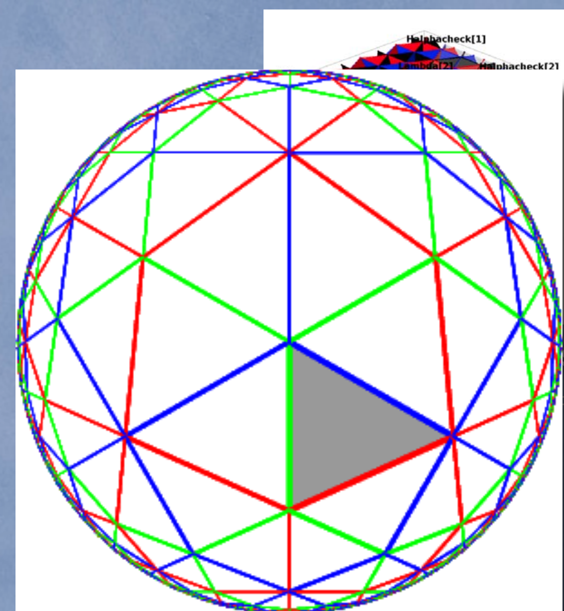
N.B: reflection hyperplanes leave in the dual here.

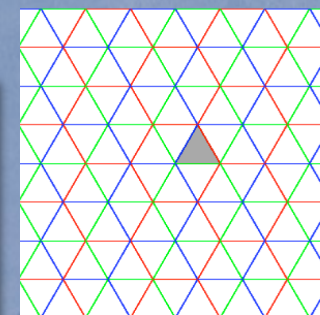
In the last Episode

In the spherical, euclidean and hyperbolic case, they are all Coxeter groups (models for these geometry exist in V or its dual; `cut' these models by the hyperplanes of reflections)



Finite





positive degenerate.
$= (n-1,0,1)$

Hyperbolic case i.e. $sgn(B) = (n-1,1,0)$ ($V = V^*$). Many models exists: projective (non conformal), hyperboloïd or the ball model

$$H^{n-1} = \{x \in V \mid B(x,x) = -1\}$$

In the last Episode

world of roots | world of words

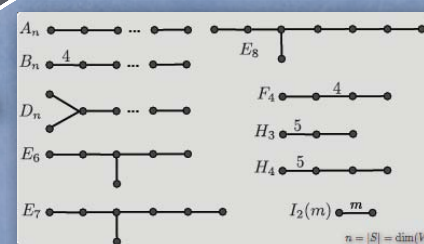
B-Reflection groups

signature (p,q,r) of B ⟵ Γ_W allowing $\infty (a \leq -1)$

words to roots

Coxeter groups/graphs

Finite Reflection Groups ⟷



roots to words

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The weak order is a meet-semilattice and

$$u \leq w \iff \mathrm{inv}(u^{-1}) \subseteq \mathrm{inv}(w^{-1})$$

$(\mathrm{inv}(w^{-1}) = \Phi^+ \cap w(\Phi^-))$

An Illustration: Words, Roots and Generalized Associahedra

ⓒsomeone on the internet

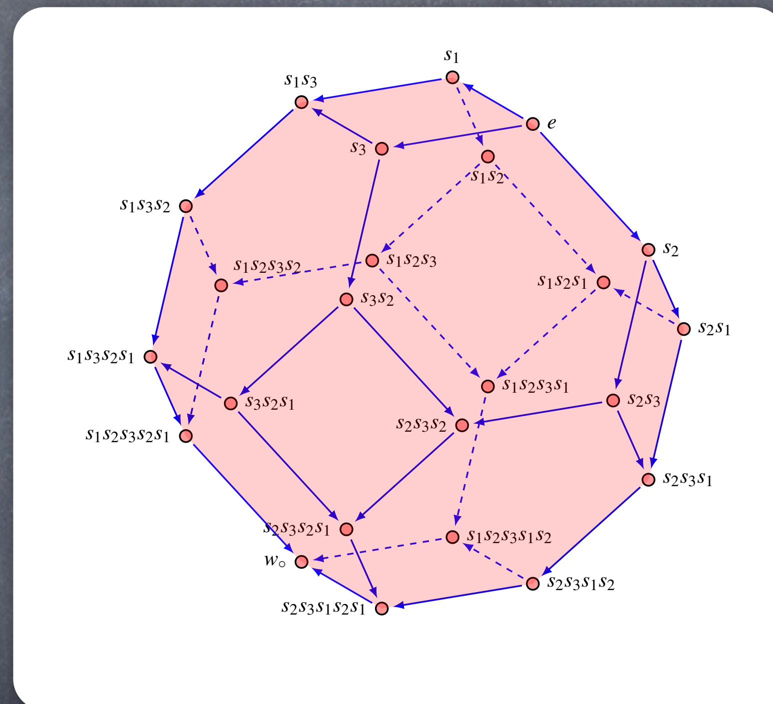
(W, S) finite Coxeter system, so $W \leq \mathrm{O}(V)$

Permutahedra

- Δ simple system;
- $S = \{s_\alpha \,|\, \alpha \in \Delta\}$;
- Choose $\boldsymbol{a}$ generic i.e.

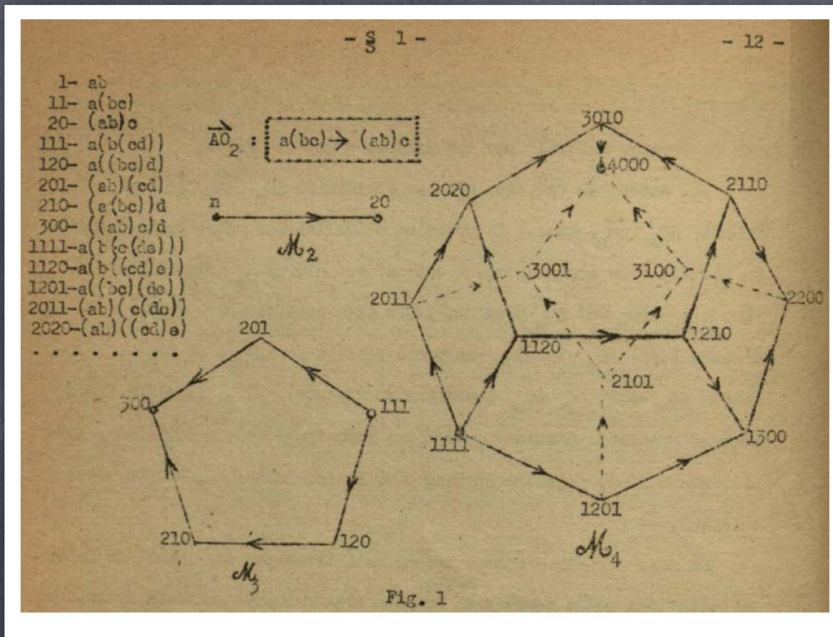
$$\langle \boldsymbol{a}, \alpha \rangle > 0, \ \forall \alpha \in \Delta$$

$$\mathrm{Perm}^{\boldsymbol{a}}(W) = \mathrm{conv}\,\{w(\boldsymbol{a}) \,|\, w \in W\}$$



Proposition. $\mathrm{Perm}^{\boldsymbol{a}}(W)$ is a simple polytope whose oriented 1-skeleton is the graph of the (right) weak order.

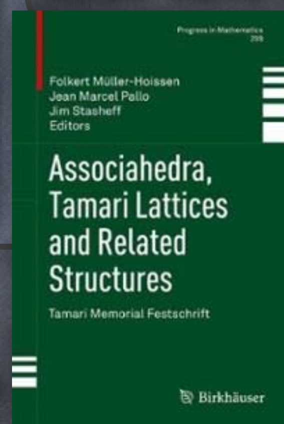
Building Generalized Associahedra



Tamari's associahedron

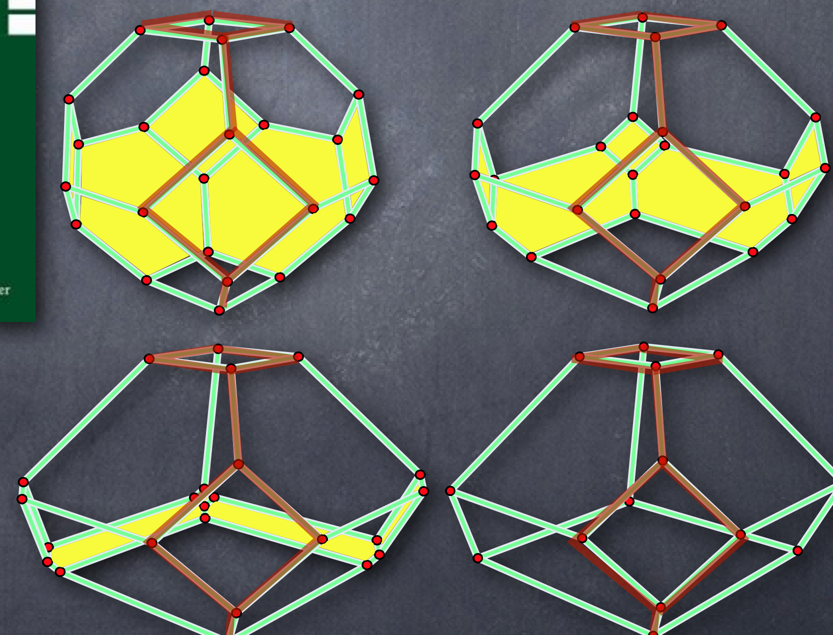
Associahedra (lattices/complexes):

- Lattice (Tamari, 1951)
- Cell complex (Stasheff, 1963)
- Cluster complex (Fomin-Zelevinsky, 2003)
- Cambrian lattices (Reading 2007, 2007)and more …



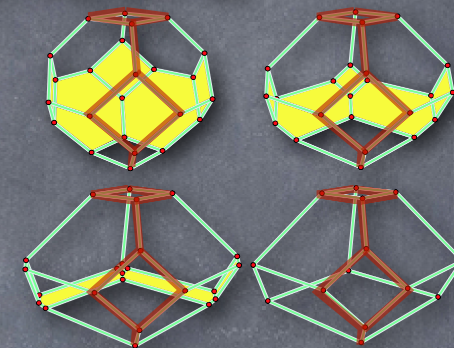
Associahedra (Convex polytopes):

- Type A (Haiman 1984, Lee, Loday, …)
- Type B - cyclohedra (Bott-Taubes 1994, …)
- Weyl groups (Chapoton-Fomin-Zelevinsky, 2003)
- from permutahedra of finite Coxeter groups (CH-Lange-Thomas 2011, …)



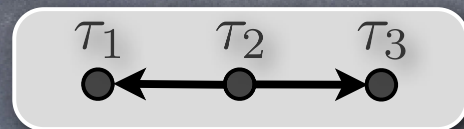
Building Generalized Associahedra

Hohlweg, C. Lange, H. Thomas (2009)



- **Data:** $\mathrm{Perm}^{a}(W)$ and an orientation of Γ_W

$W = S_4$



- c Coxeter element associated to this orientation i.e product without repetition of all the simple reflections;

$$c = \tau_2\tau_3\tau_1$$

- $c_{(I)}$ subword with letters $I \subseteq S$

$$I = \{\tau_1, \tau_2\} \subseteq S \Rightarrow c_{(I)} = \tau_2\tau_1$$

- c - word of $w_\circ$: $\boldsymbol{w_o}(c) = c_{(K_1)}c_{(K_2)}\dots c_{(K_p)}$ reduced expression s.t. $S \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_p \neq \emptyset$

$$\boldsymbol{w_o}(\tau_1\tau_2\tau_3) = \tau_1\tau_2\tau_3.\tau_1\tau_2.\tau_1 = c_{(S)}c_{(\{\tau_1,\tau_2\})}c_{(\{\tau_1\})}$$

$$\boldsymbol{w_o}(\tau_2\tau_3\tau_1) = \tau_2\tau_3\tau_1.\tau_2\tau_3\tau_1 = c_{(S)}c_{(S)}.$$

Building Generalized Associahedra

Hohlweg, C. Lange, H. Thomas (2009)

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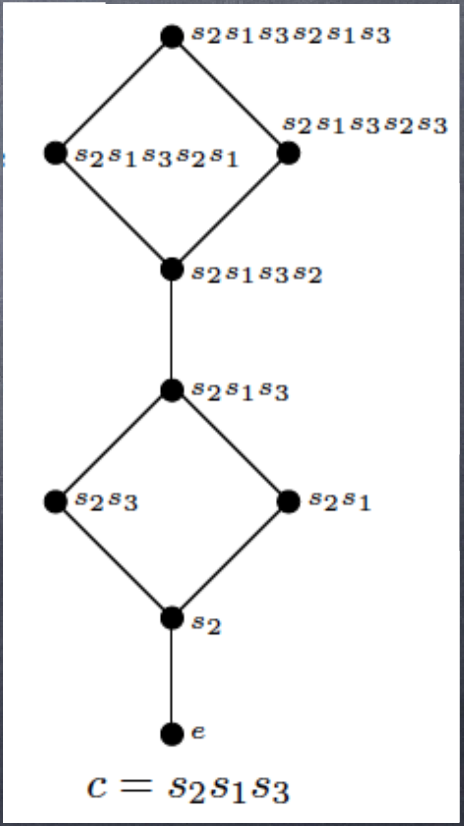
$$\boldsymbol{w_o}(\tau_2\tau_3\tau_1) = \tau_2\tau_3\tau_1.\tau_2\tau_3\tau_1 = c_{(S)} c_{(S)}.$$

- c - singletons are the prefixes of $\boldsymbol{w_o}(c)$ up to commutations

e, $\tau_2\tau_3$, $\tau_2\tau_3\tau_1\tau_2\tau_3$,
τ_2, $\tau_2\tau_3\tau_1$, $\tau_2\tau_3\tau_1\tau_2\tau_1$, and
$\tau_2\tau_1$, $\tau_2\tau_3\tau_1\tau_2$, $w_o = \tau_2\tau_1\tau_3\tau_2\tau_1\tau_3$.

Proposition. c - singletons form a distributive sublattice of the weak order.



Christophe Hohlweg, 2013

Building Generalized Associahedra

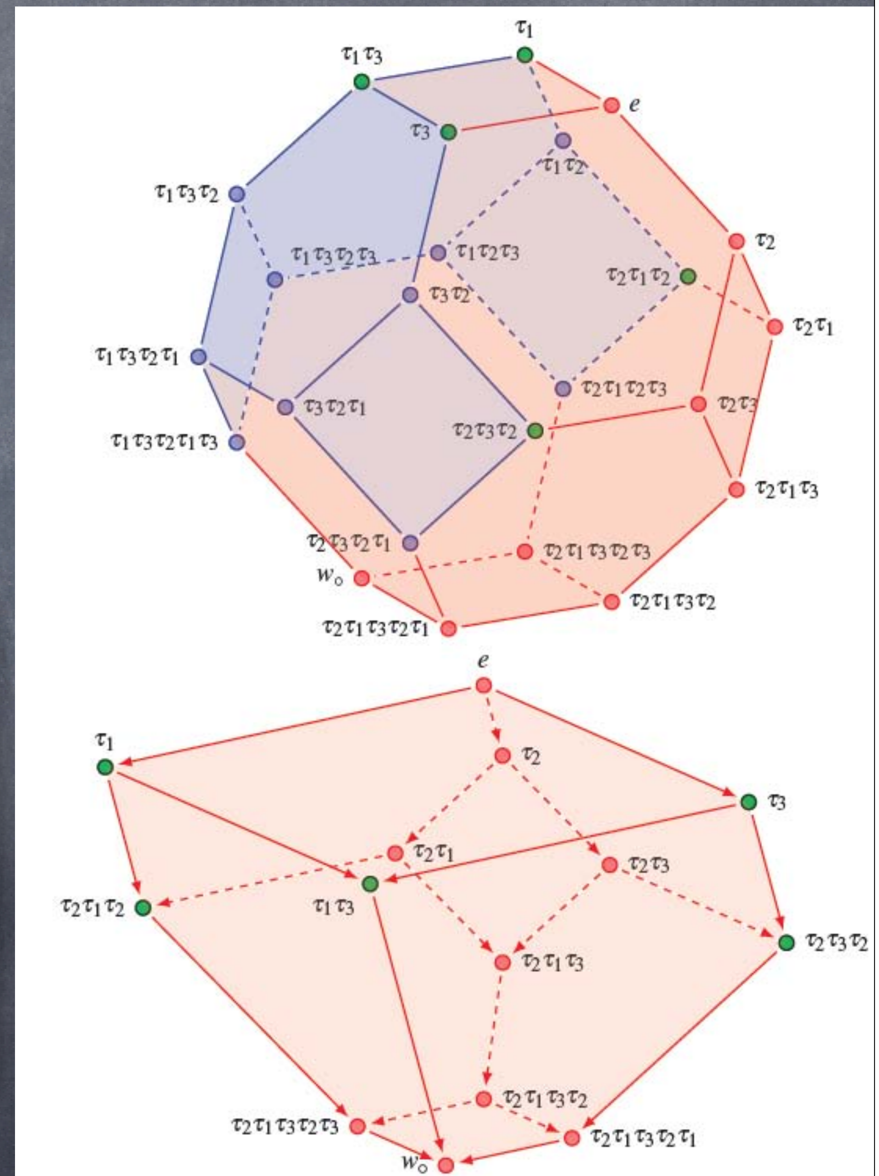
Hohlweg, C. Lange, H. Thomas (2009)

- c - generalized associahedron is the polytope $\mathrm{Asso}_c^{\boldsymbol{a}}(W)$ obtained from $\mathrm{Perm}^{\boldsymbol{a}}(W)$ by keeping only the facets containing a c - singleton

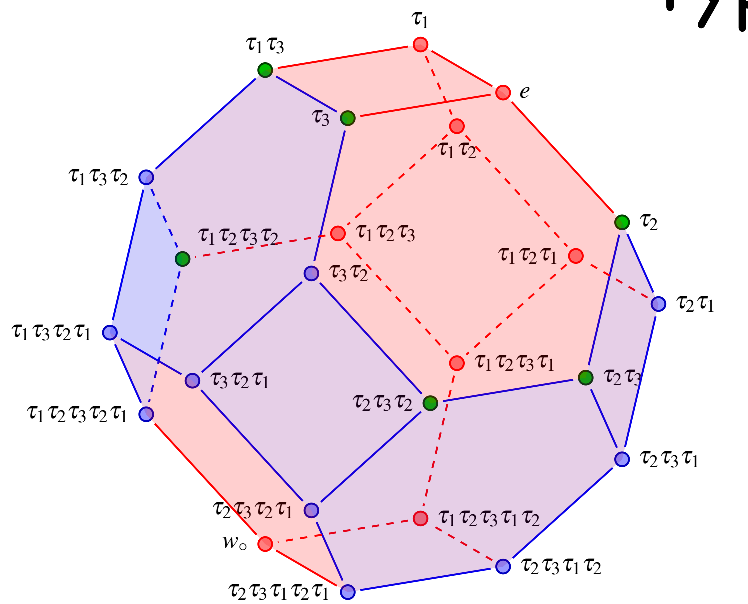
Theorem. The 1-skeleton of

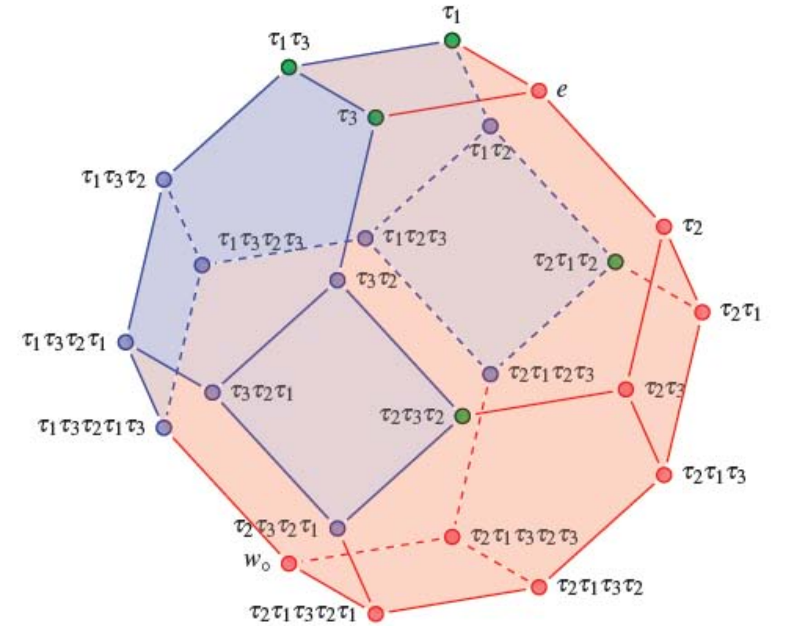
$$\mathrm{Asso}_c^{\boldsymbol{a}}(W)$$

is N. Reading's c- Cambrian lattice; its normal fan is the corresponding Cambrian fan studied in detailed by N. Reading & D. Speyer.



Type A


τ_1
$\tau_1\tau_3$
e
τ_3
$\tau_1\tau_2$
$\tau_1\tau_3\tau_2$
τ_2
$\tau_1\tau_2\tau_3\tau_2$
$\tau_1\tau_2\tau_3$
$\tau_1\tau_2\tau_1$
$\tau_3\tau_2$
$\tau_2\tau_1$
$\tau_1\tau_3\tau_2\tau_1$
$\tau_3\tau_2\tau_1$
$\tau_1\tau_2\tau_3\tau_1$
$\tau_2\tau_3\tau_2$
$\tau_2\tau_3$
$\tau_1\tau_2\tau_3\tau_2\tau_1$
$\tau_2\tau_3\tau_1$
$\tau_2\tau_3\tau_2\tau_1$
$\tau_1\tau_2\tau_3\tau_1\tau_2$
$w_\circ$
$\tau_2\tau_3\tau_1\tau_2$
$\tau_2\tau_3\tau_1\tau_2\tau_1$


τ_1
$\tau_1\tau_3$
e
τ_3
$\tau_1\tau_2$
$\tau_1\tau_3\tau_2$
τ_2
$\tau_1\tau_3\tau_2\tau_3$
$\tau_1\tau_2\tau_3$
$\tau_2\tau_1\tau_2$
$\tau_3\tau_2$
$\tau_2\tau_1$
$\tau_1\tau_3\tau_2\tau_1$
$\tau_2\tau_1\tau_2\tau_3$
$\tau_3\tau_2\tau_1$
$\tau_2\tau_3\tau_2$
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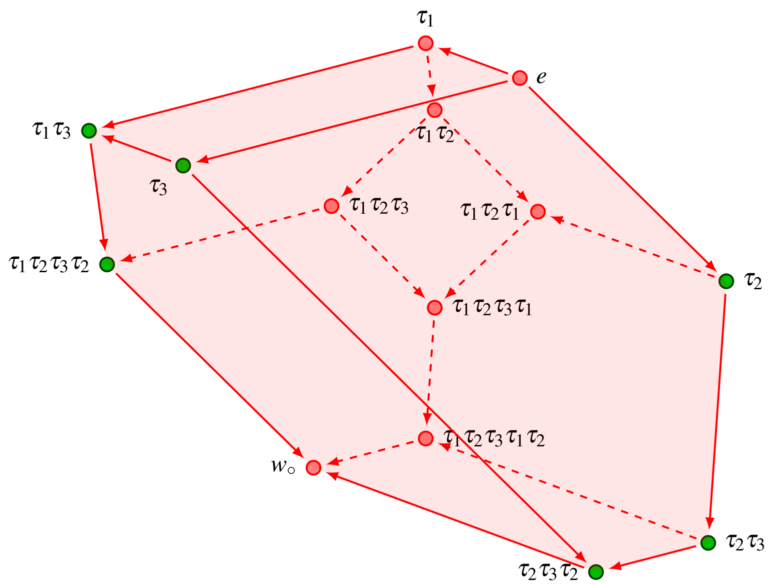
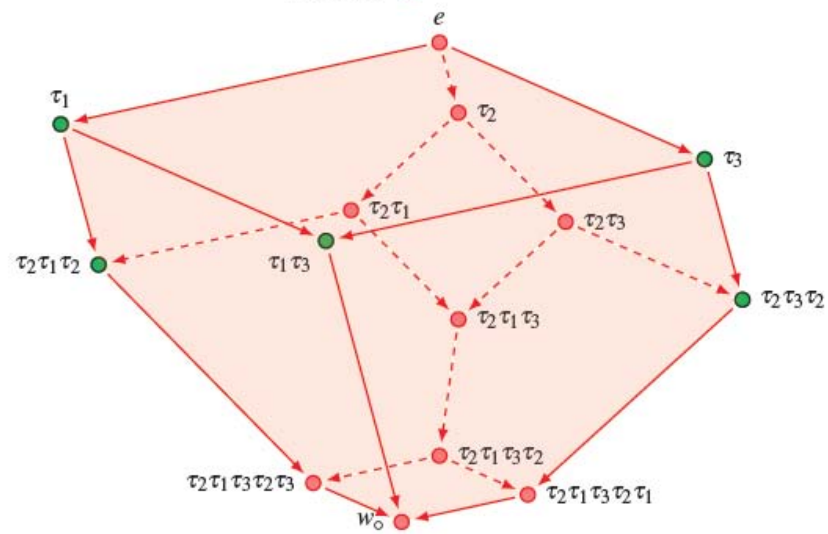
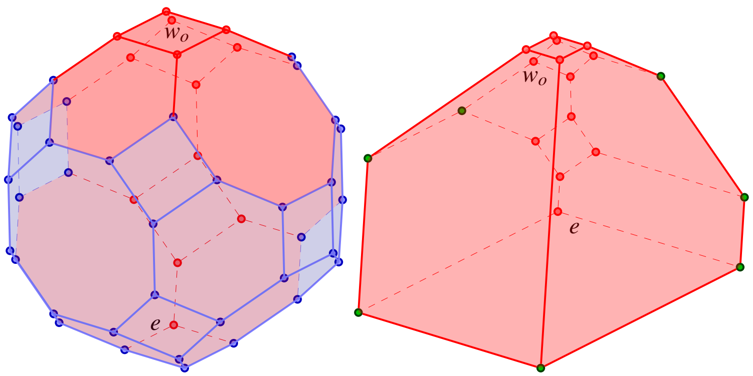
$\tau_1\tau_3\tau_2\tau_1\tau_3$
$\tau_2\tau_1\tau_3$
$\tau_2\tau_3\tau_2\tau_1$
$\tau_2\tau_1\tau_3\tau_2\tau_3$
$w_\circ$
$\tau_2\tau_1\tau_3\tau_2$
$\tau_2\tau_1\tau_3\tau_2\tau_1$

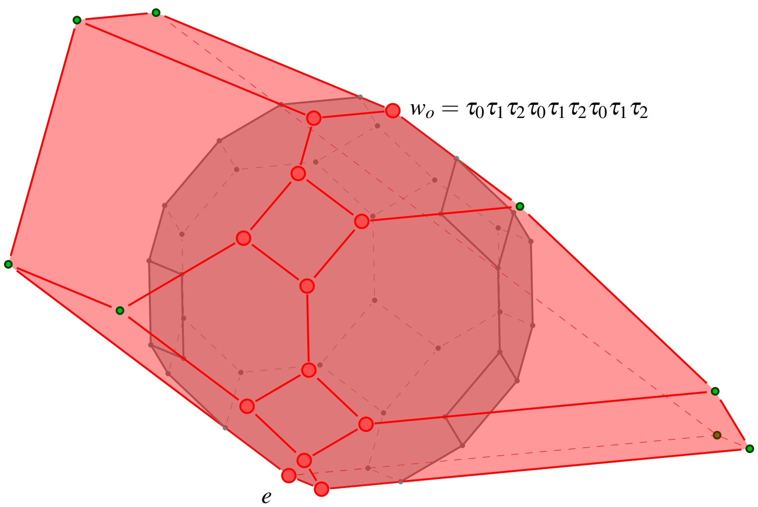

τ_1
e
$\tau_1\tau_3$
$\tau_1\tau_2$
τ_3
$\tau_1\tau_2\tau_3$
$\tau_1\tau_2\tau_1$
$\tau_1\tau_2\tau_3\tau_2$
τ_2
$\tau_1\tau_2\tau_3\tau_1$
$\tau_1\tau_2\tau_3\tau_1\tau_2$
$w_\circ$
$\tau_2\tau_3$
$\tau_2\tau_3\tau_2$


e
τ_1
τ_2
τ_3
$\tau_2\tau_1$
$\tau_2\tau_3$
$\tau_2\tau_1\tau_2$
$\tau_1\tau_3$
$\tau_2\tau_3\tau_2$
$\tau_2\tau_1\tau_3$
$\tau_2\tau_1\tau_3\tau_2$
$\tau_2\tau_1\tau_3\tau_2\tau_3$
$\tau_2\tau_1\tau_3\tau_2\tau_1$
$w_\circ$

Type B

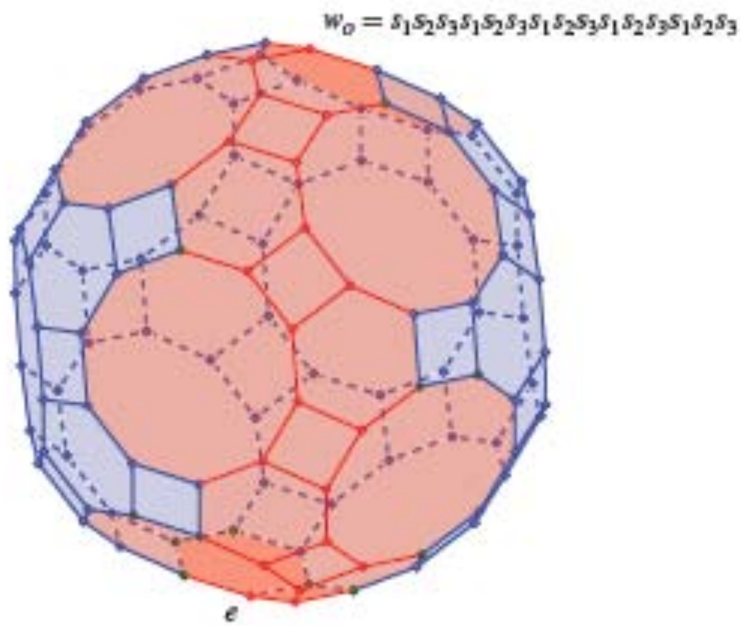


w_o
e
w_o
e

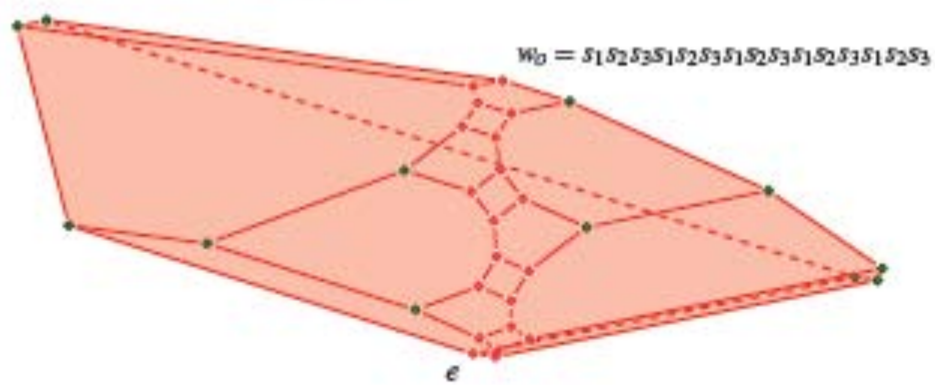


$w_o = \tau_0\tau_1\tau_2\tau_0\tau_1\tau_2\tau_0\tau_1\tau_2$
e

Type H



$w_o = s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3$
e



$w_o = s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3s_1s_2s_3$
e

Selected developements on the subject

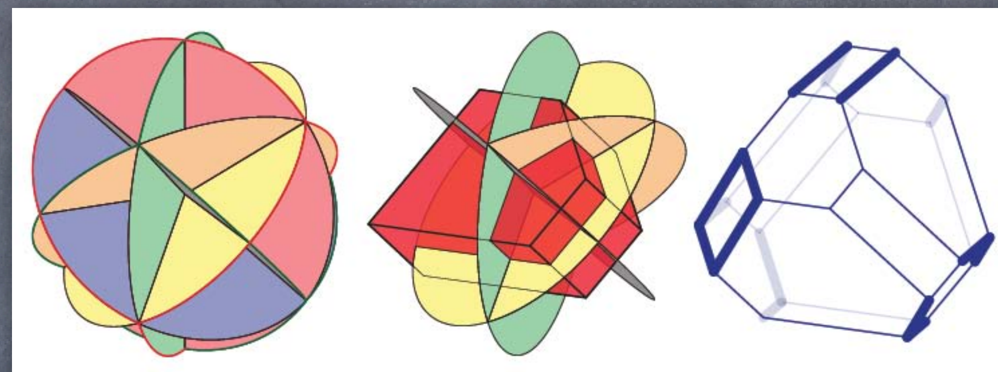
- Convex hull of the vertices: brick polytopes. Barycenter identical to the permutahedron:

V. Pilaud and C. Stump:

1. Brick polytopes of spherical subword complexes: A new approach to generalized associahedra (2012)

2. Vertex barycenter of generalized associahedra (2012)

© Pilaud-Stump

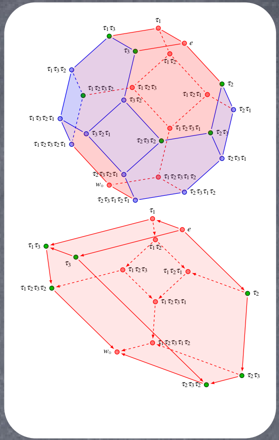


- Classification of isometry classes in term of the lattices of $c-$singletons (N. Bergeron, Hohlweg, C. Lange, H. Thomas, 2009)
- Recovering the corresponding cluster algebra:

S. Stella, Polyhedral models for generalized associahedra via Coxeter elements (2013)

ASSOCIAHEDRA IN INFINITE CASE ?

ASSOCIAHEDRA IN INFINITE CASE ?



My original motivation (2010): to generalize this approach in the infinite case ...

- Infinite case: Cambrian meet-semilattices (Sortable Elements in Infinite Coxeter Groups, N. Reading and D. Speyer, 2011) are not big enough ...

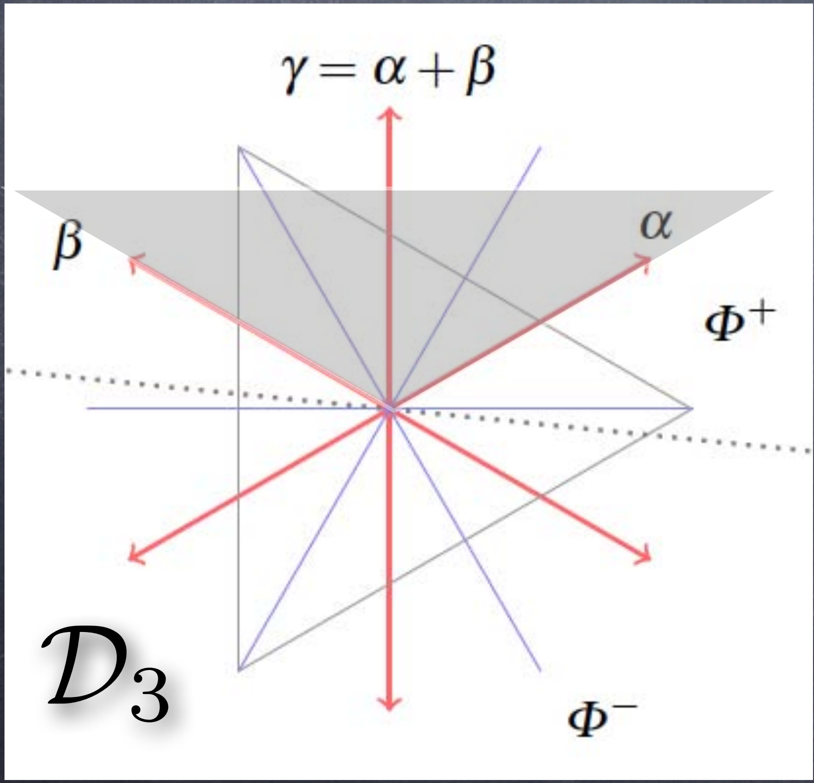
Problem: is it possible to «enlarge» Coxeter groups to have reasonable candidates with a weak order that is a complete lattice ? An answer may lie on the side of inversion sets!

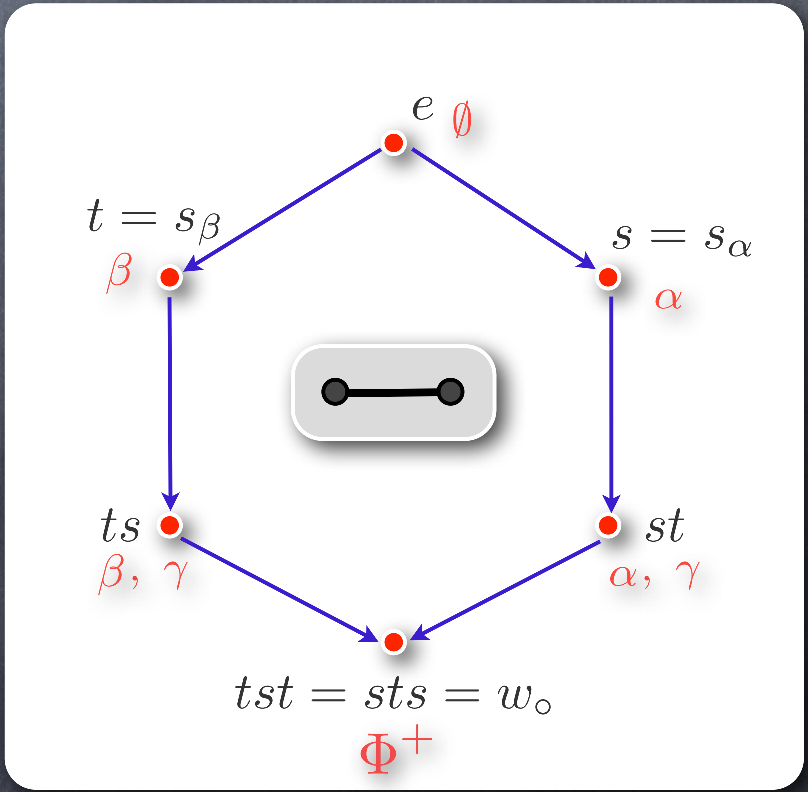
More on the weak order

Weak order: write $N(w) = \mathrm{inv}(w^{-1})$ then $u \leq v \iff N(u) \subseteq N(v)$

Proposition. The map $N : W \to \mathcal{P}(\Phi^+)$ is an injective morphism of meet-semilattice. Reduced expressions `are' chains in intervals.

What is $\mathrm{Im}(N)$?





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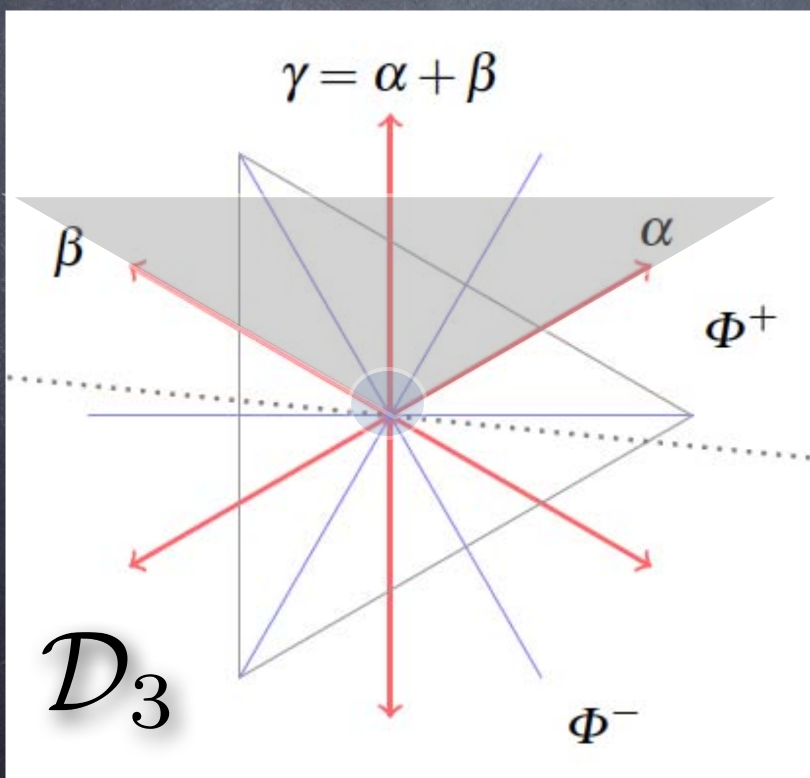

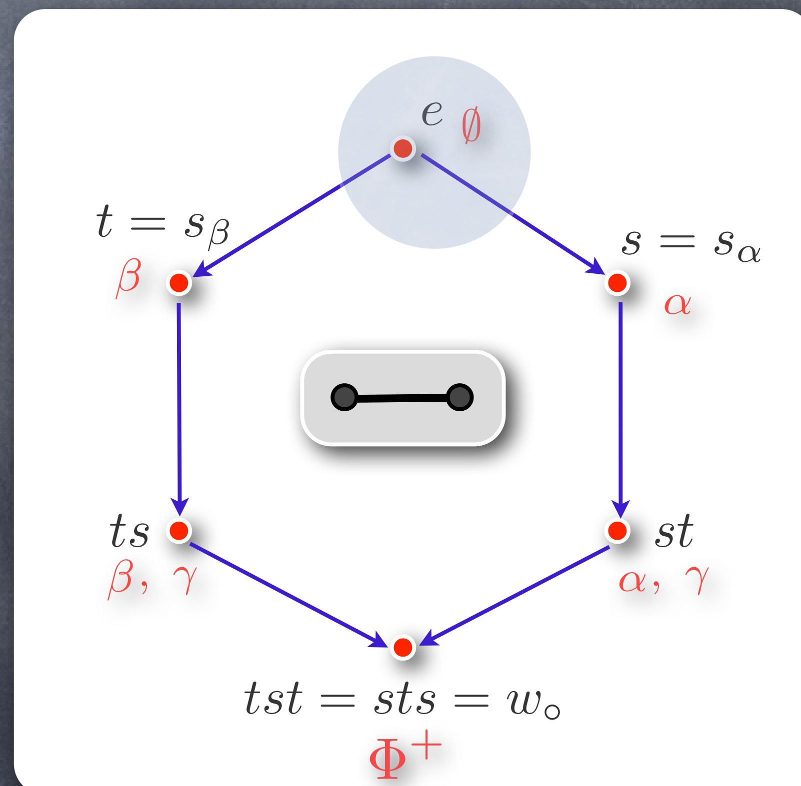

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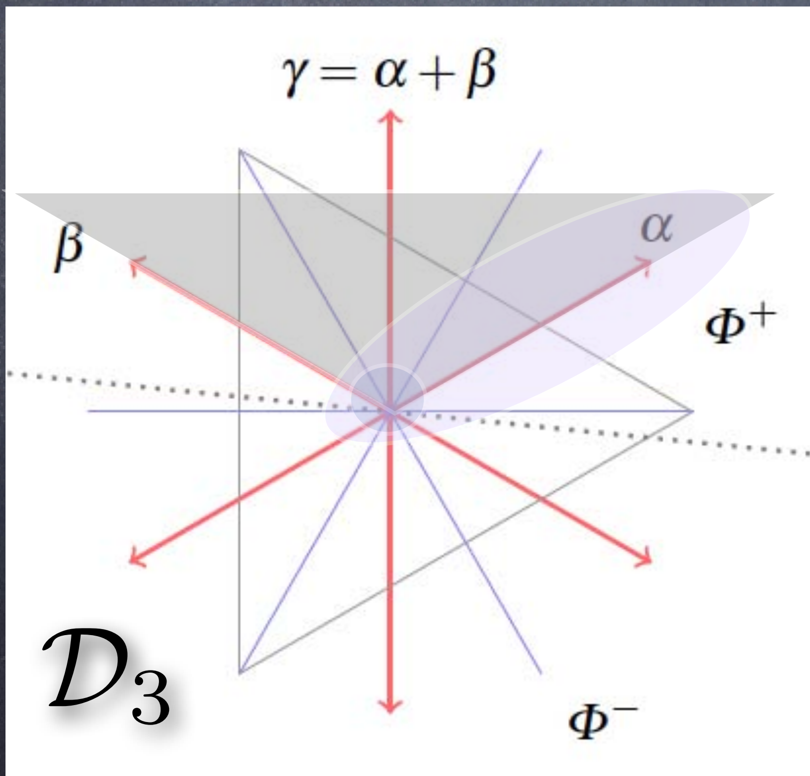

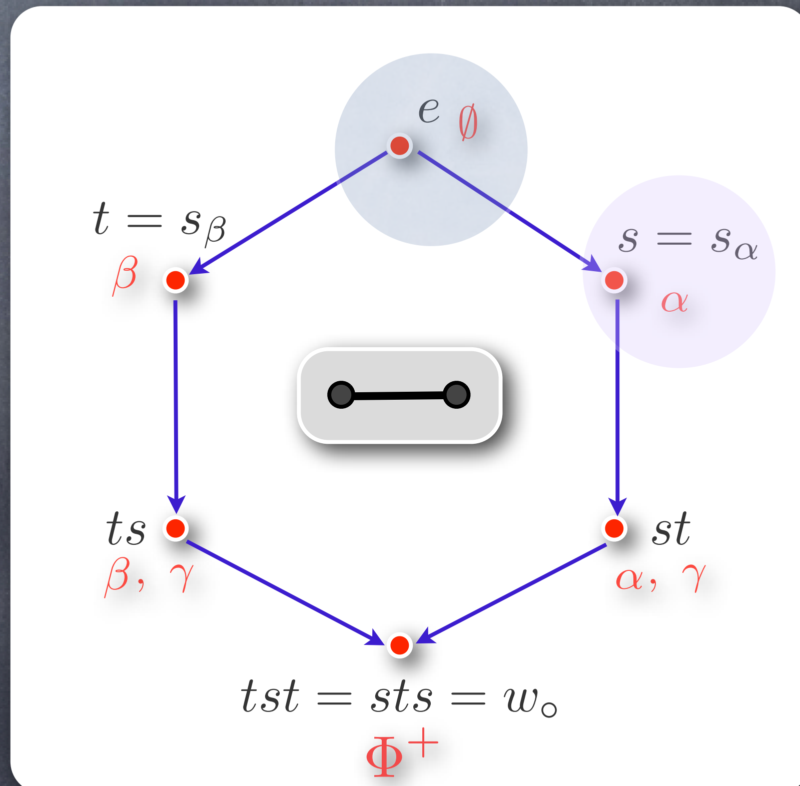

Christophe Hohlweg, 2013

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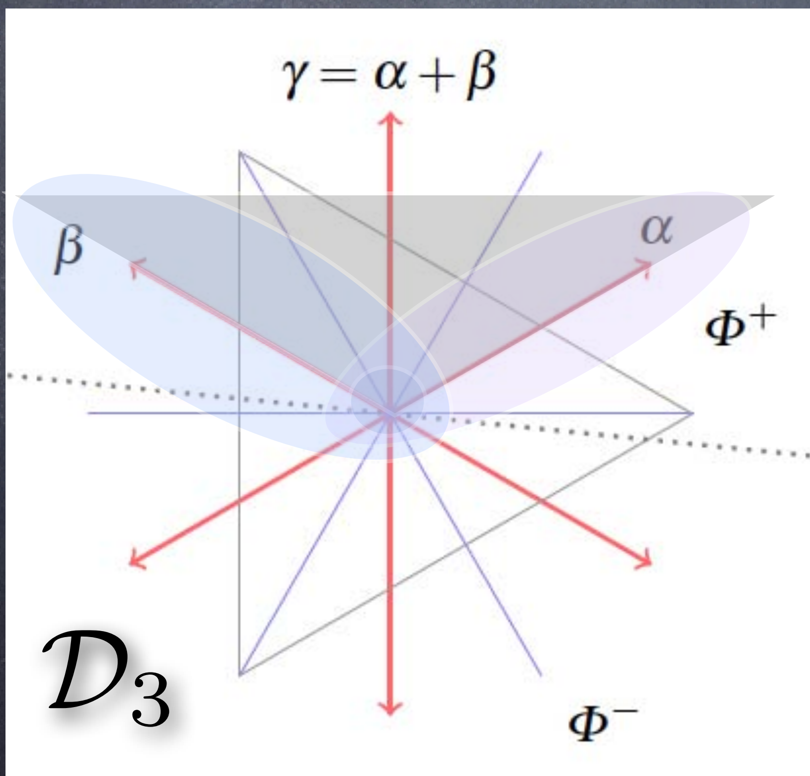

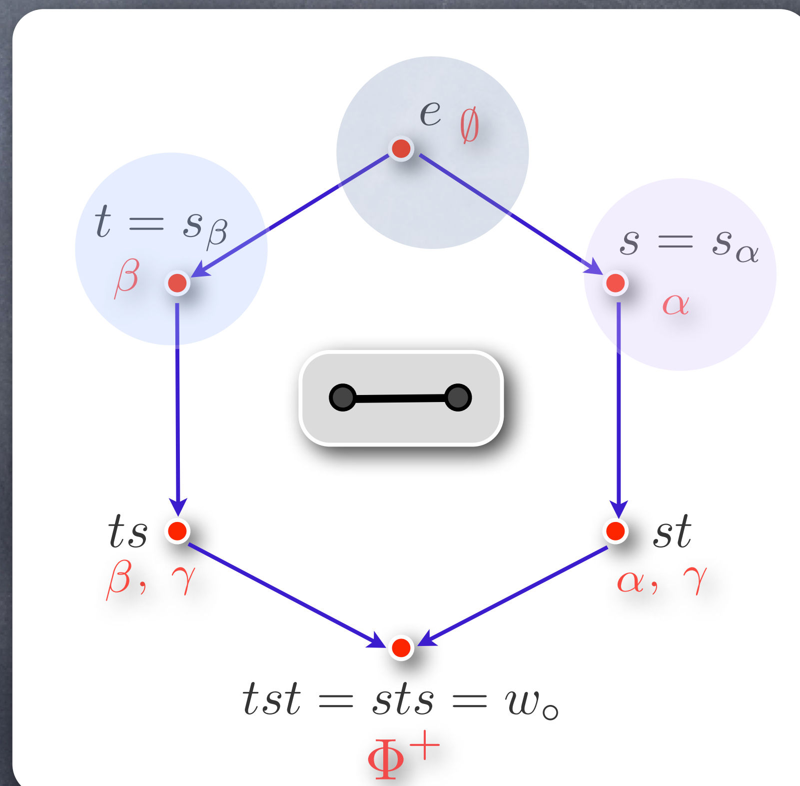

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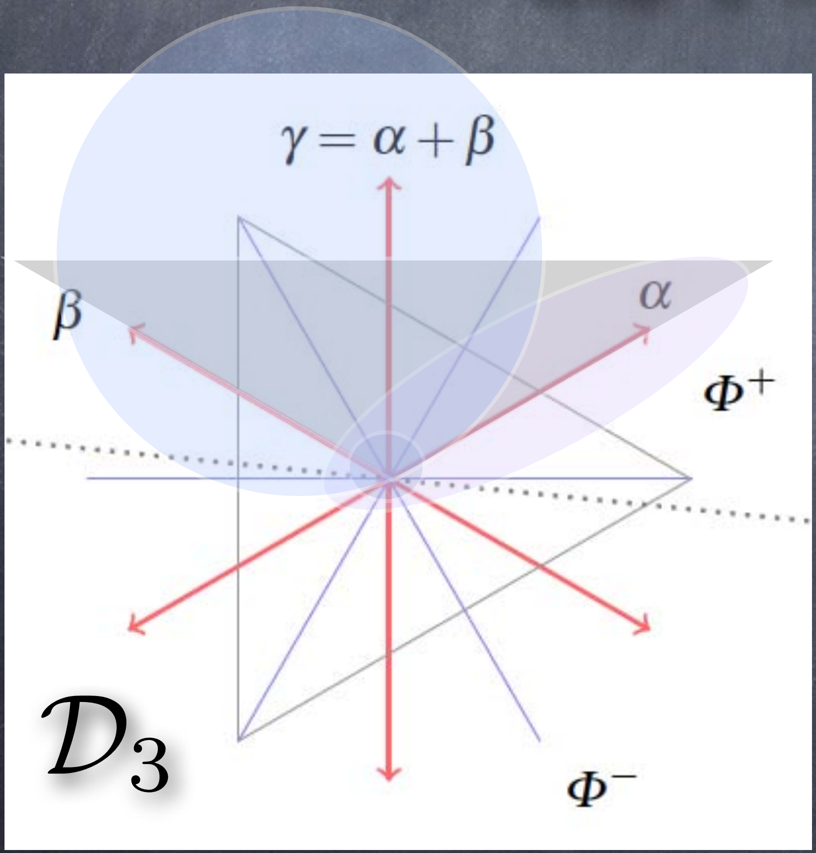

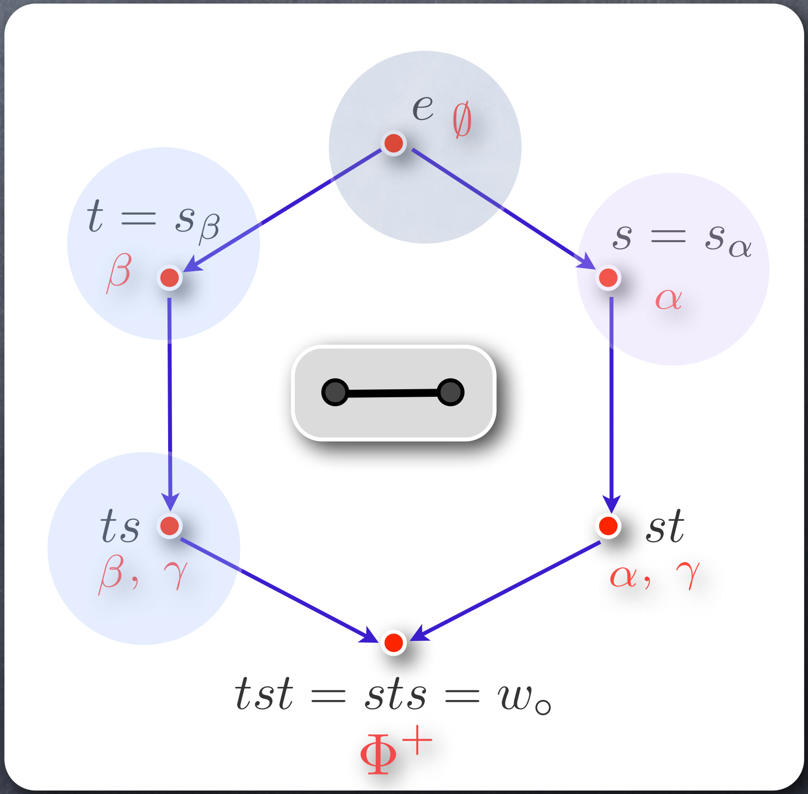

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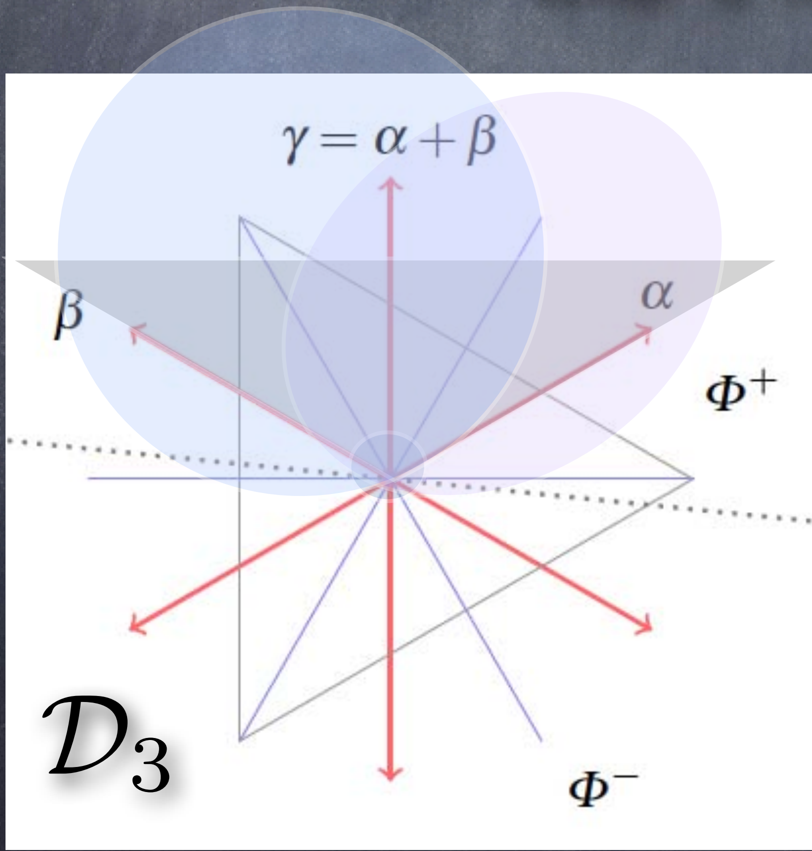

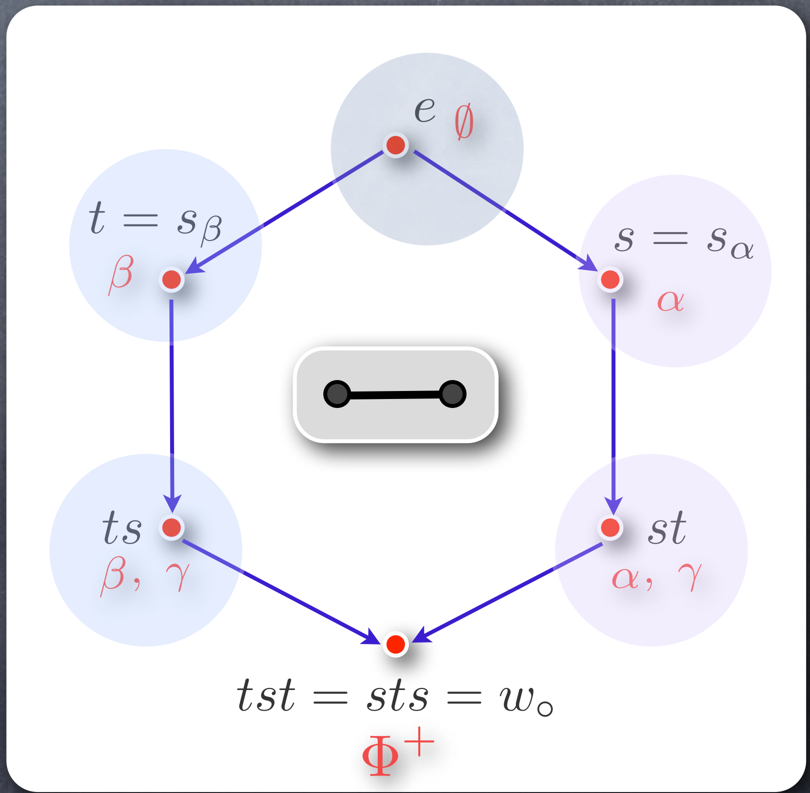

A recap from the other way around

Weak order: write $N(w) = \mathrm{inv}(w^{-1})$ then $u \leq v \iff N(u) \subseteq N(v)$

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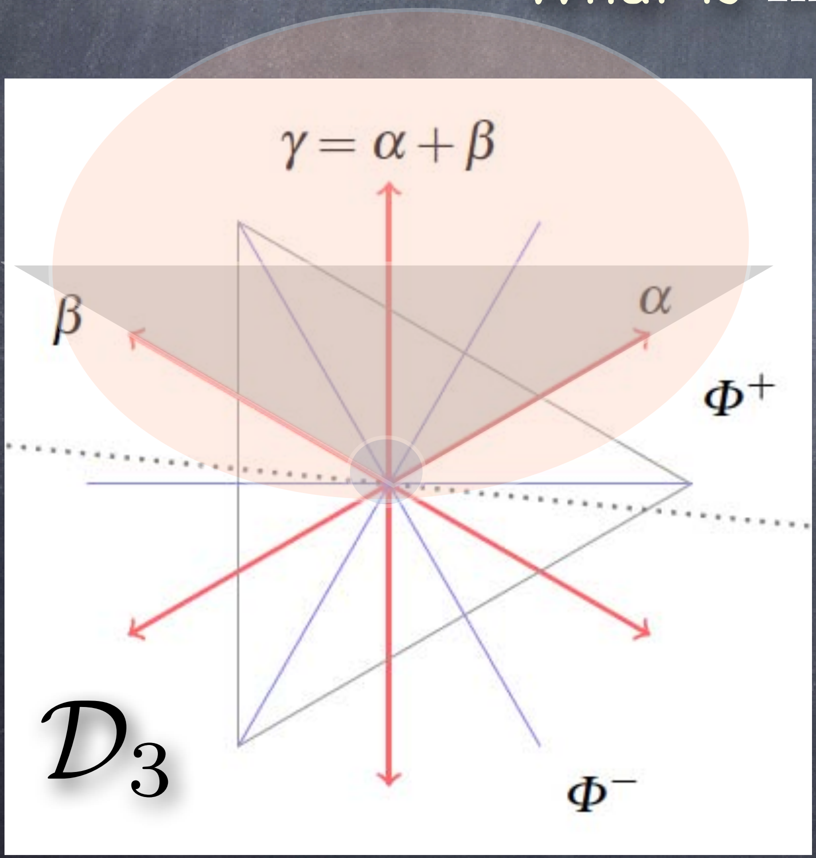
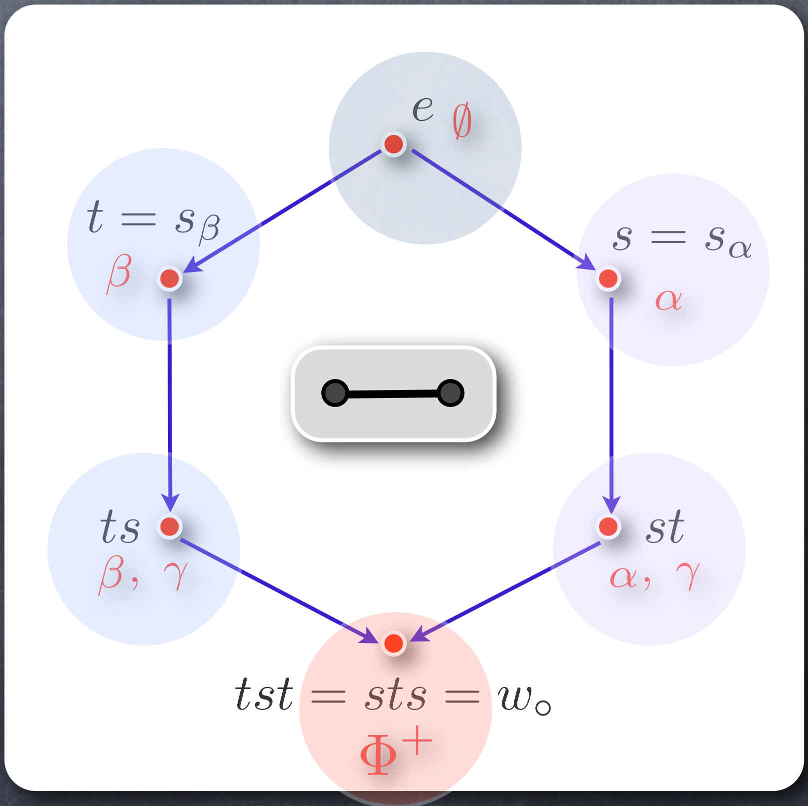
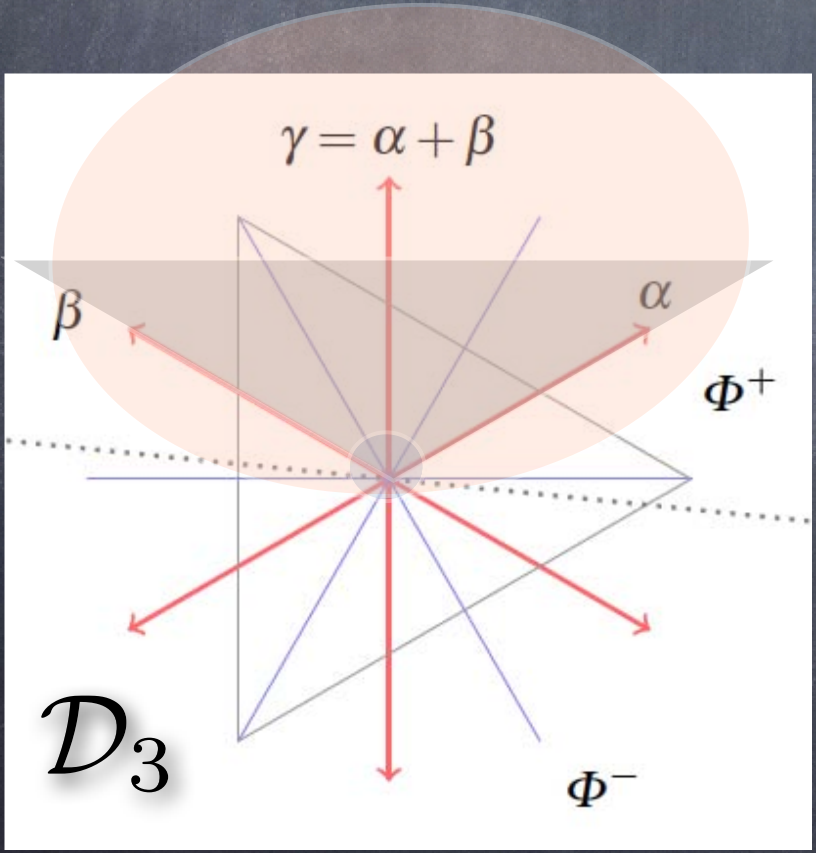


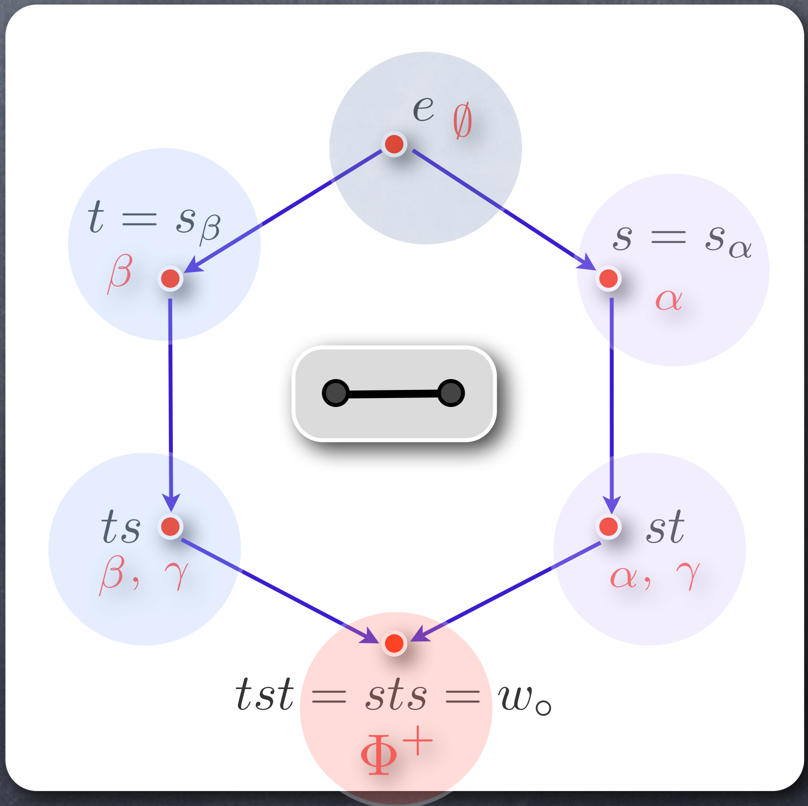


A 'weak order lattice' in general?

Proposition. $\mathrm{Im}(N) = \{\text{finite Biclosed sets in } \Phi^+\}$

- $A \subseteq \Phi^+$ is closed if for all $\alpha, \beta \in A$, $\mathrm{cone}(\alpha,\beta) \cap \Phi \subseteq A$;
- $A \subseteq \Phi^+$ is biclosed if A, $\Phi^+ \setminus A$ are closed.
- $\mathcal{B}(W) = \{\text{biclosed sets}\}$





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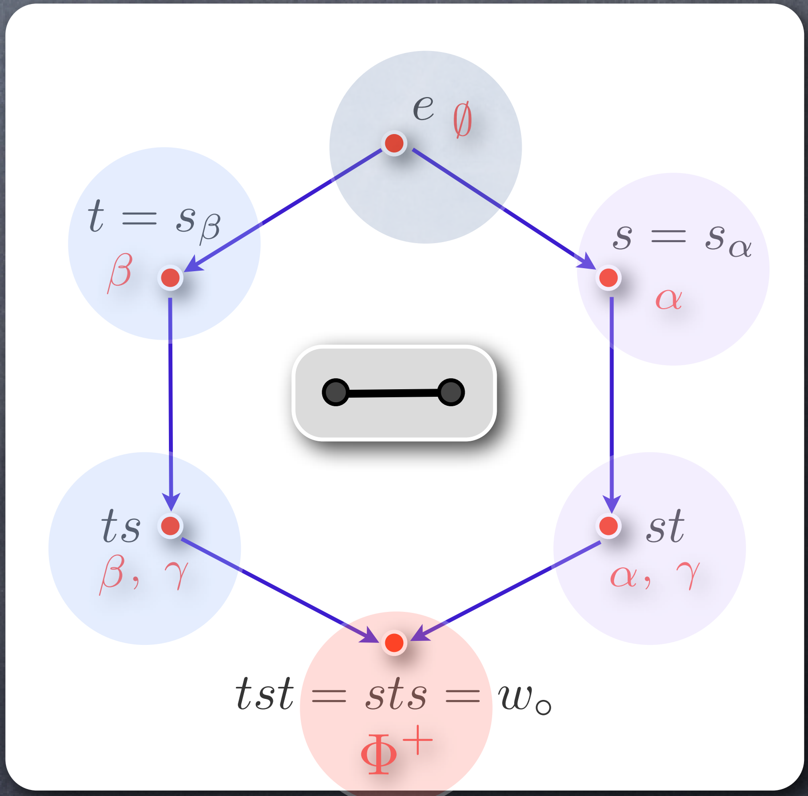
- $\mathcal{B}(W) = \{\text{biclosed sets}\}$

Conjecture (M. Dyer, 2011). $(\mathcal{B}(W), \subseteq)$ is a lattice (with minimal element $\emptyset$ and maximal element Φ^+.)

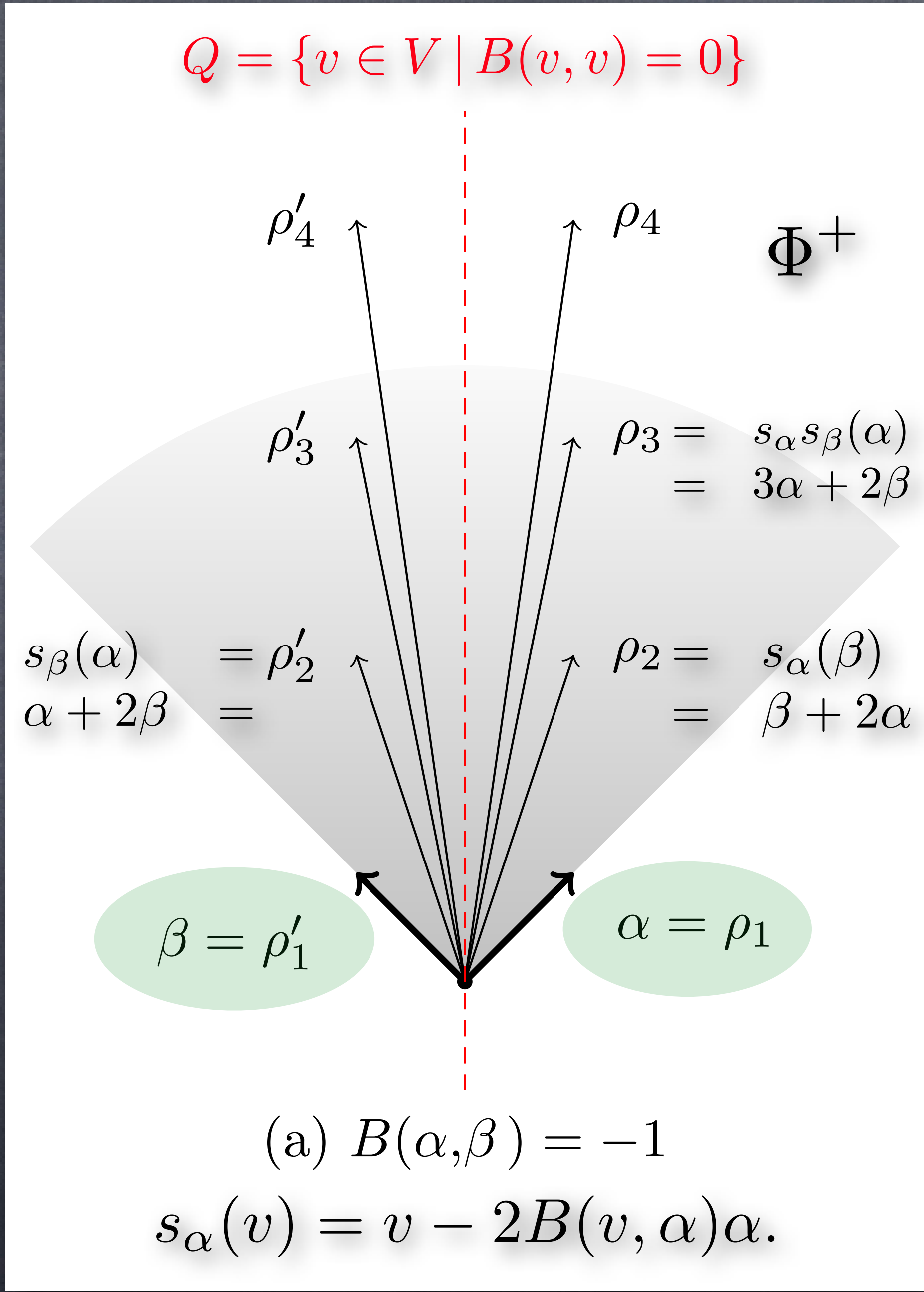
- $\vee \neq \cup$; $\wedge \neq \cap$ so how to understand them geometrically?
- Biclosed sets are the candidate for «generalized words»

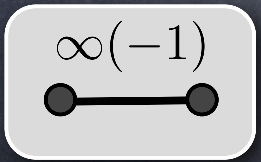


The example of the infinite dihedral group

$\rho'_n = n\alpha + (n+1)\beta$ $\rho_n = (n+1)\alpha + n\beta$

Infinite dihedral group I





The biclosed are:
- the finite ones;
- their complements;
- and two infinite ones: the left and right side of Q!

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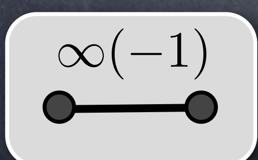
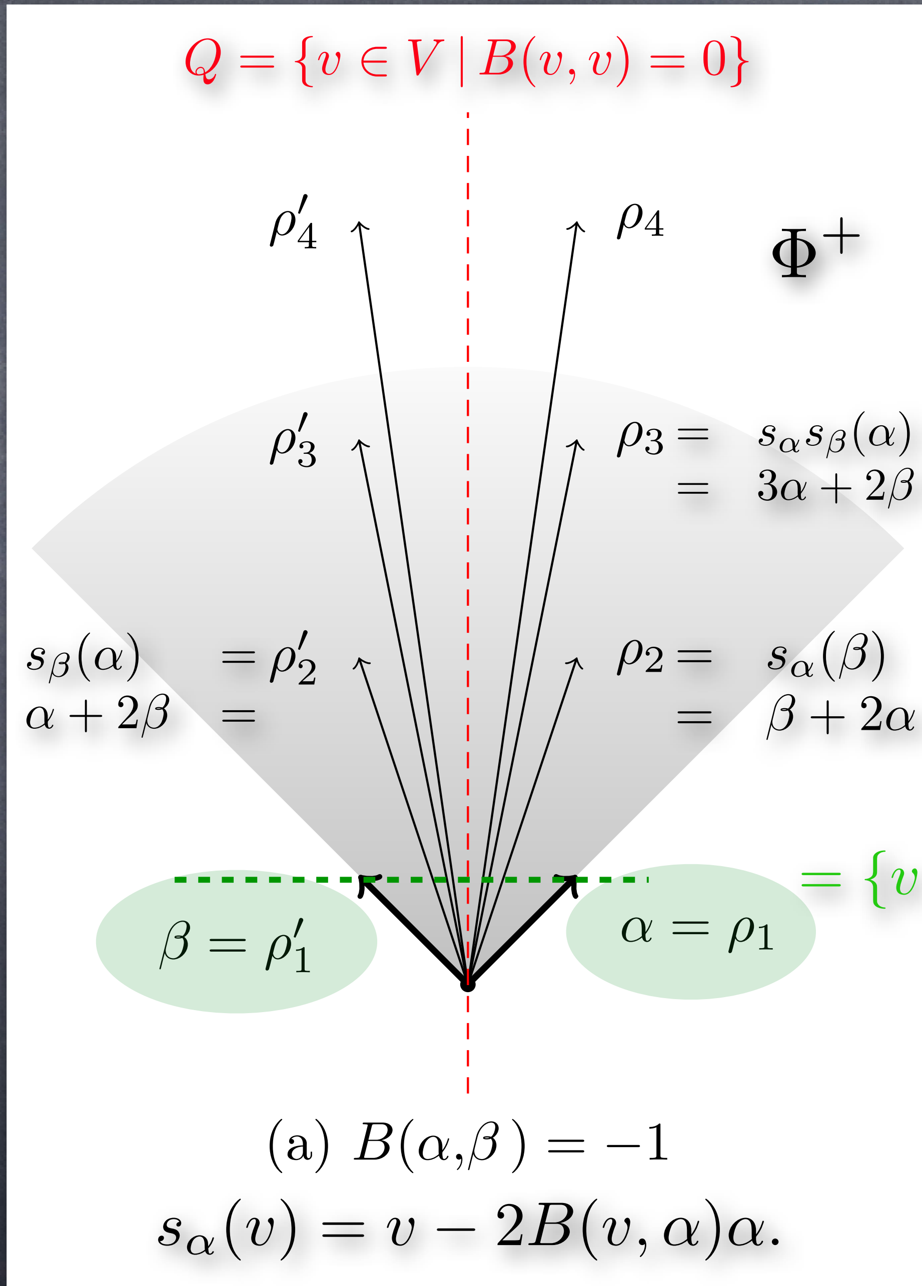
$\rho_n = (n+1)\alpha + n\beta$

Infinite dihedral group I



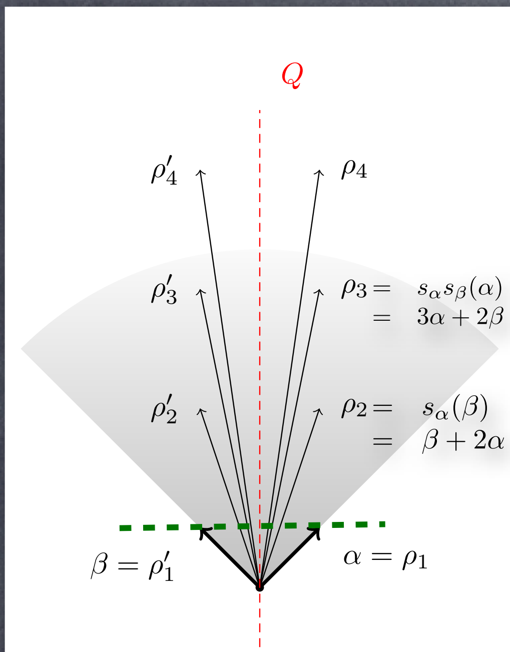


$= \{v \in V \mid \sum_{\alpha\in\Delta} v_\alpha = 1\}$

More examples: 'Cut' Φ^+ by an affine hyperplane

Christophe Hohlweg, 2013

Other examples of infinite root systems?



(a) $B(\alpha,\beta) = -1$

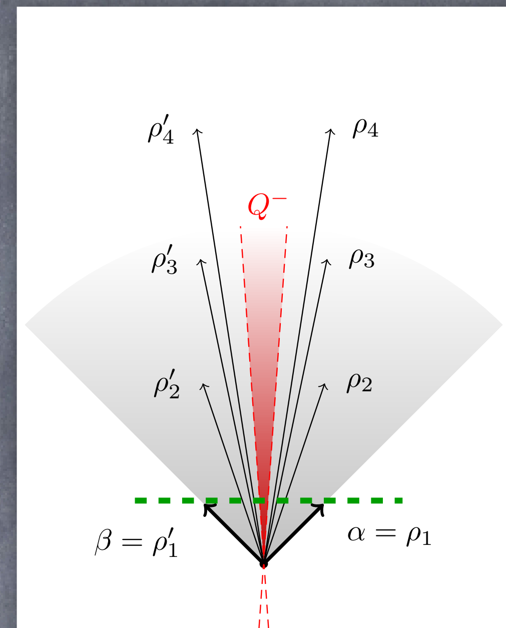
Affine hyperplane

$$V_1 = \{v \in V \mid \sum_{\alpha\in\Delta} v_\alpha = 1\}$$

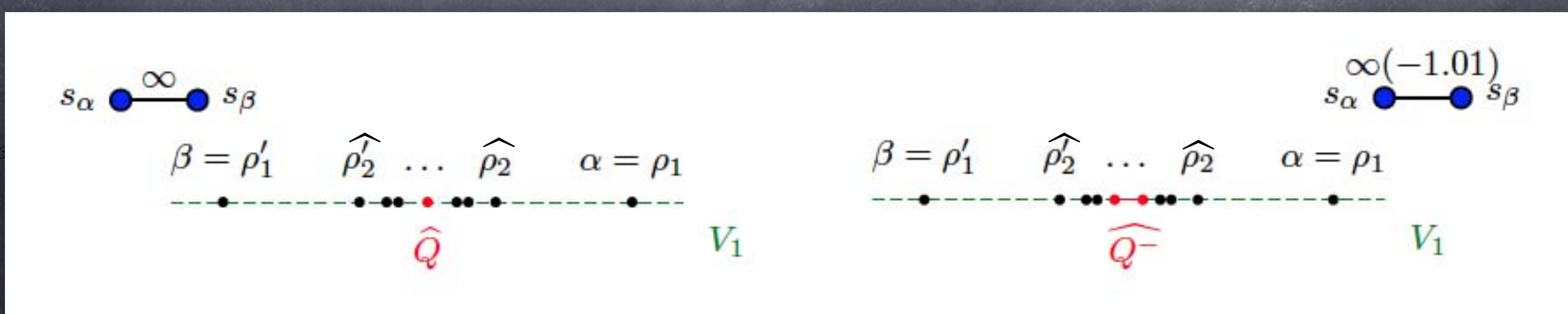
Normalized isotropic cone: $\widehat{Q} := Q \cap V_1$

Normalized roots

$$\widehat{\rho} := \rho / \sum_{\alpha\in\Delta} \rho_\alpha$$



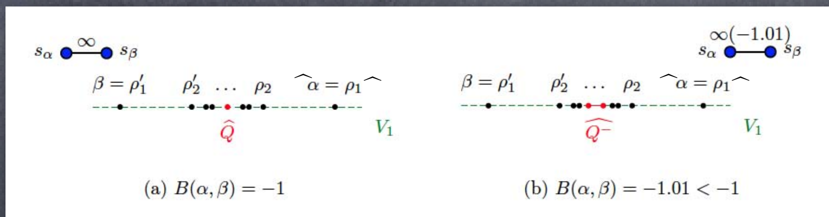
(b) $B(\alpha,\beta) = -1.01 < -1$



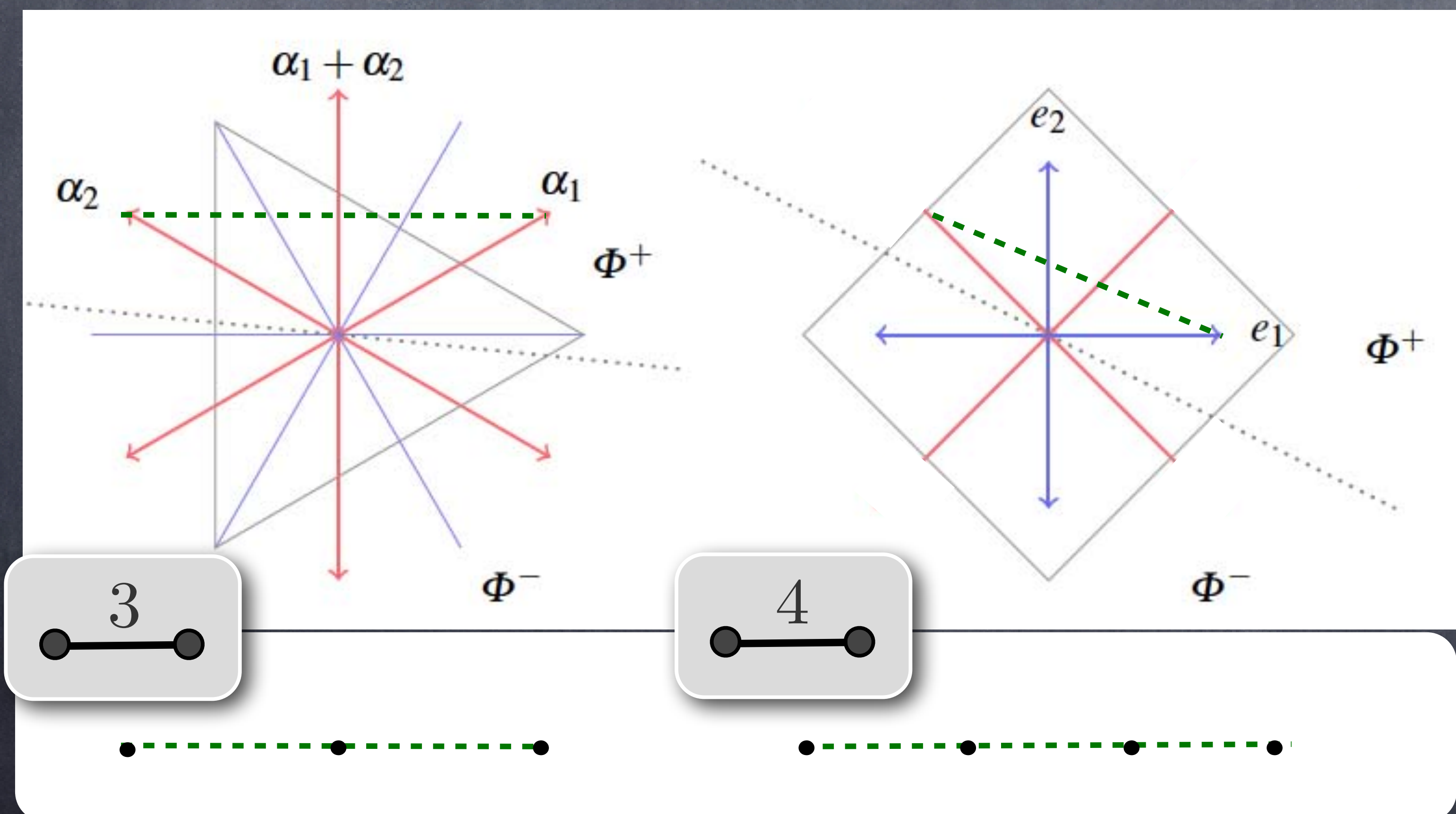
(a) $B(\alpha, \beta) = -1$ (b) $B(\alpha, \beta) = -1.01 < -1$

Christophe Hohlweg, 2013

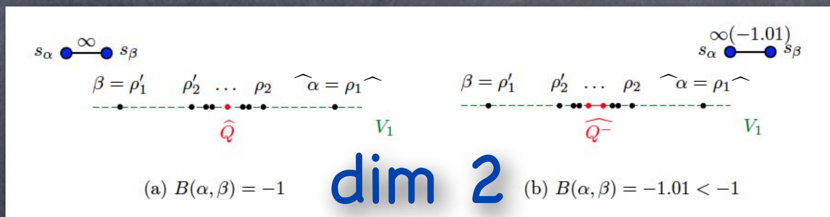
Other examples of infinite root systems ...

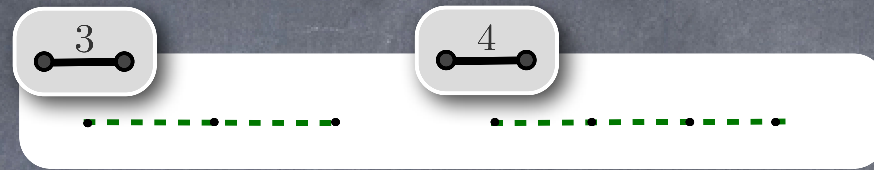


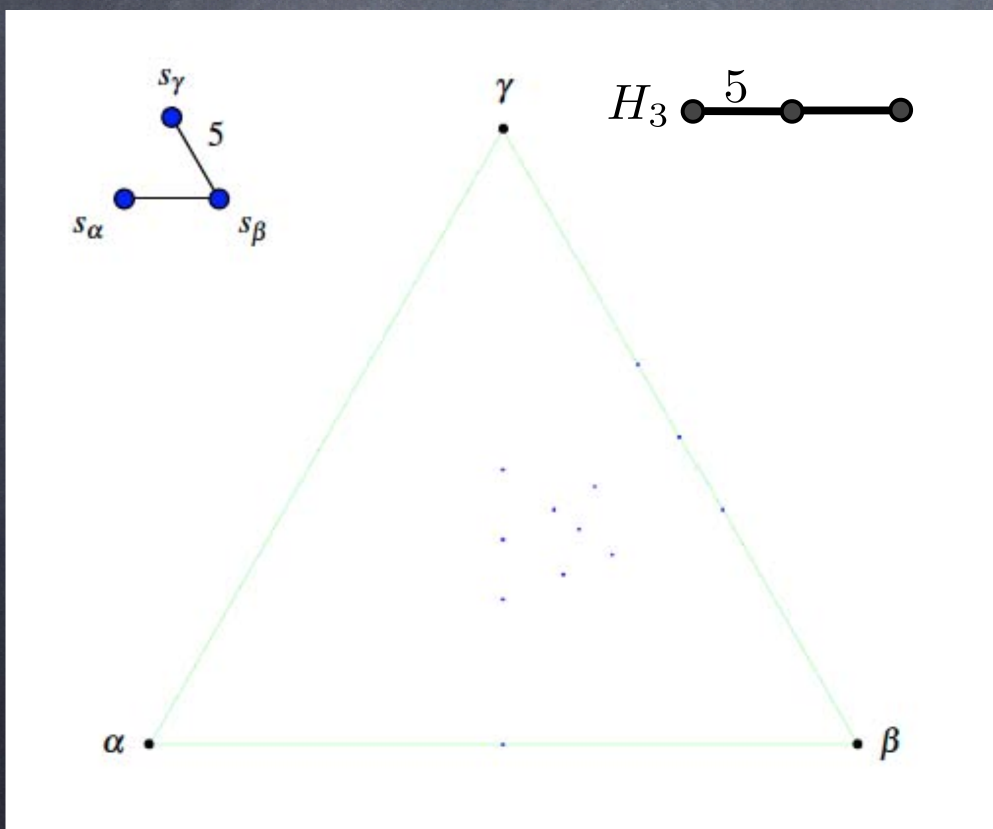
finite type of rank 2

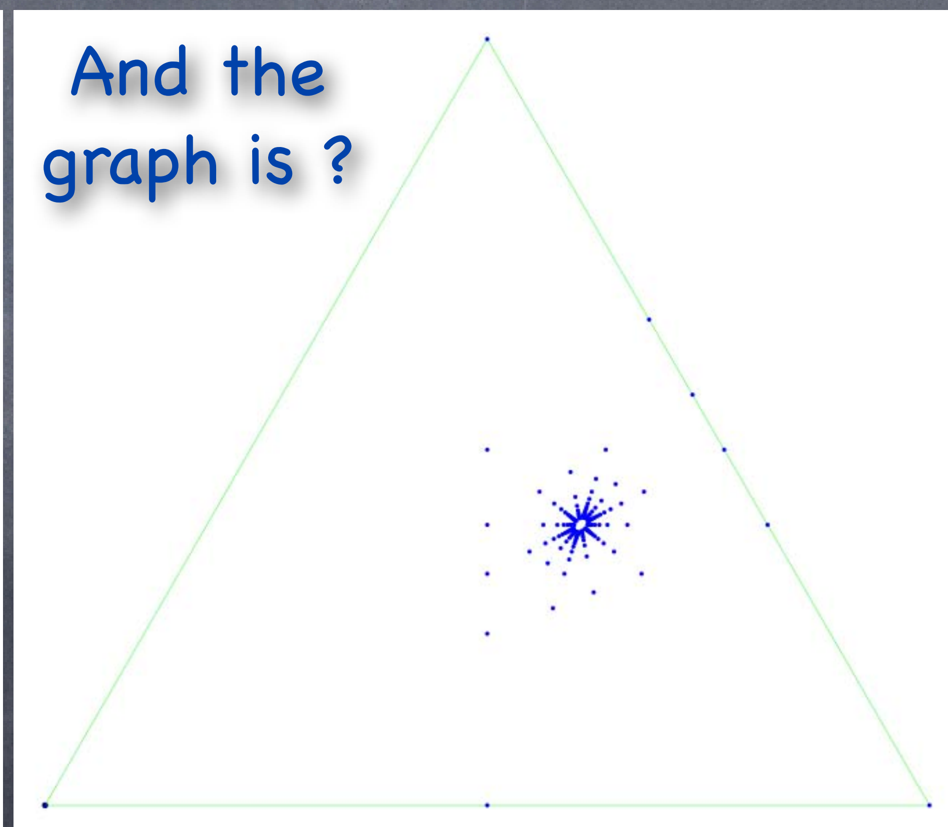


Other examples of infinite root systems (with SAGE)

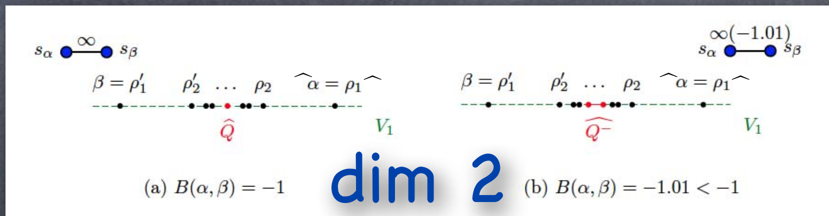


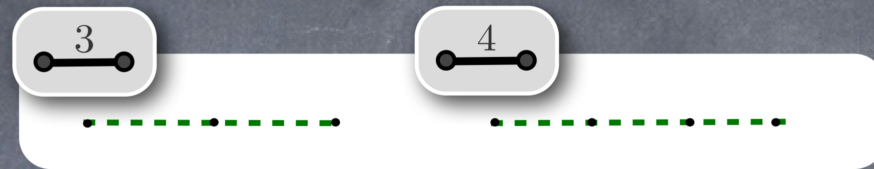


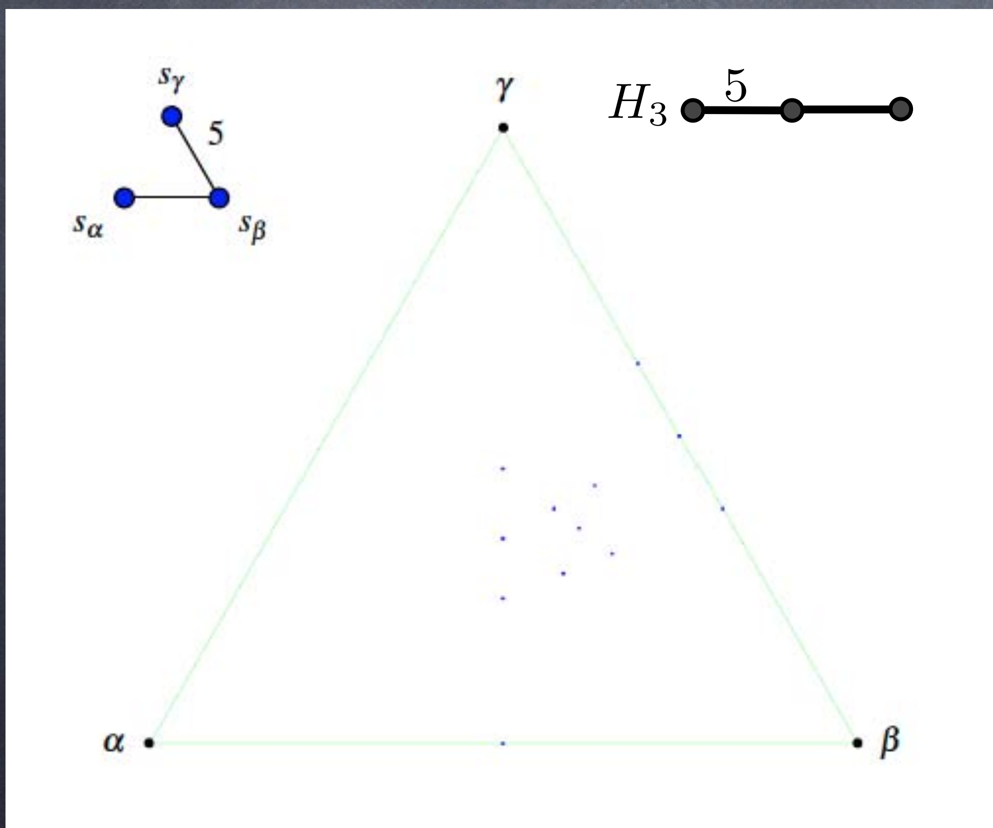


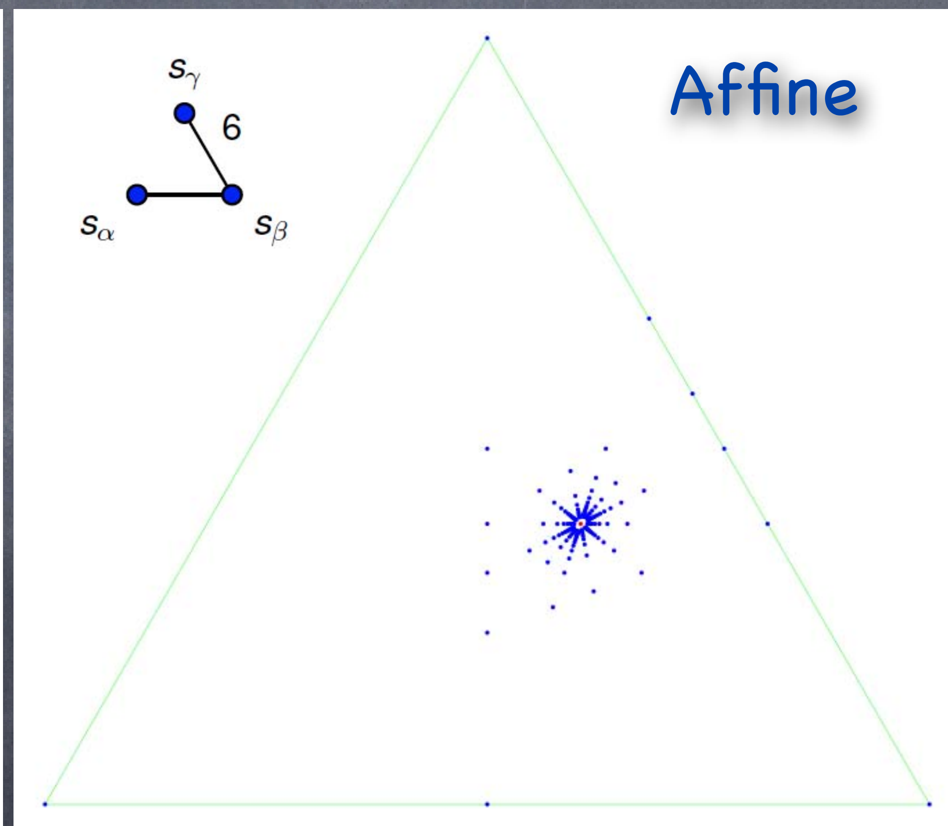


Other examples of infinite root systems (with SAGE)









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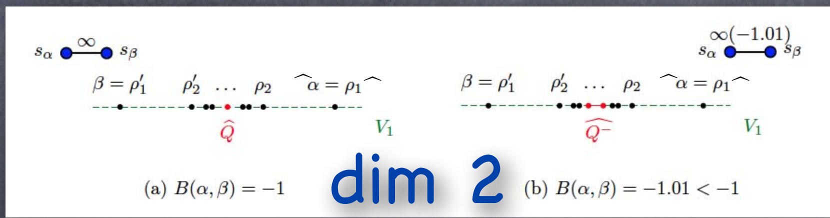

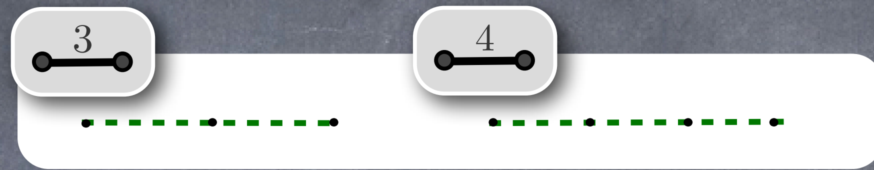

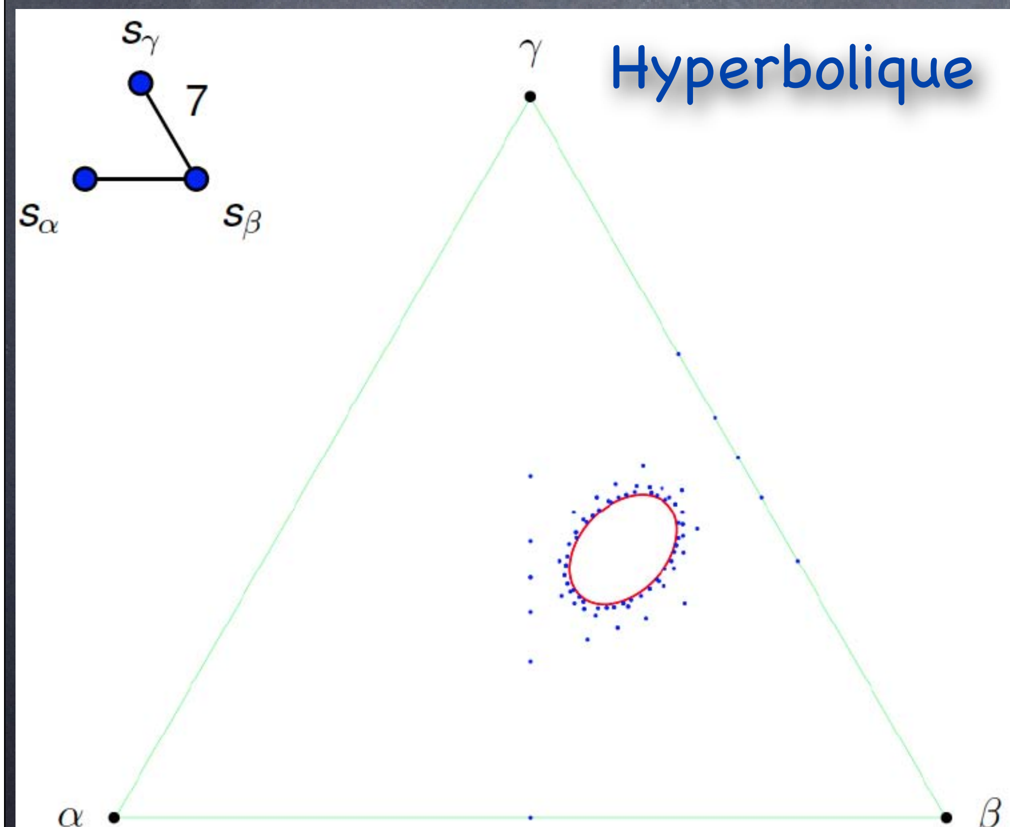

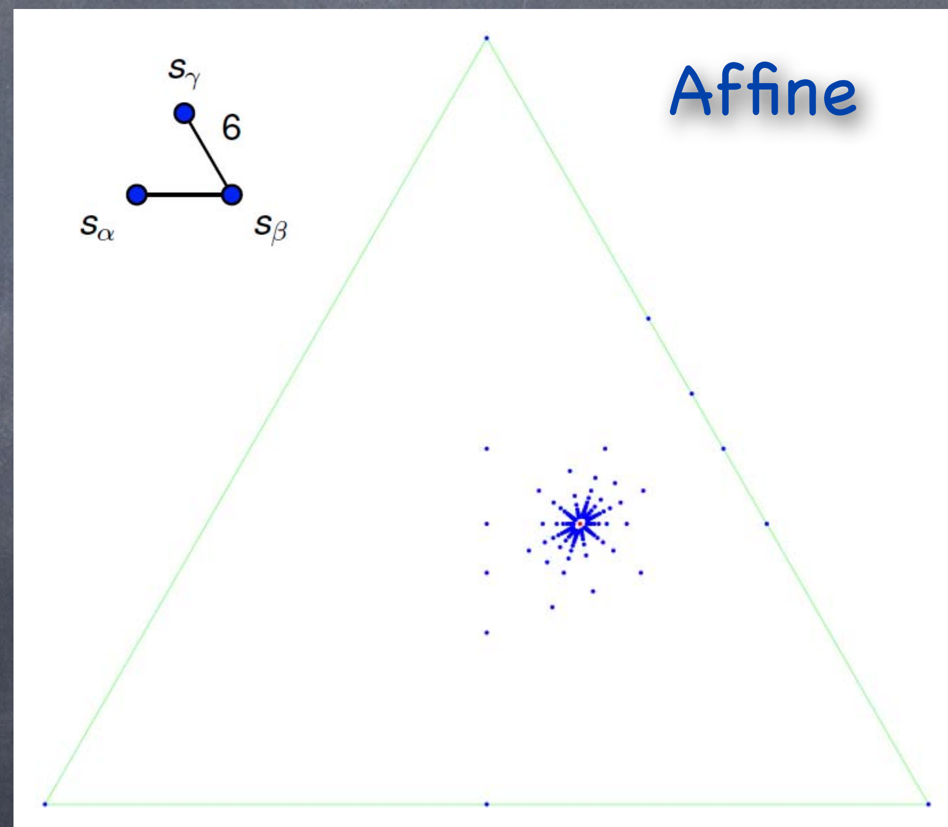

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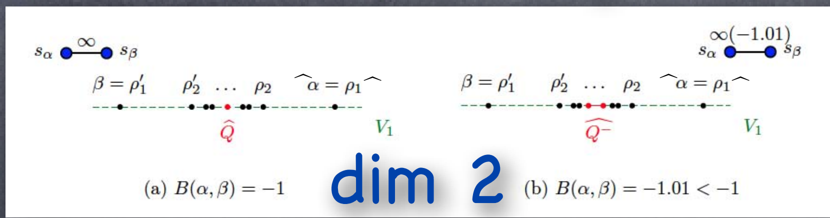
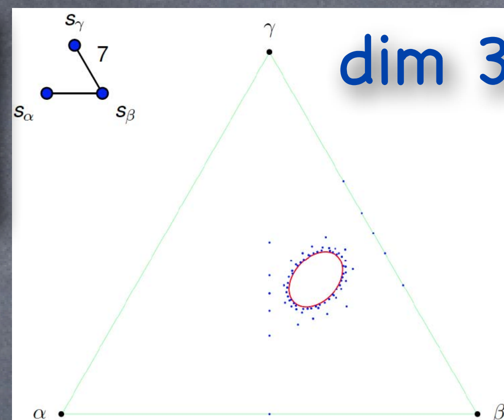
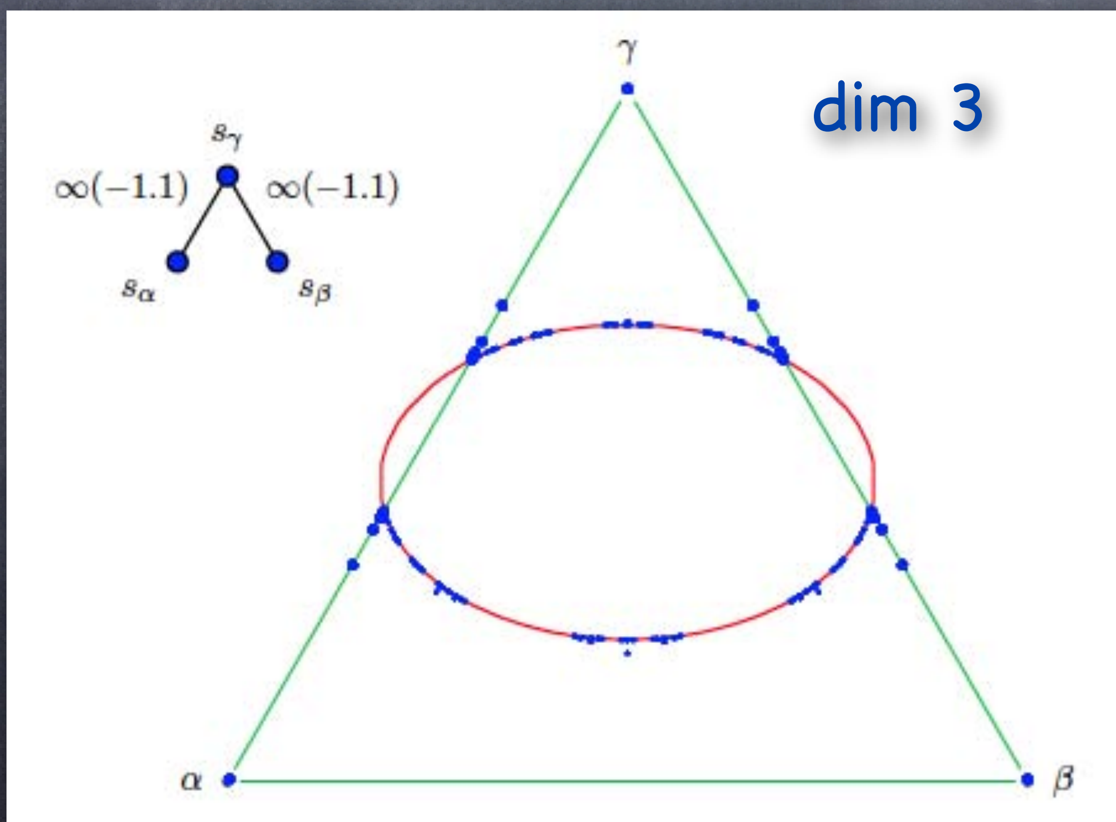
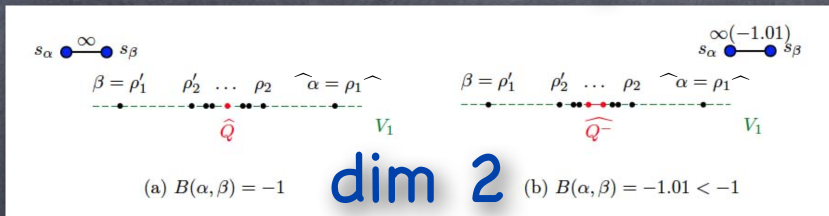


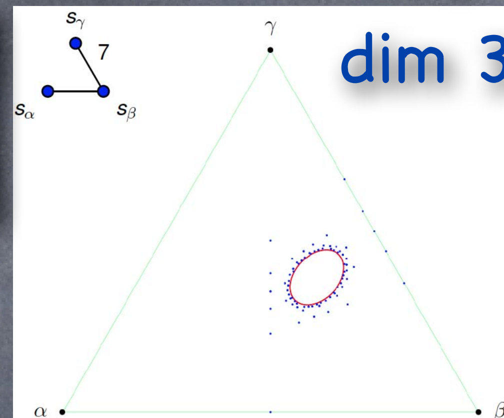


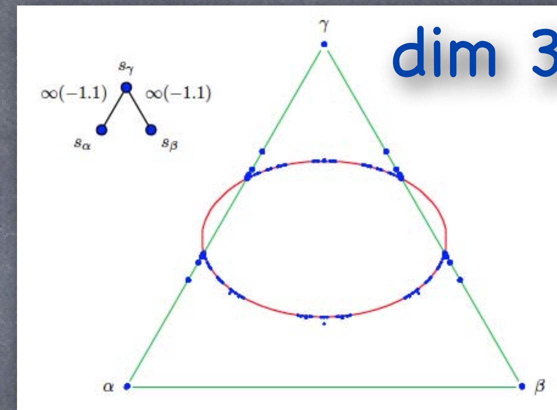


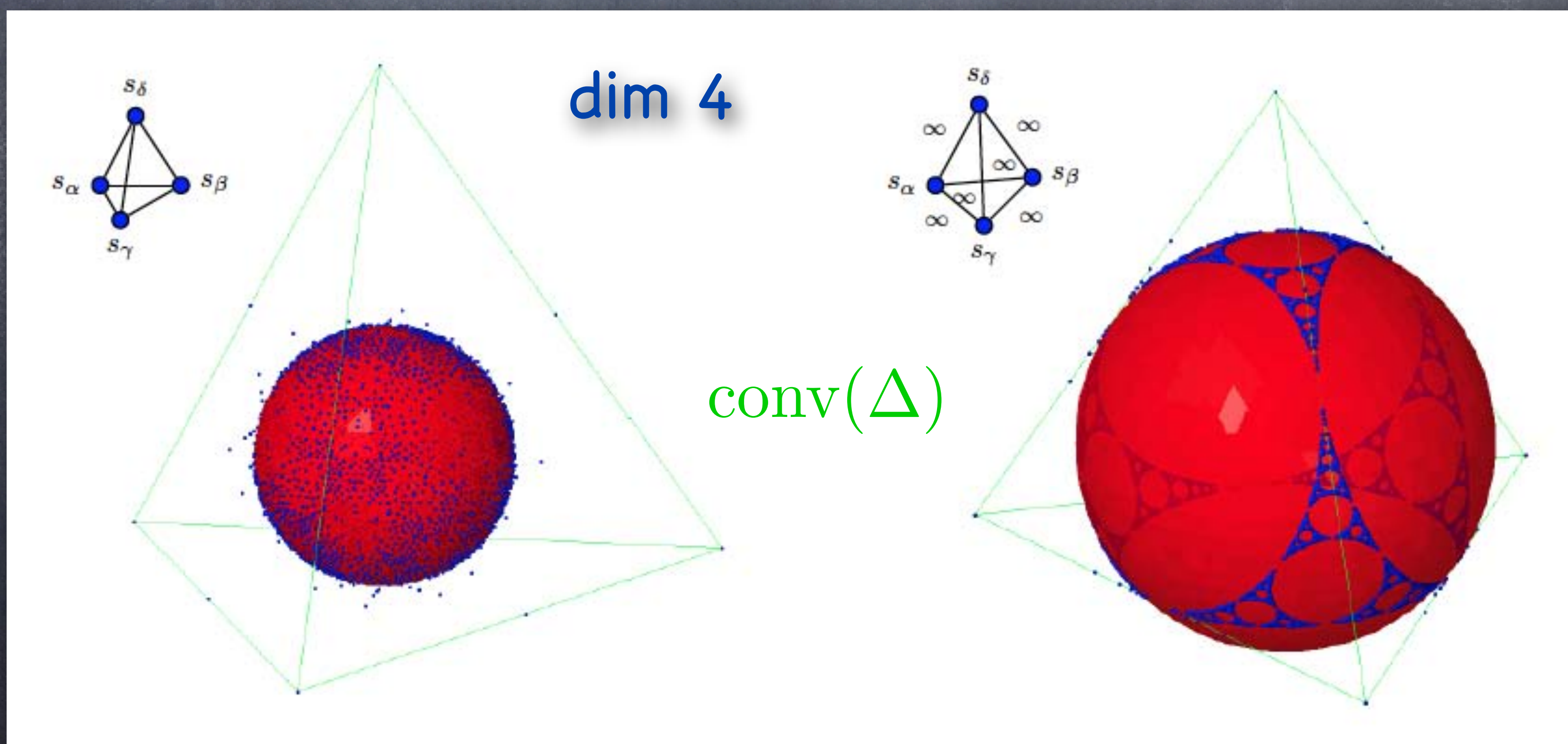
Observation: a dihedral subgroup group is infinite iff the associated line cuts Q

Other examples of infinite root systems (with SAGE)









Christophe Hohlweg, 2013

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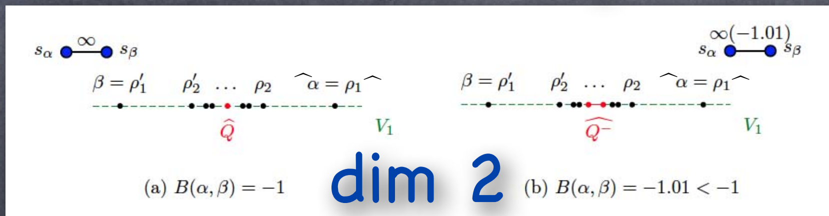

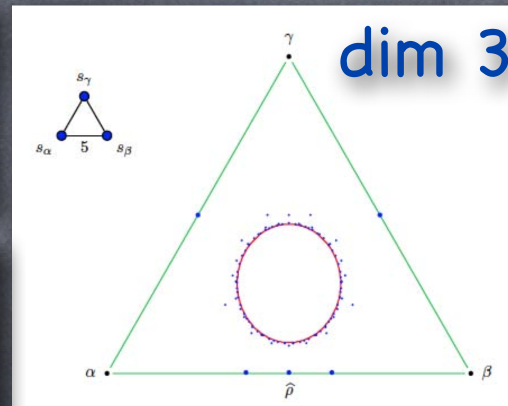

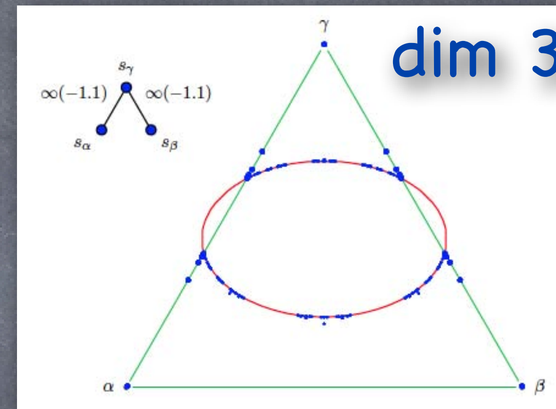

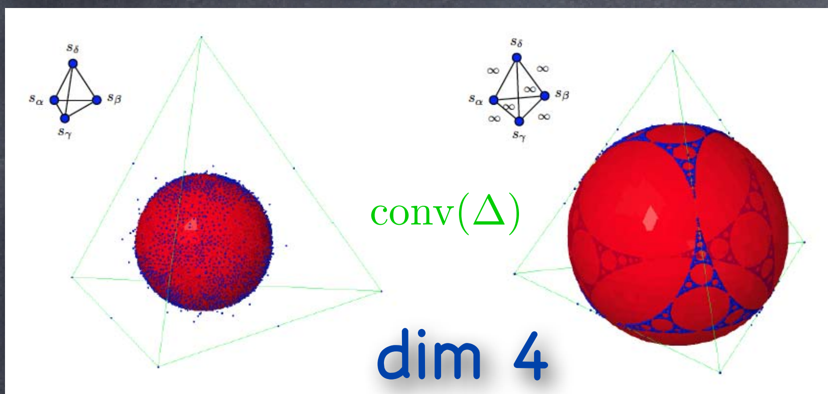

From joint works with:

- J.P Labbé and V. Ripoll (2012)
- M. Dyer and V. Ripoll (2013)

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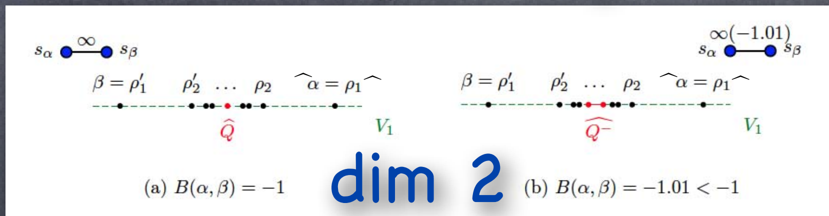

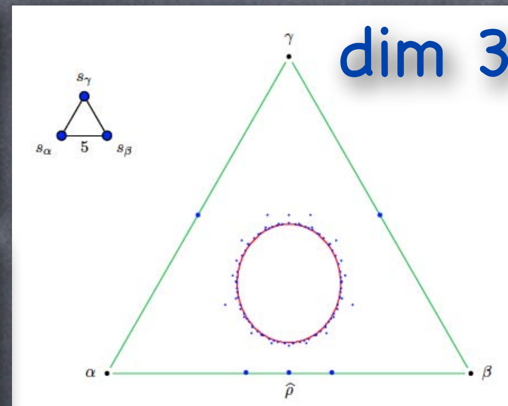

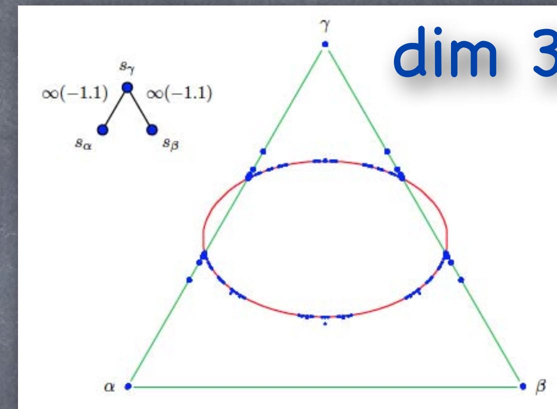

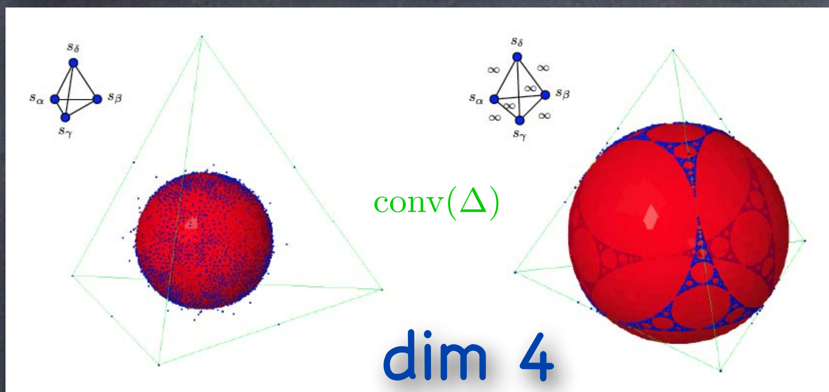

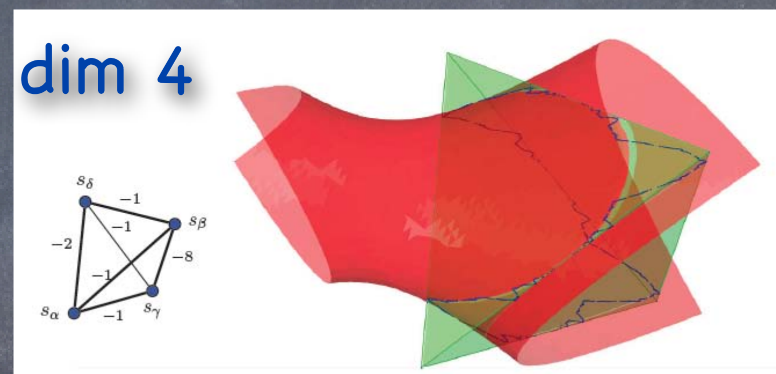

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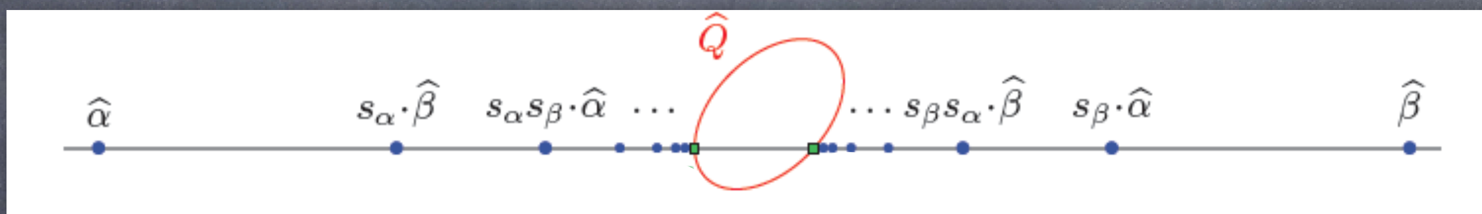
Problem still there: what can we say about these pictures that help understand biclosed sets?

Actually, at this point, not that much about biclosed but...

Illustration of some known combinatorics on infinite root systems

How to see the action of W on $\widehat{\Phi}$: $s_\alpha \cdot \beta = \widehat{s_\alpha(\beta)} \in L(\hat{\alpha}, \hat{\beta})$ is a barycenter of $\hat{\alpha}$ and $\hat{\beta}$.



Depth of a root is $\mathrm{dp}(\rho) = 1 + \min\{k \mid \rho = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}(\alpha_{k+1}), \alpha_1, \dots, \alpha_k, \alpha_{k+1} \in \Delta\}.$

Root poset on Φ^+ : transitive closure of the relation

$$\beta < s_\alpha(\beta) \iff \mathrm{dp}(\beta) < \mathrm{dp}(s_\alpha(\beta)); \quad (\alpha \in \Delta)$$

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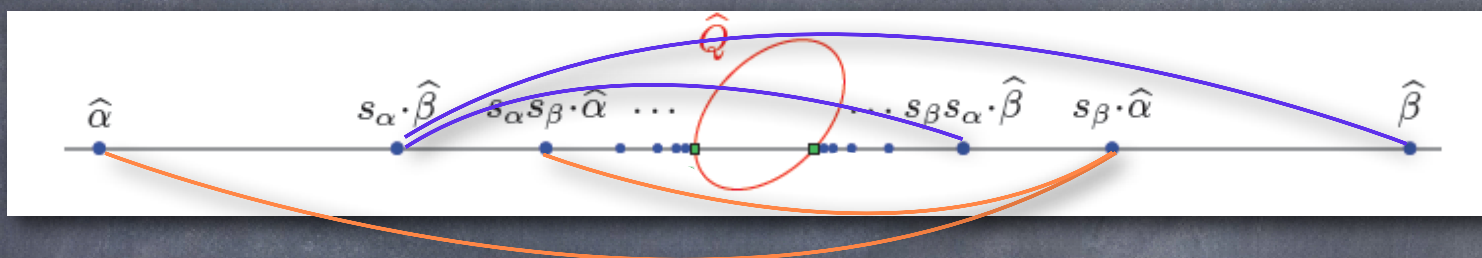

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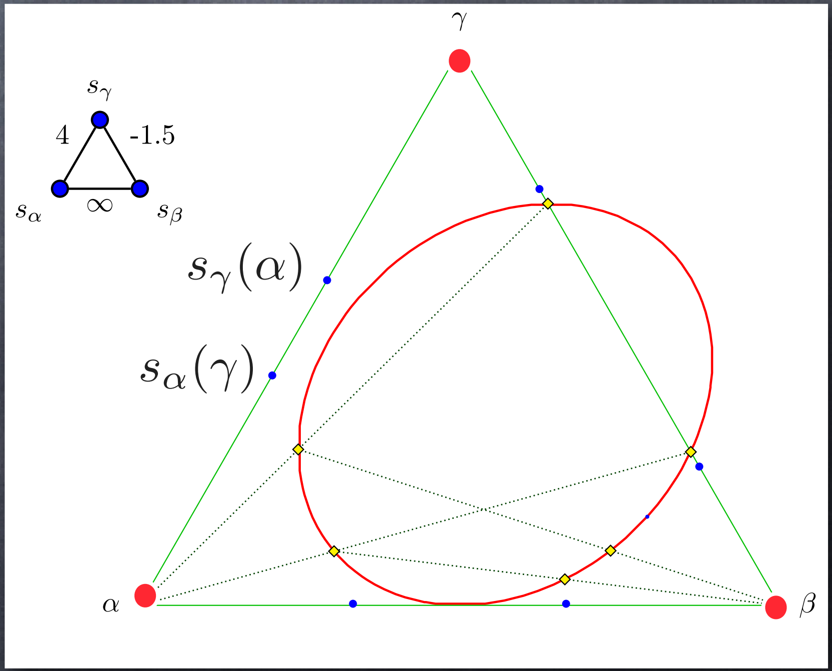
Small root: roots obtained from Δ along a path in the root poset corresp. to finite dihedral reflection subgroups (i.e. the lines does not cut Q).

Theorem (Brink-Howlett, 1993)
The set Σ of small roots is finite.

Problem:
$|\Sigma|$ from Γ_W ?

A finite state automaton that recognize reduced expressions:

- everything depends of the combinatorics of the small descent set $D_\Sigma(w) = \mathrm{inv}(w) \cap \Sigma$
- The nodes of a finite automaton that recognized the set of reduced words is: $\{D_\Sigma(w) \mid w \in W\}$



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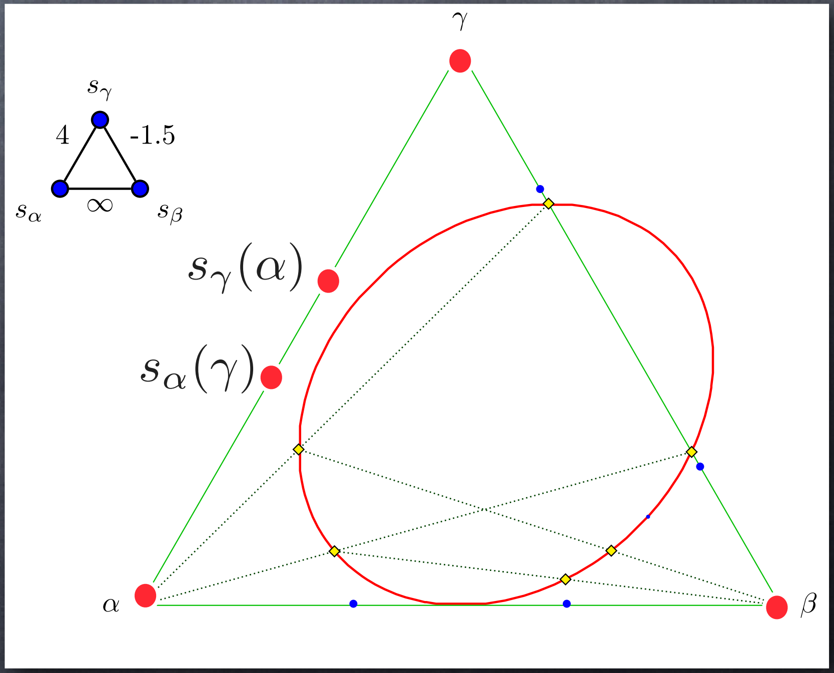

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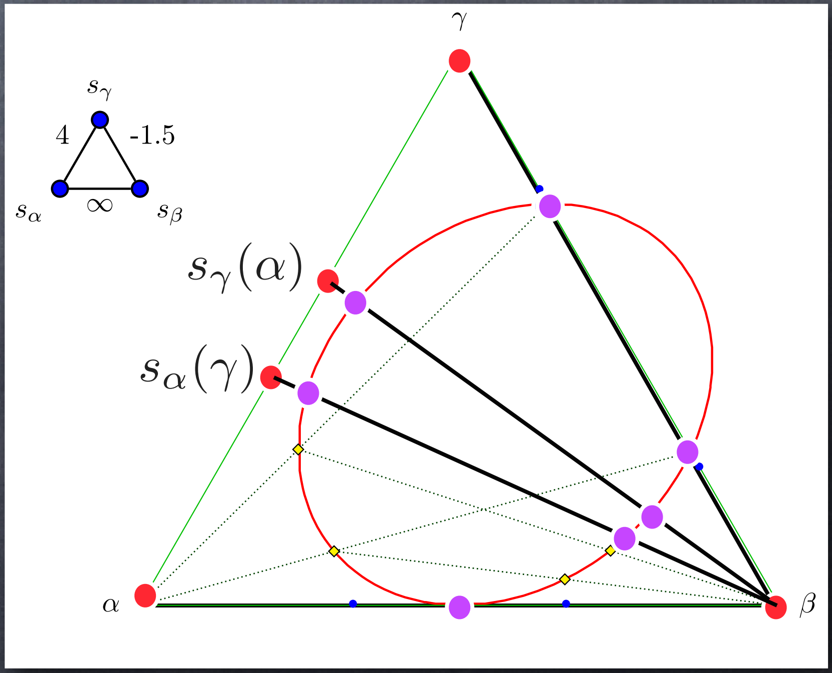

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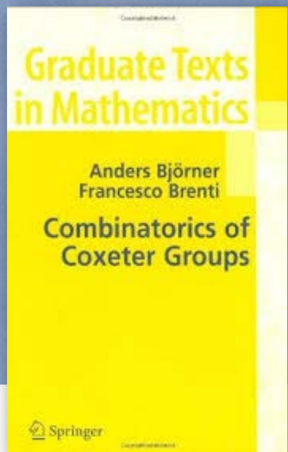
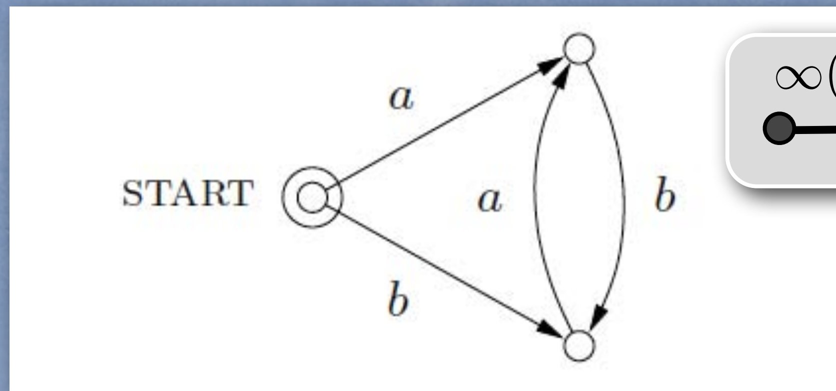
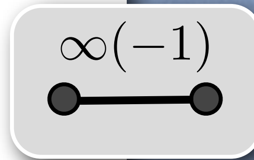
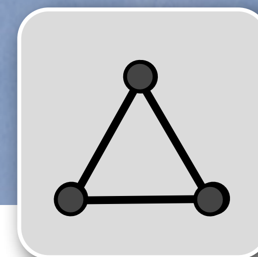
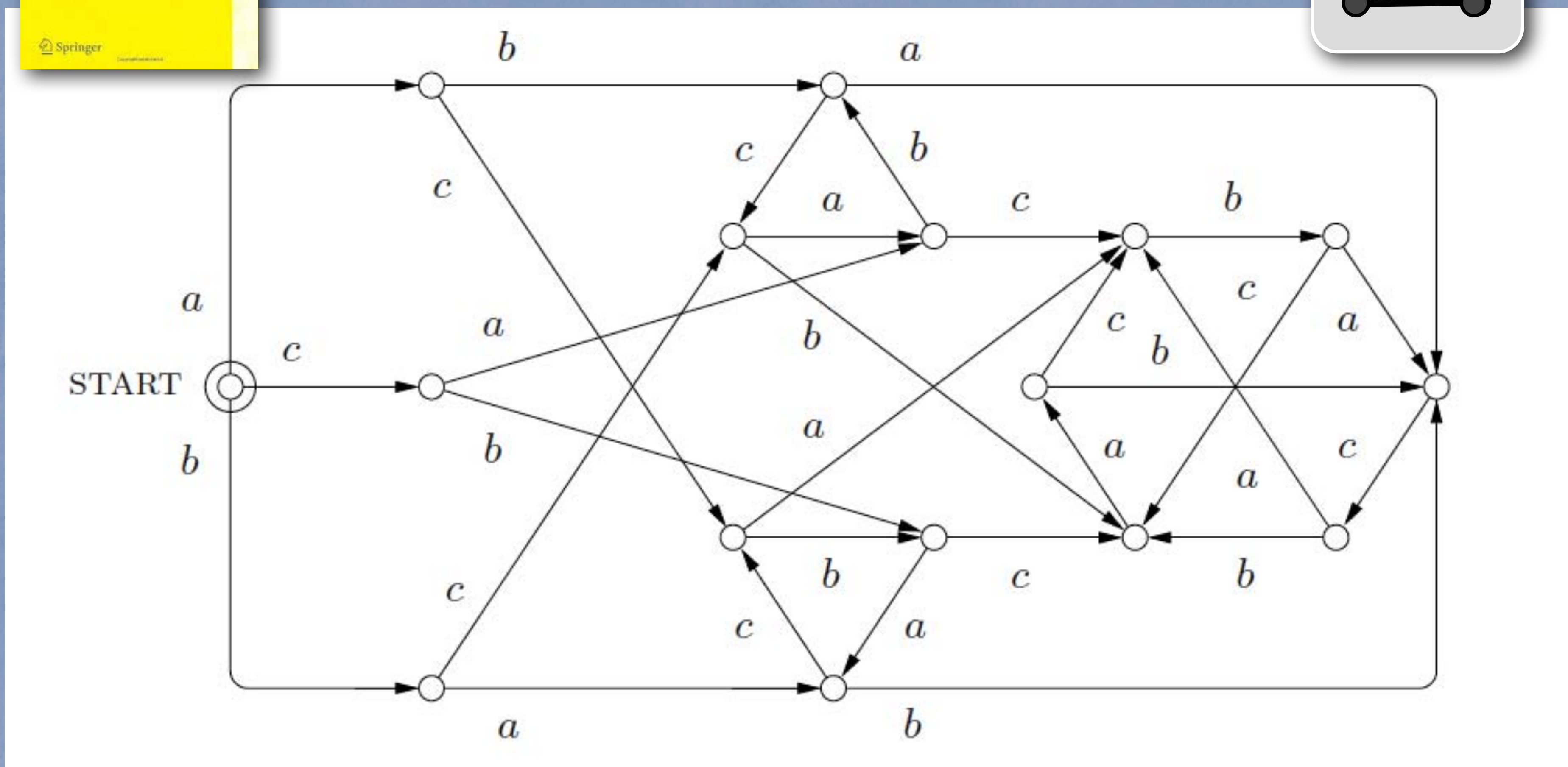

Graduate Texts in Mathematics
Anders Björner
Francesco Brenti
Combinatorics of Coxeter Groups
Springer


START
a
a
b
b


$\infty(-1)$




START

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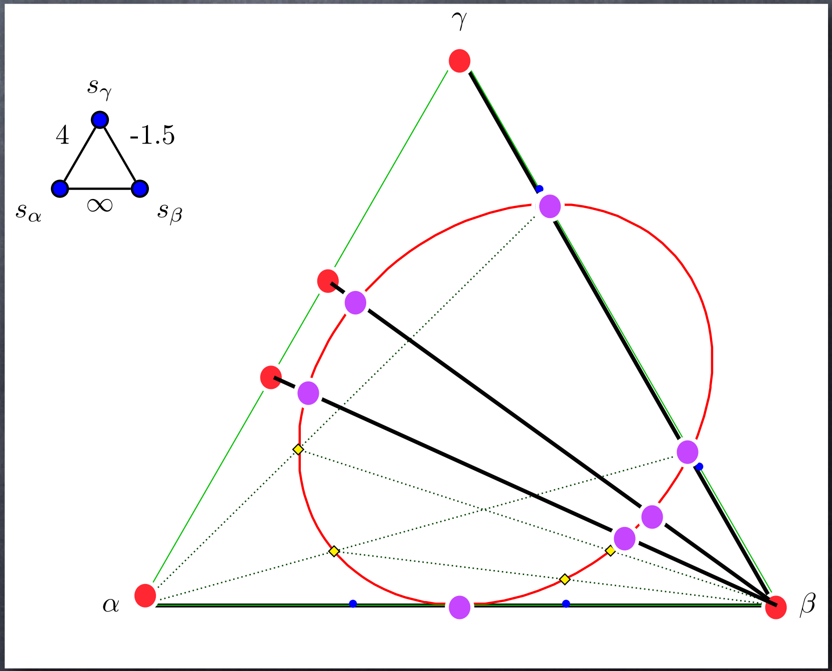
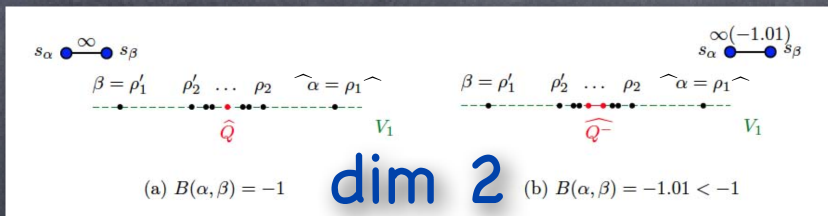
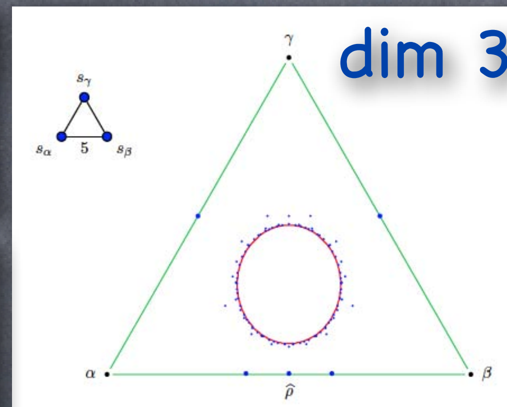
For building a finite state automaton that recognize reduced expressions:

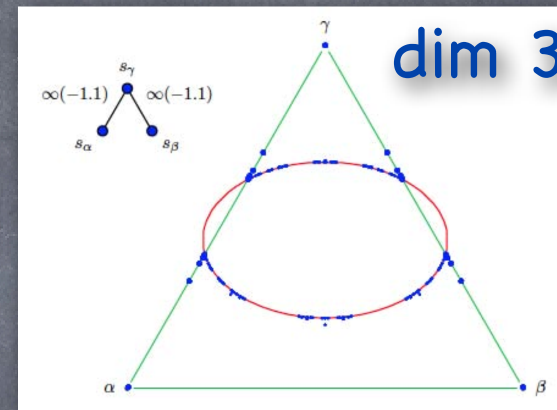
Question:
Is it possible to recognize biclosed sets???
If A is biclosed, properties of $\sum_{\beta\in A} q^{\mathrm{dp}(\beta)}$?

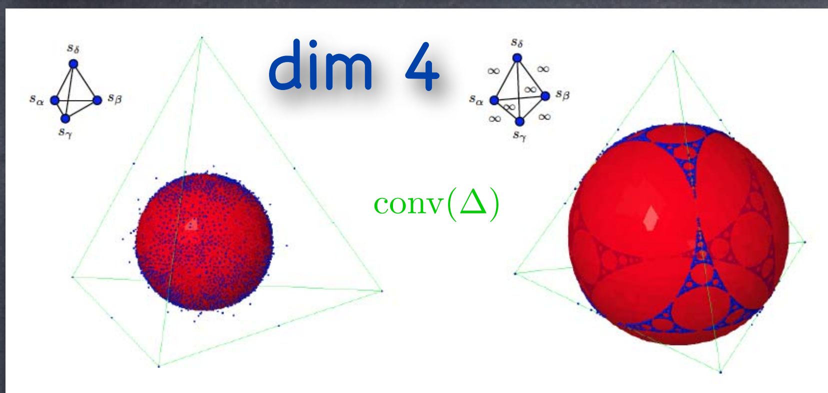


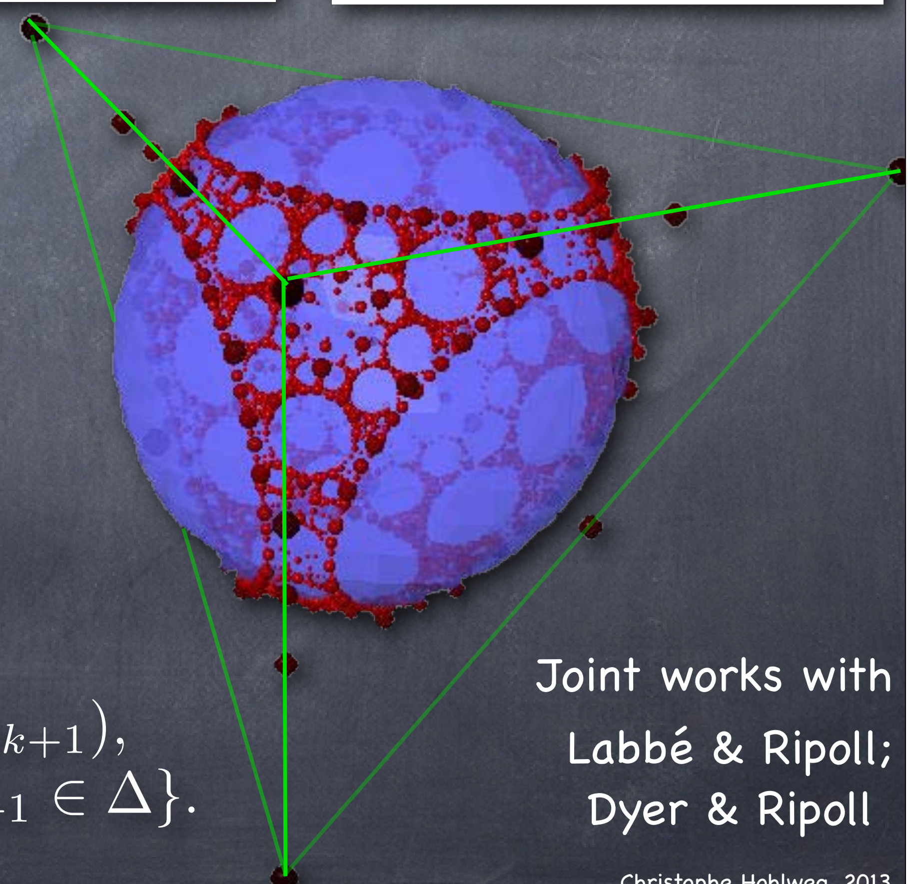
What can we say about these pictures?











Obervation: The `size' of a generalized root (in red in this last picture) is decreasing as the depth of the root is increasing.

$$\mathrm{dp}(\rho) = 1 + \min\{k \,|\, \rho = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}(\alpha_{k+1}), \; \alpha_1, \dots, \alpha_k, \alpha_{k+1} \in \Delta\}.$$

Joint works with Labbé & Ripoll; Dyer & Ripoll

Christophe Hohlweg, 2013

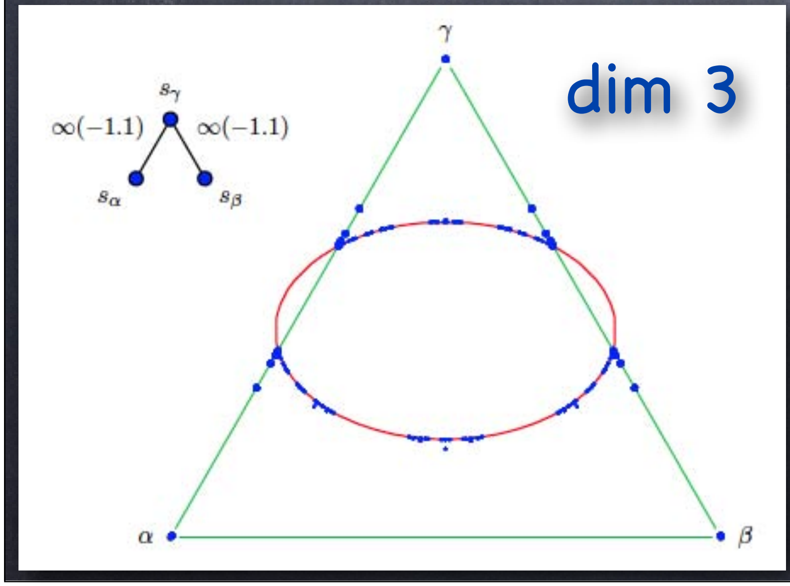
A look at limits of roots

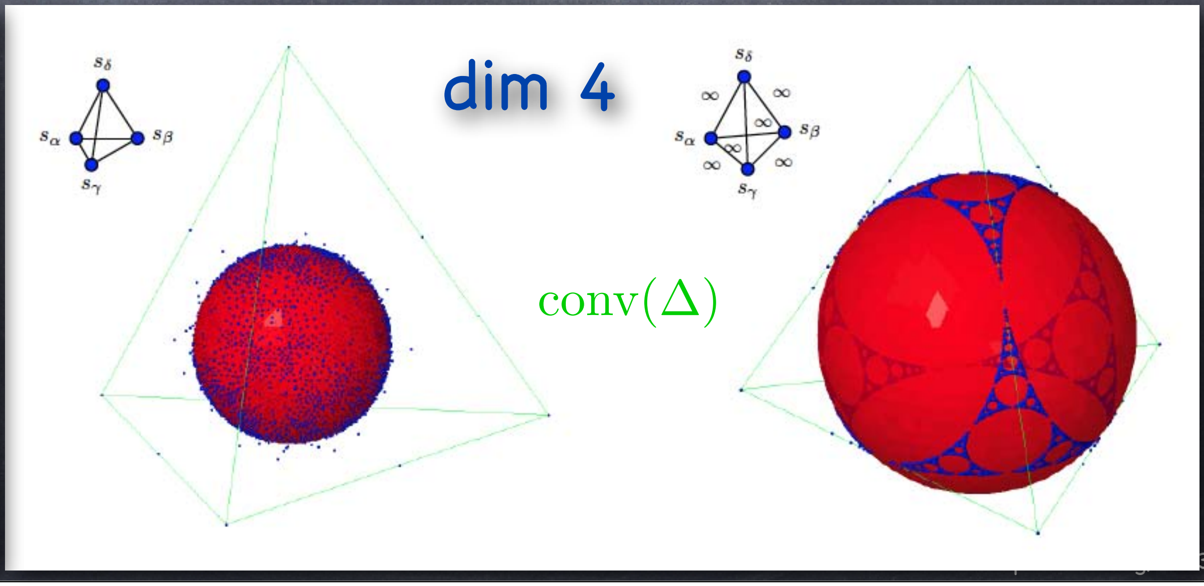
Joint works with :
Labbé & Ripoll;
Dyer & Ripoll

Definition/Proposition: the set of limit roots is:

$$E(\Phi) = \mathrm{Acc}(\widehat{\Phi}) \subseteq Q \cap \mathrm{conv}(\Delta)$$

- A `fractal phenomenon'? How W acts on $E(\Phi)$?
- Link with hyperbolic geometry (hyperbolic reflection groups) and with Apollonian gasket (Kleinian groups) – story in CH, JP-Préaux and V. Ripoll (2013)



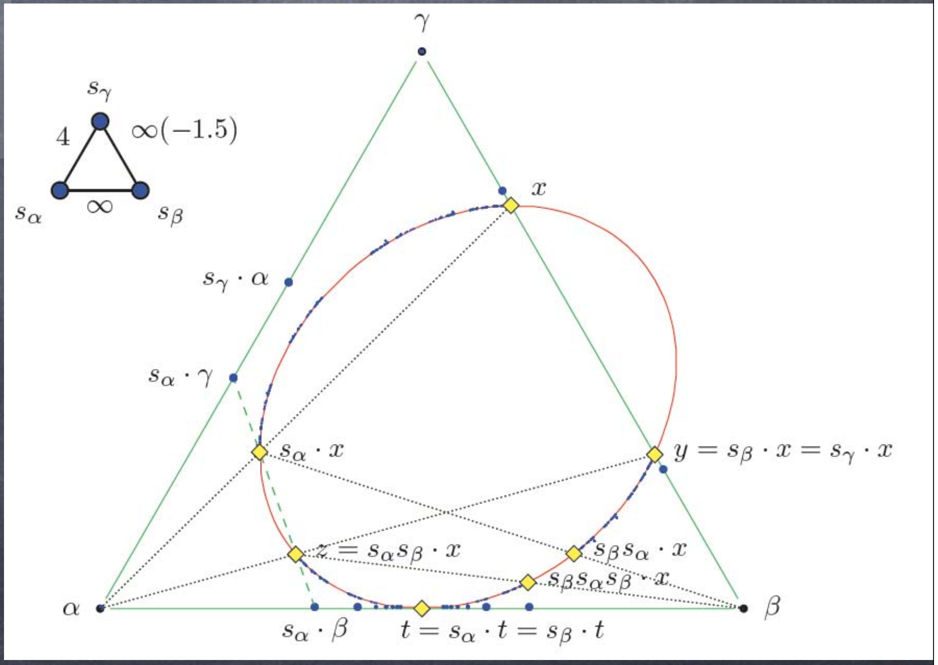


A geometric action on $E(\Phi) = \mathrm{Acc}(\widehat{\Phi})$

Extending the `barycentric' action $W \cdot \widehat{\Phi}$

- New action: $w \cdot v = \widehat{w(v)}$ on the set $\widehat{\Phi} \sqcup E$ given on E by: $\widehat{Q} \cap L(\alpha, x) = \{x, s_\alpha \cdot x\}$

Theorem (Dyer, CH, Ripoll 2013)
Action on E faithful if irreducible not affine nor finite of rank > 2.

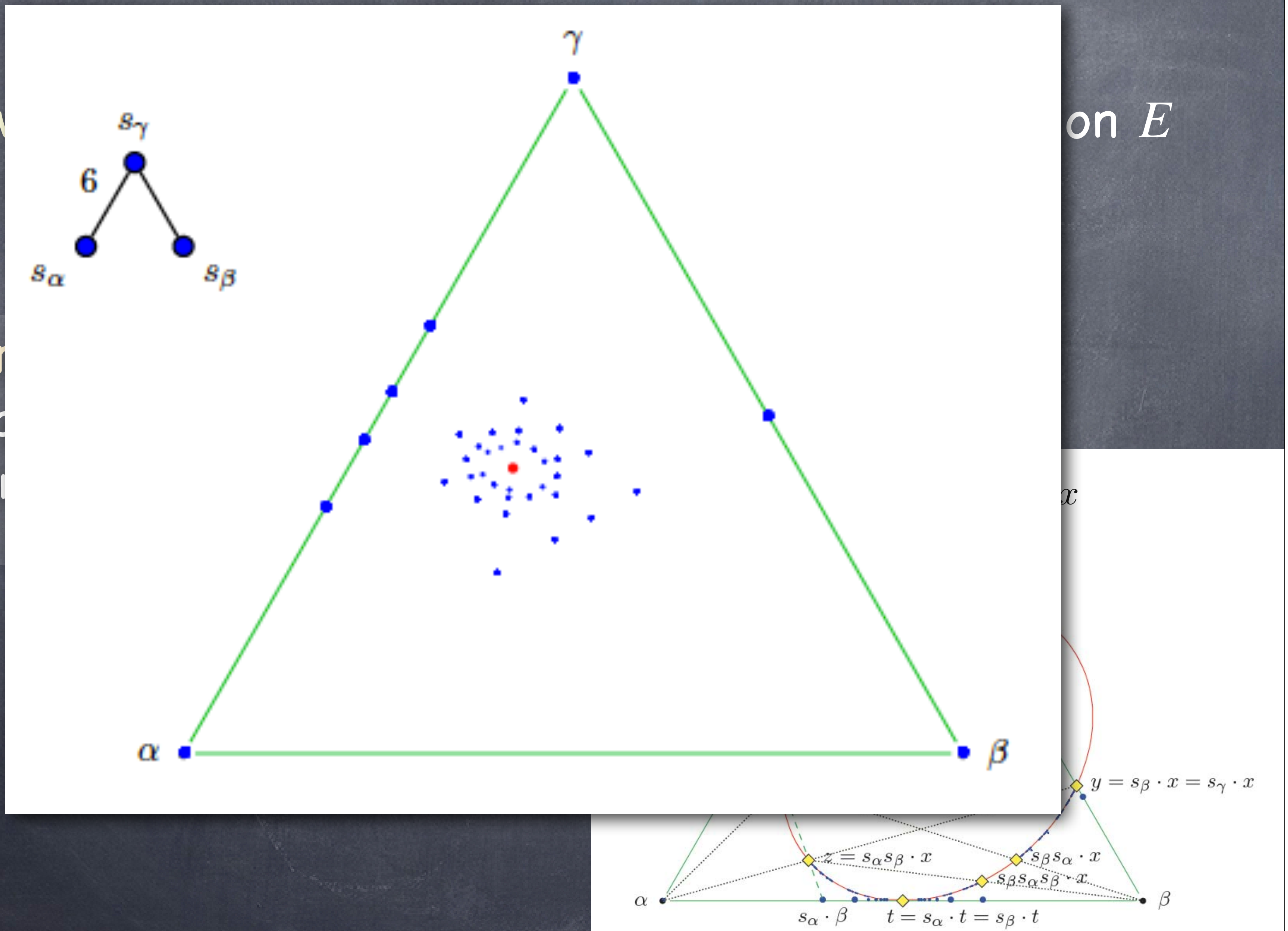


A geometric action on $E(\Phi) = \mathrm{Acc}(\widehat{\Phi})$

- New ... on E

  by:

Theorem ...
Action ...
not affi...



A geometric action on $E(\Phi) = \mathrm{Acc}(\widehat{\Phi})$

Remark: V_1 is not stable under W.

- New action: $w \cdot v = \widehat{w(v)}$ on the set $\widehat{\Phi} \sqcup E$ given on E by: $\widehat{Q} \cap L(\alpha, x) = \{x, s_\alpha \cdot x\}$

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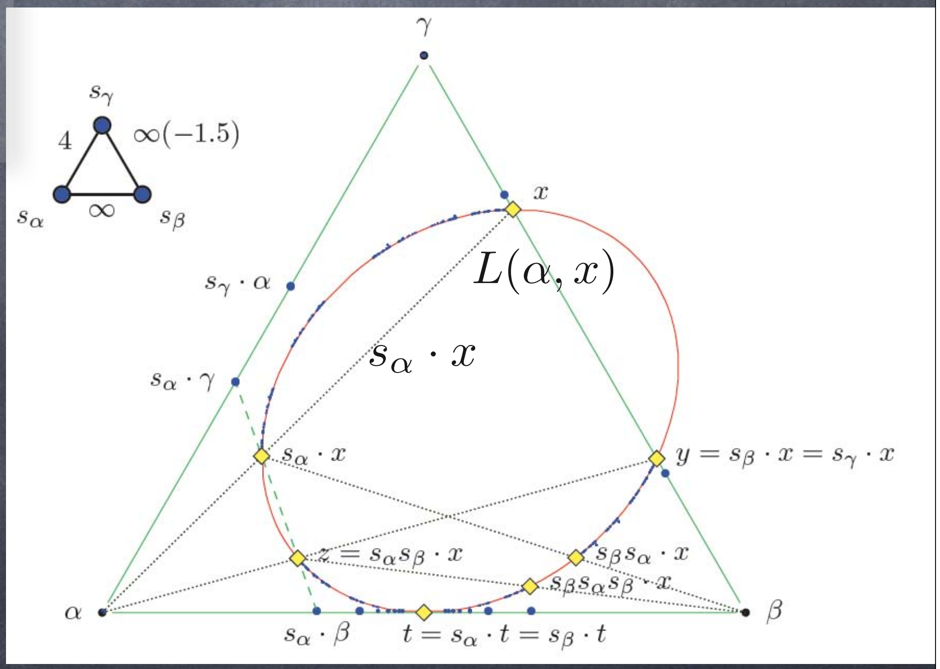
Corollary: to build E ...

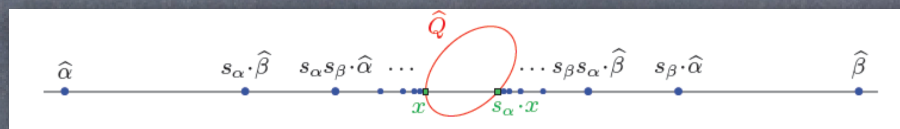


Remarkable dense subsets of $E(\Phi) = \mathrm{Acc}(\widehat{\Phi})$

Dihedral reflection subgroups: $W' = \langle s_\rho, s_\gamma \rangle$, $\rho, \gamma \in \Phi^+$

Associated root system: $\Phi' = W'(\{\rho, \gamma\})$

Observation: $E(\Phi') = \widehat{Q} \cap L(\widehat{\rho}, \widehat{\gamma})$



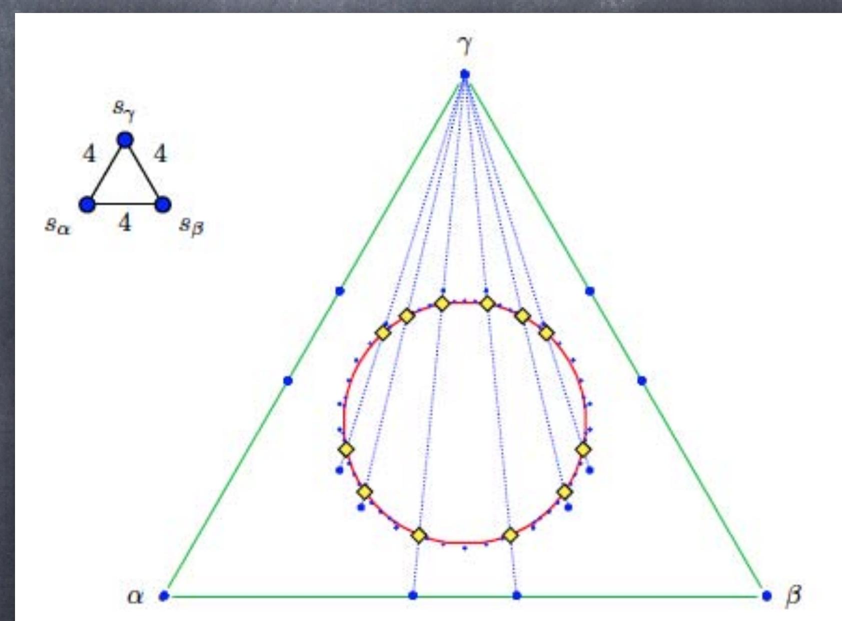
Limits of roots of dihedral reflection subgroups:

- $E_2 = W \cdot E_2^\circ$ where

$$E_2^\circ := \bigcup_{\substack{\alpha \in \Delta \\ \rho \in \Phi^+}} L(\alpha, \widehat{\rho}) \cap \widehat{Q}$$

Theorem (CH, Labbé, Ripoll 2012)

The sets E_2 and E_2° are dense in $E(\Phi)$.



The action on E is minimal

Theorem (Dyer, CH, Ripoll 2013)
The closure of $W \cdot x$ is dense in $E(\Phi)$ for $x \in E(\Phi)$

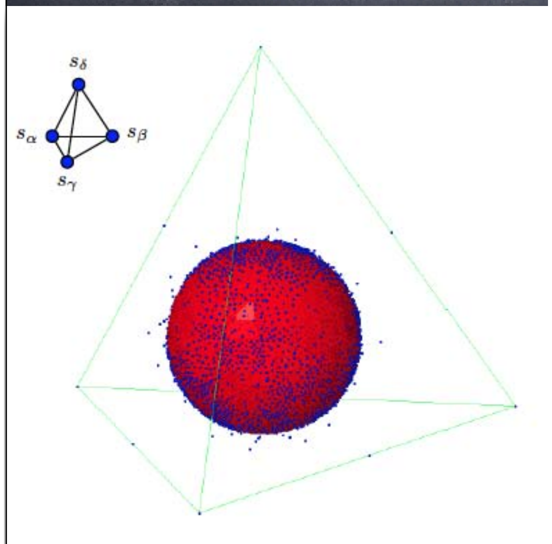
Theorem (Dyer, CH, Ripoll, 2013)

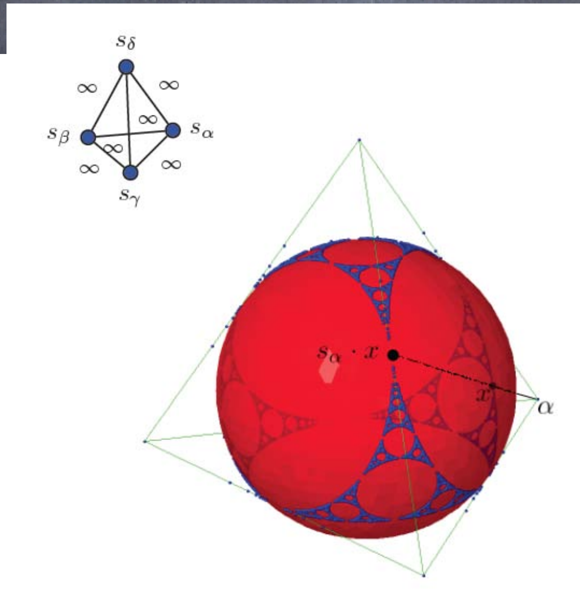
$$E = \hat{Q} \iff \hat{Q} \subseteq \operatorname{conv}(\Delta)$$

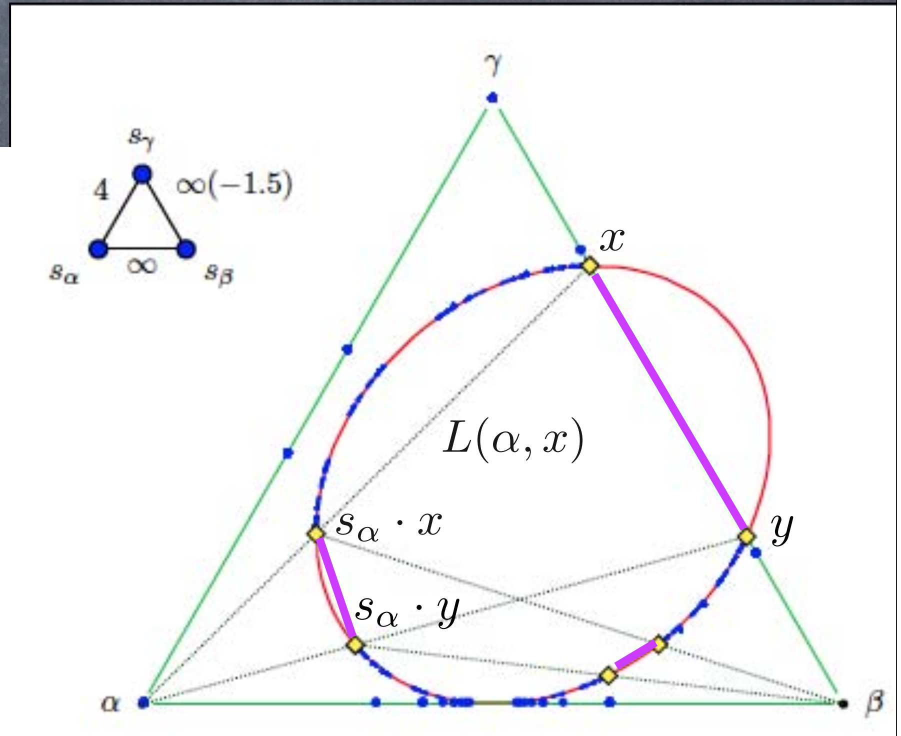
Morever, in this case,

$$\operatorname{sgn}(B) = (n, 1, 0)$$

Corollary (Dyer, CH, Ripoll, 2013)
A first fractal Phenomenon.



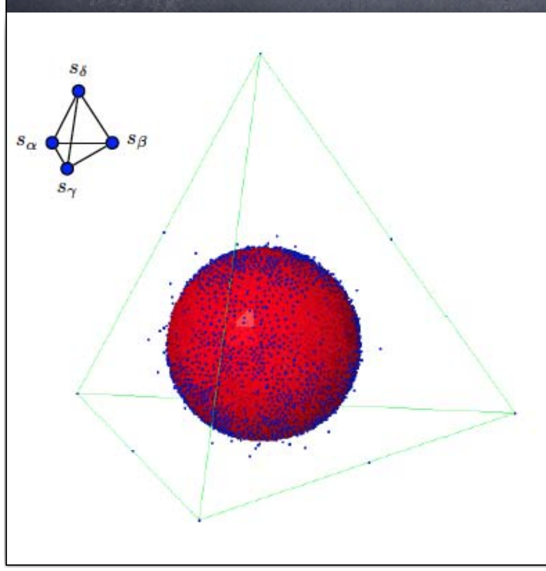


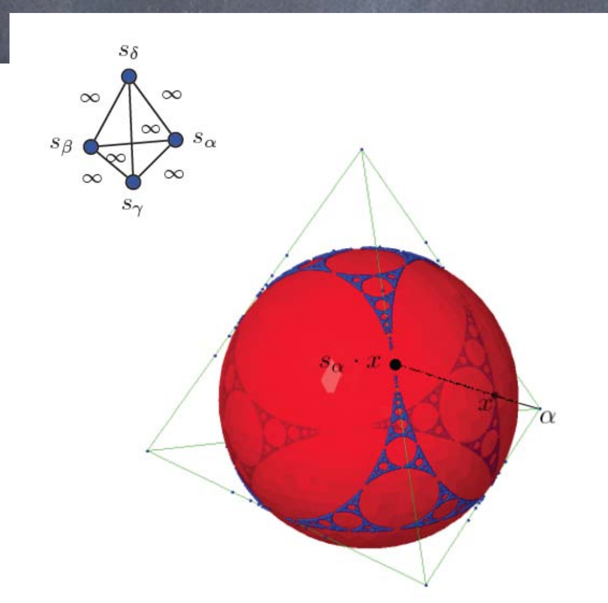


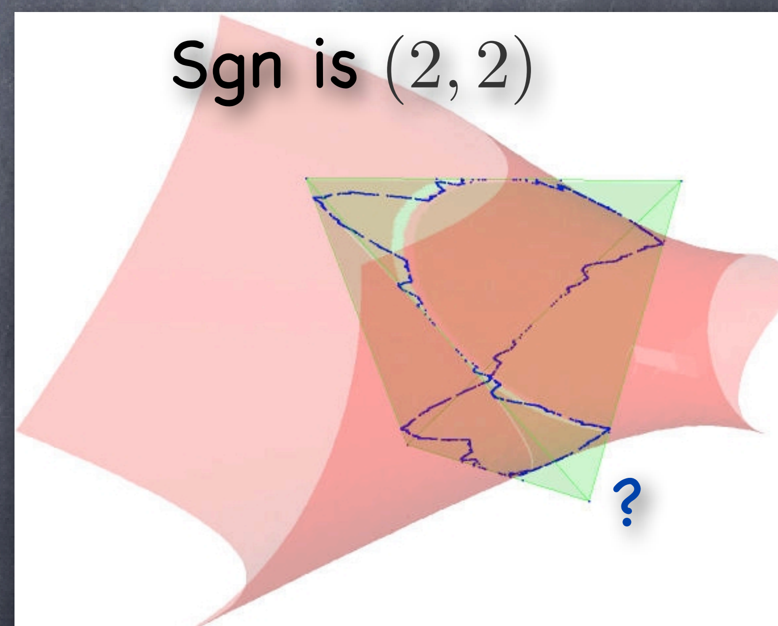
A second fractal phenomenon

Theorem (Dyer, CH, Ripoll 2013) For irreducible root of signature $(n,1,0)$ we have: $E = \mathrm{conv}(E) \cap Q$

Problem (second fractal phenomenon): is it true for other indefinite types?







Imaginary cone and tiling of $\mathrm{conv}(E)$

Proposition (Dyer, CH, Ripoll 2013). The action of W on E extends to an action of W on $\mathrm{conv}(E)$. So W acts on $\widehat{\Phi} \sqcup \mathrm{conv}(E)$

- Imaginary convex body $\mathcal{I}$ is the $W-$orbit of the polytope

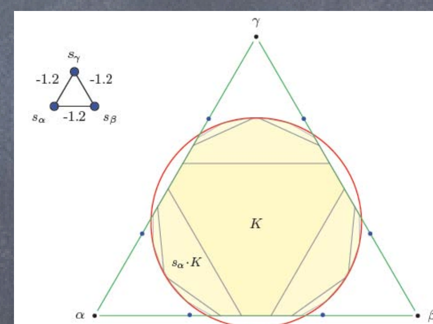
$$K = \{v \in \mathrm{conv}(\Delta) \mid B(v, \alpha) \leq 0, \forall \alpha \in \Delta\}$$

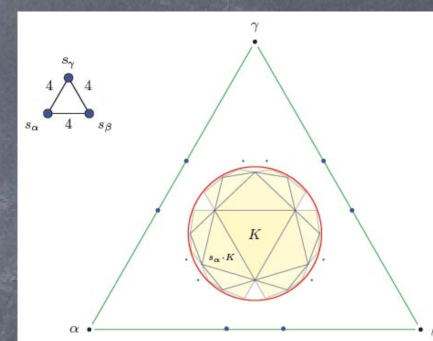
Theorem (Dyer, 2012). $\overline{\mathcal{I}} = \mathrm{conv}(E)$

Theorem (Dyer, CH, Ripoll 2013).

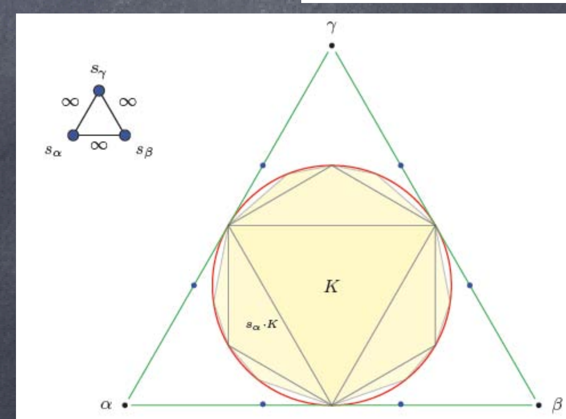
$$E \subseteq \overline{W \cdot z}, \ \forall z \in \overline{\mathcal{I}}$$

with equality for sgn $(n, 1, 0)$

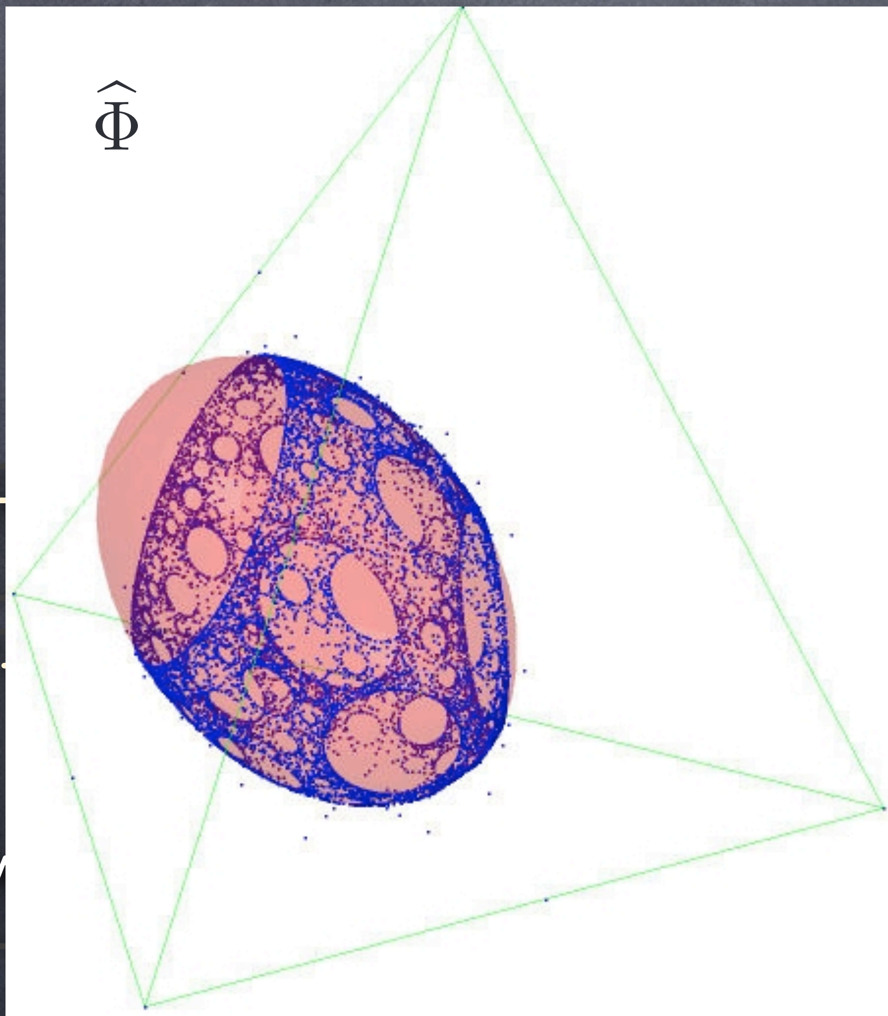


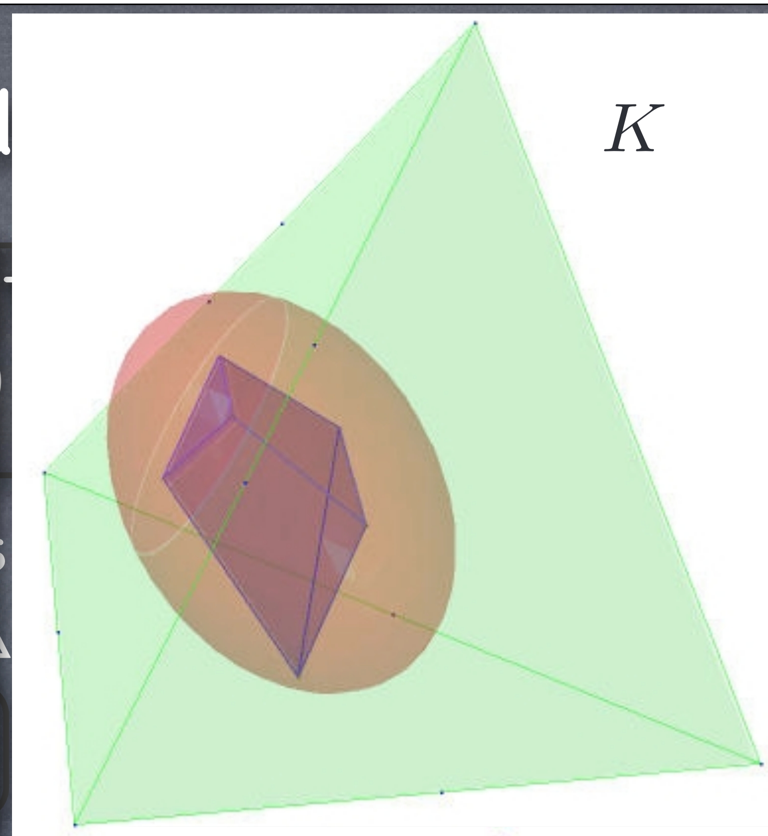


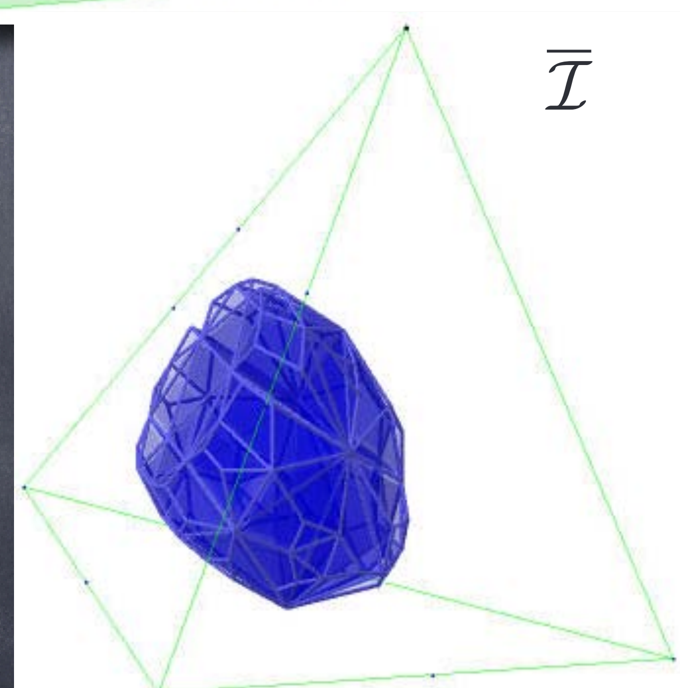
Problem: prove equality in general!

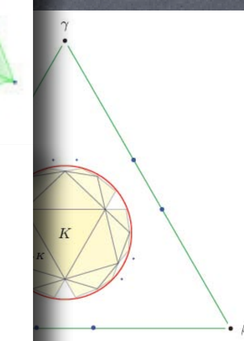


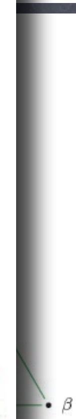
Imaginary cone and











Problem: prove equality in general!

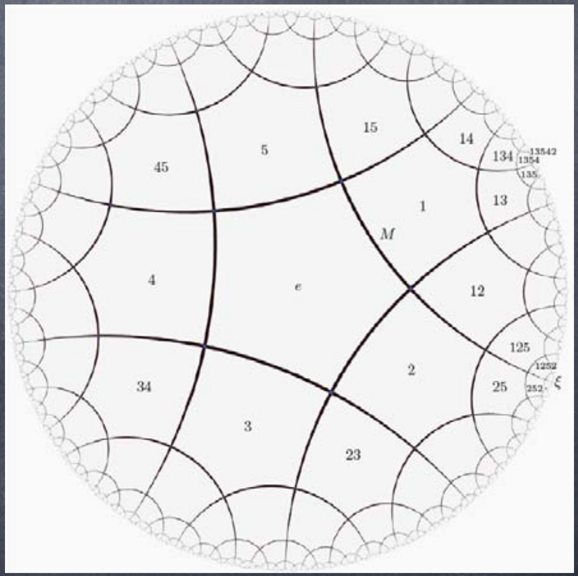
A step toward biclosed sets: Infinite words, their inversion sets and limit weak order

- Thomas Lam & Anne Thomas: in ``Infinite Reduced Words and the Tits Boundary of a Coxeter Group"
  - Limit weak order on infinite words (modulo braid relations) as the topology of the visual boundary of the Davis complex

Propostion. The inversion sets of infinite words are biclosed.

$$N(s_\alpha s_\beta s_\alpha \dots) = \{\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \dots\}$$

Work in progress (CH 2013). The imaginary convex body is a geometric realization of the Davis complex and E is the visual boundary. Biclosed and their boundary!



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A step toward biclosed sets: Infinite words, their inversion sets and limit weak order

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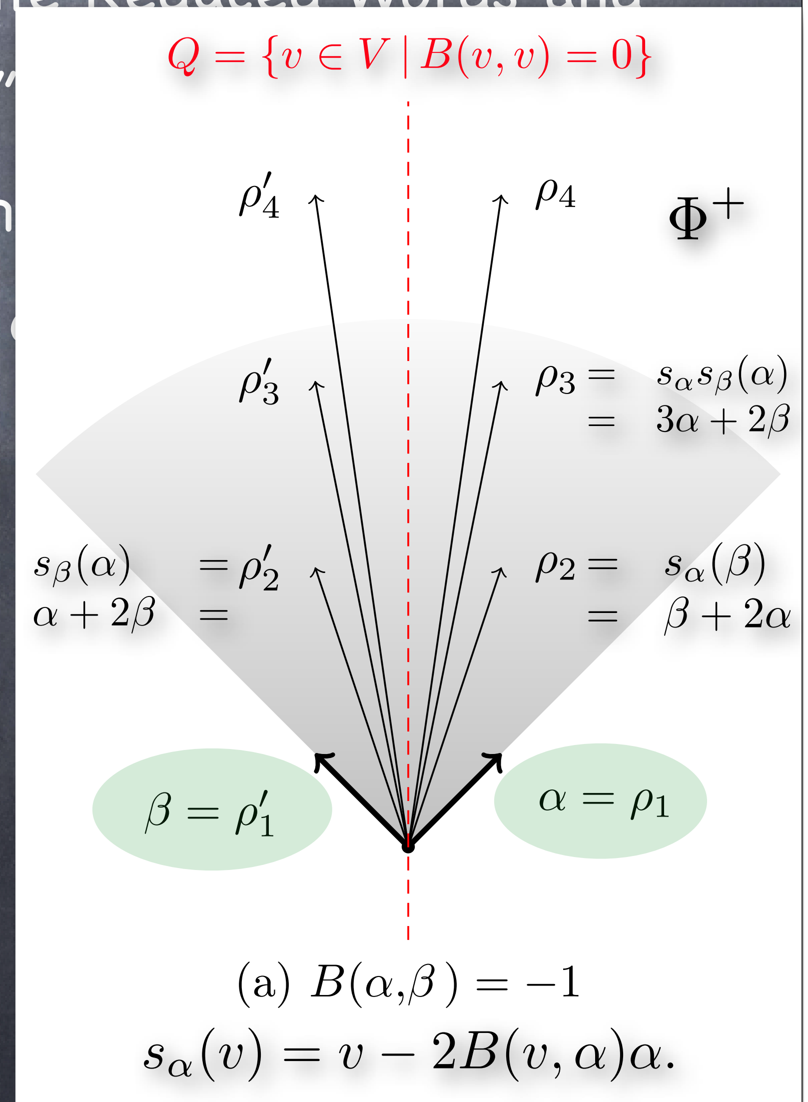
  - Limit weak order on infinite words (m
    the topology of the visual boundary



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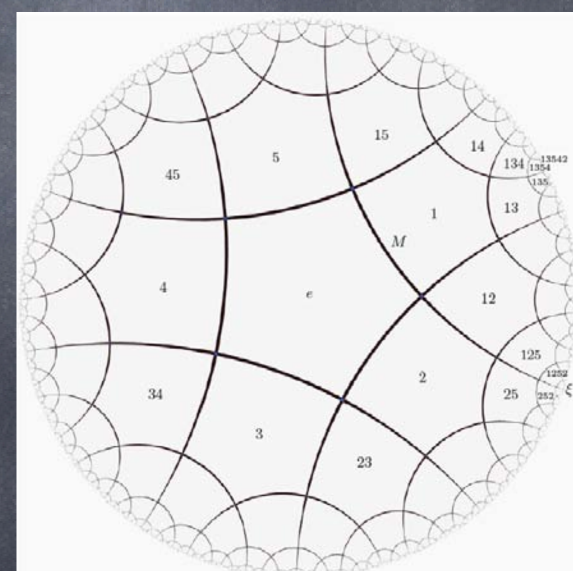

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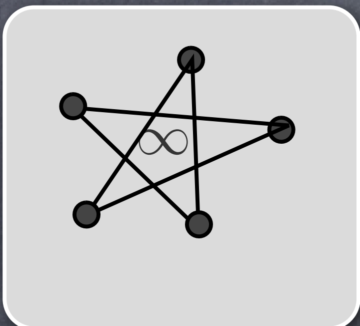
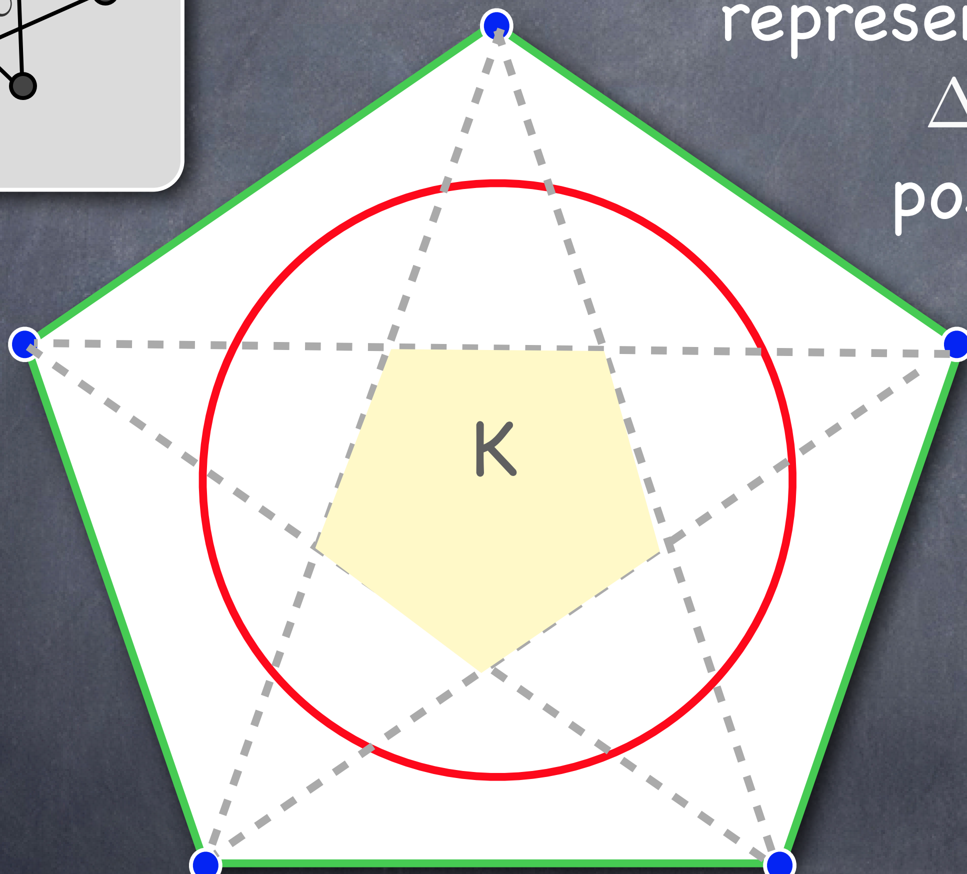
A step toward biclosed sets: Infinite words, their inversion sets and limit weak order

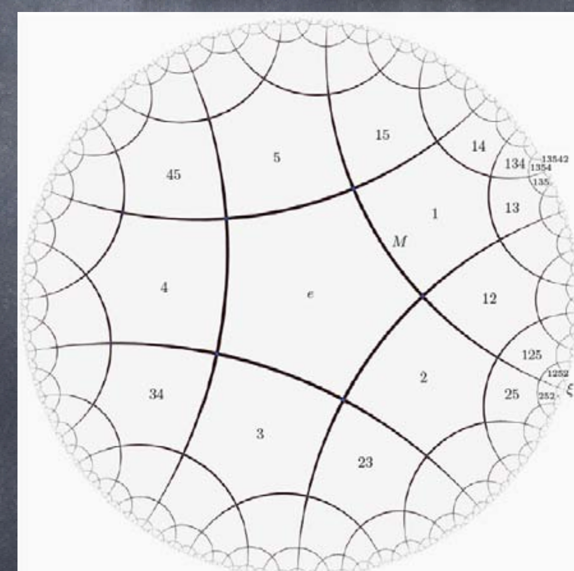


Here a rank 5 Coxeter group is represented in dim 3:
Δ is not a basis but is postivily independant.



Roots and imaginary convex body model

Ball model



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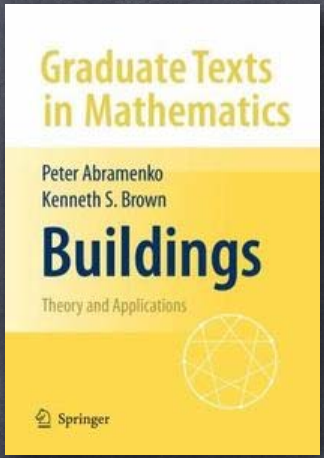
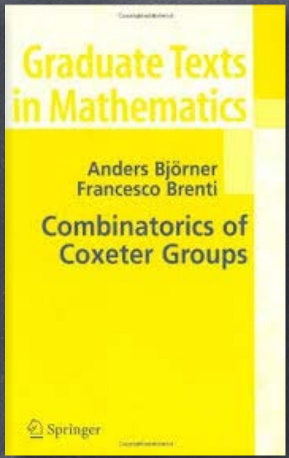
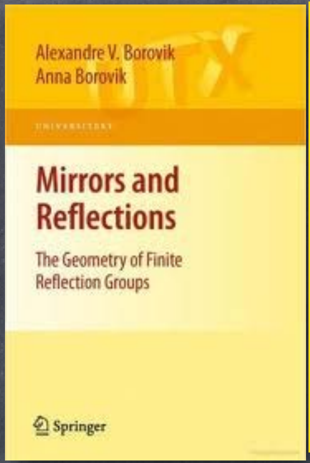
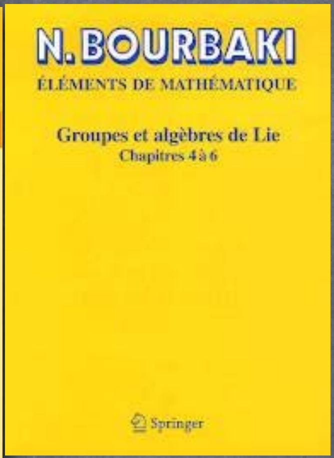
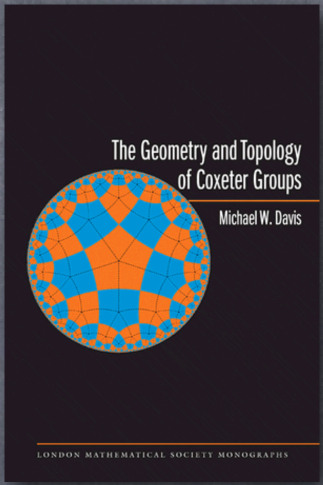
Selected bibliography and other readings

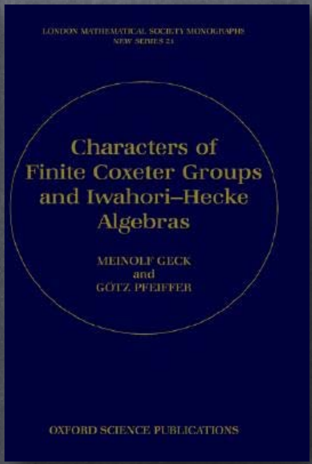


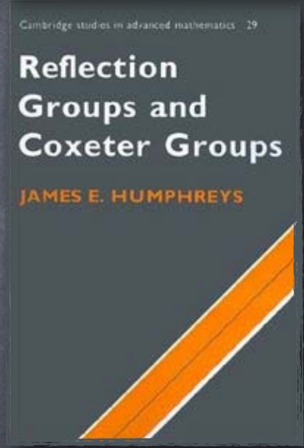












And articles already cited + from

- Brigitte Brink, Bill Casselman, Fokko du Cloux, Bob Howlett, Xiang Fu (regarding automaton and comb.)
- Matthew Dyer (imaginary cones, weak order(s))
- CH & coauthors (Matthew Dyer, Jean-Philippe Labbé, Jean-Philippe Préaux, Vivien Ripoll). A good start for limit of roots and imaginary convex bodies is the survey of the case of Lorentzian spaces (CH, Ripoll, Préaux)
- Paolo Pappi and Ken Ito (limit weak order)
- ...