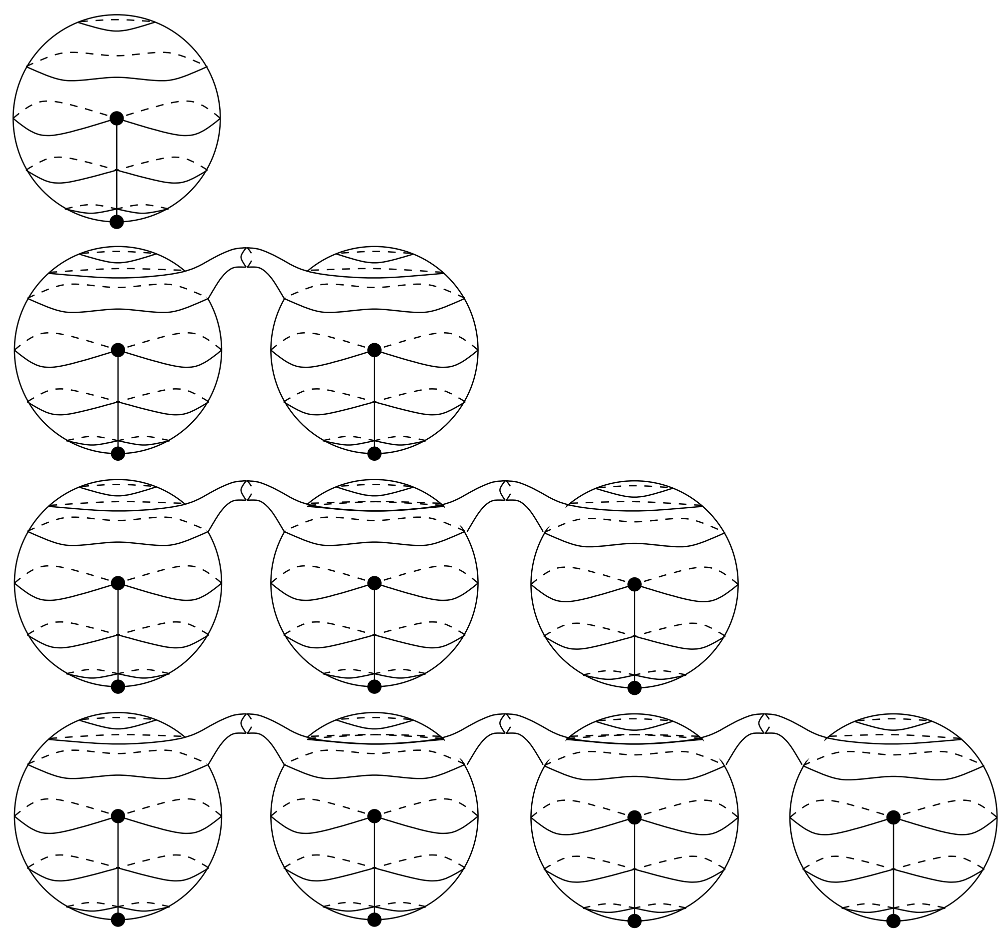
ARRANGEMENTS OF DOUBLE PSEUDOLINES

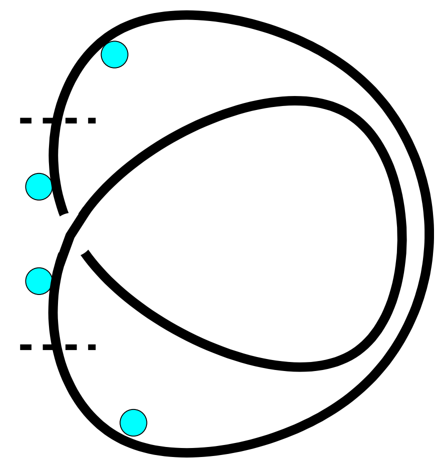
Michel Pocchiola, IMJ, U. Pierre & Marie Curie
(pocchiola@math.jussieu.fr, http://people.math.jussieu.fr/ pocchiola/)



SUMMARY

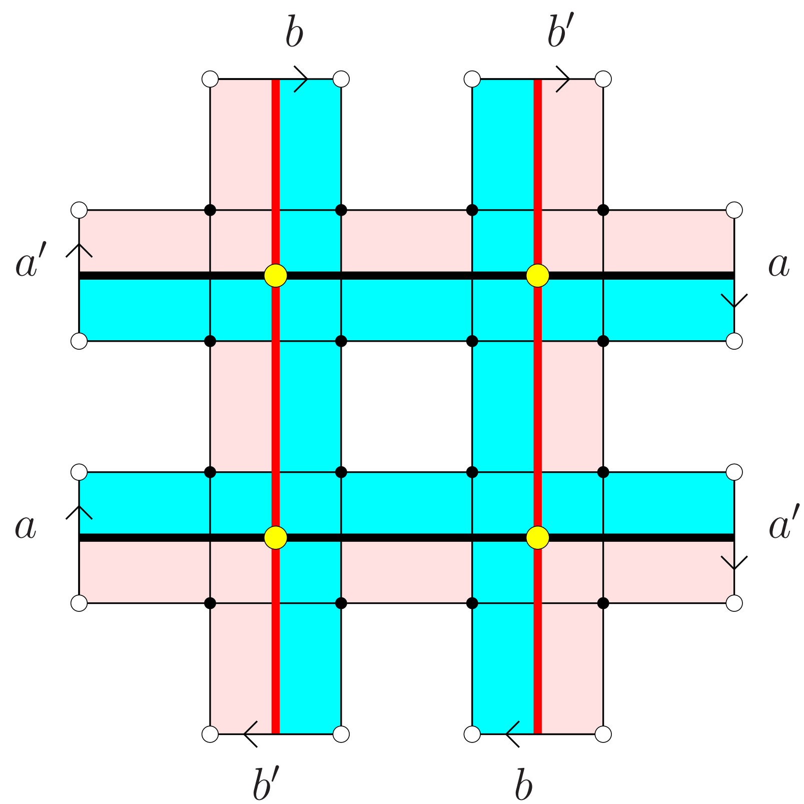
- Arrangements of DP-ribbons
- Arrangements of DP-ribbons of genus 1
- Connection with the algorithmics of (two-dimensional) visibility graphs
- Combinatorics of arrangements of DP-ribbons of genus 1
- Further research and open problems

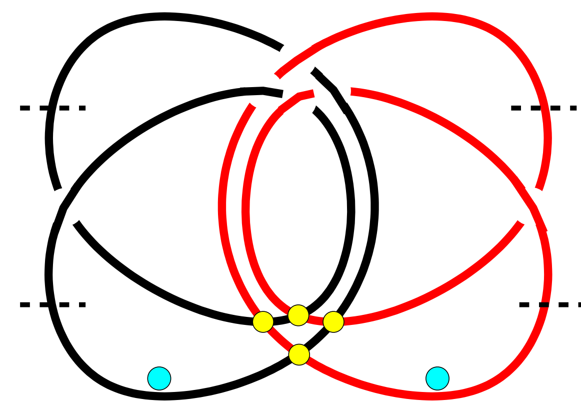
DP-RIBBONS



DF 1. *A DP-ribbon is a cylinder with a distinguished core circle with a distinguished side.*

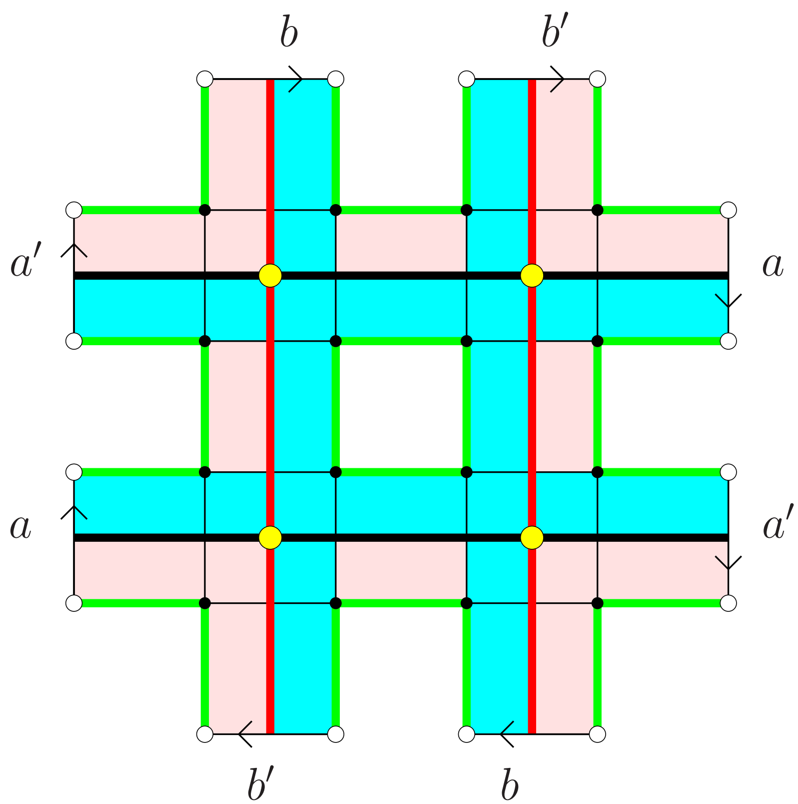
ARRANGEMENTS OF DP-RIBBONS

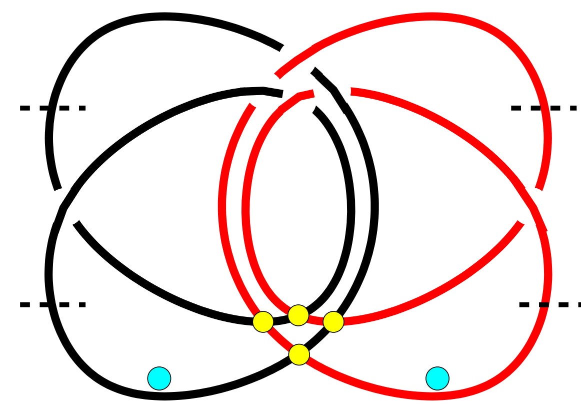




DF 2. *An arrangement of DP-ribbons is a finite family of DP-ribbons pairwise attached as shown in the above figure.*

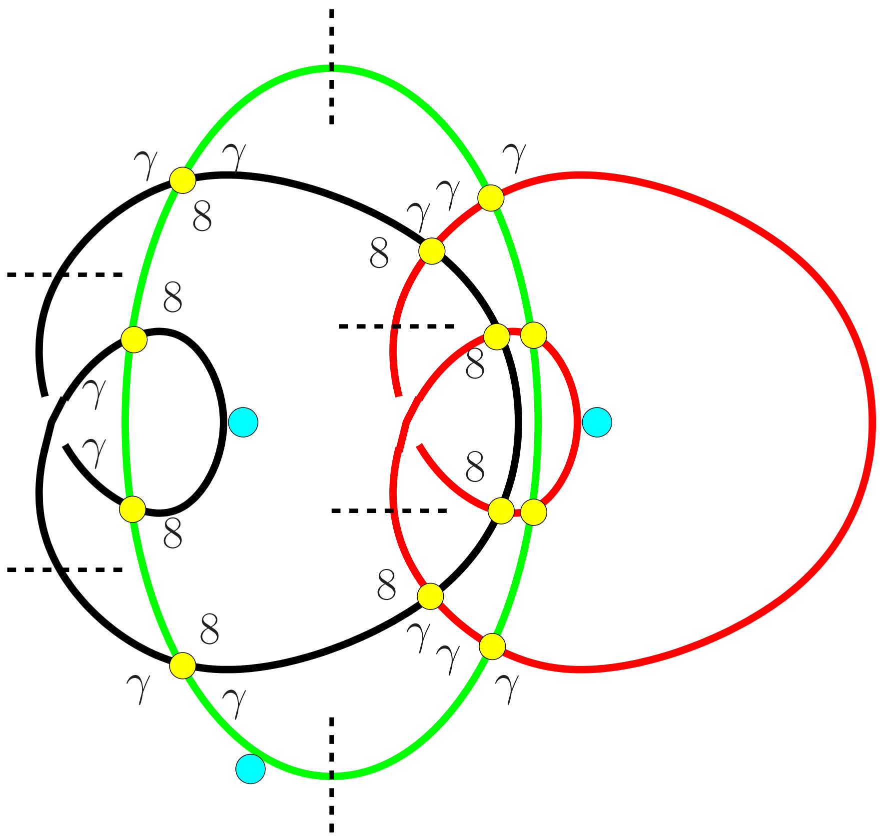
ARRANGEMENTS OF DP-RIBBONS





PP 1. *An arrangement of two DP-ribbons lives in a sphere with 1 crosscap and 5 boundaries (3 tetragons and 2 digons).*

AN ARRANGEMENT OF THREE DP-RIBBONS

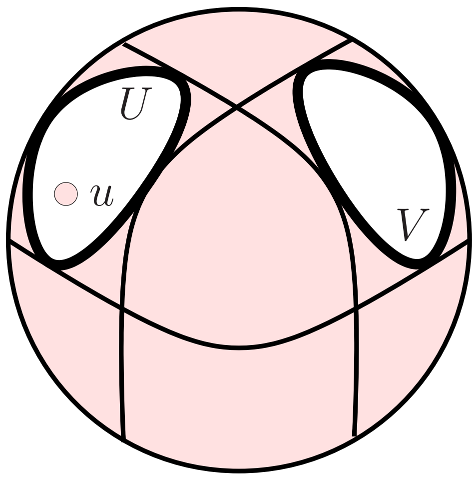


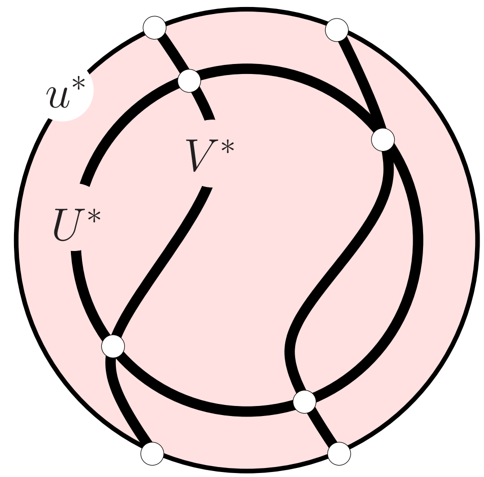
10 boundaries
- 2 digons
- 6 tetragons
- 1 octogon
- 1 dodecagon

12 vertices
double Klein bottle

$$\text{genus} \;=\; 2 - \#\text{boundaries} + \#\text{vertices}$$

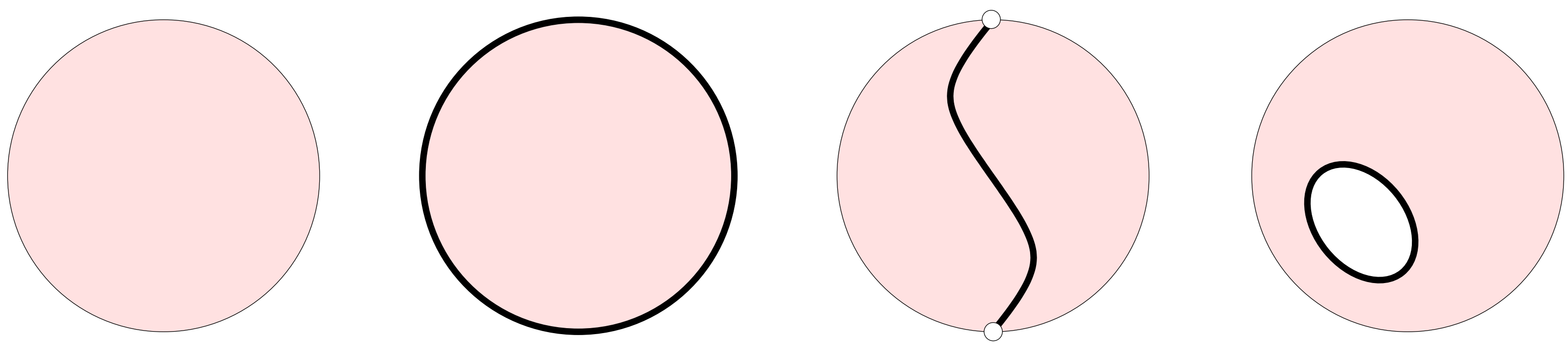
ARRANGEMENTS OF DOUBLE PSEUDOLINES





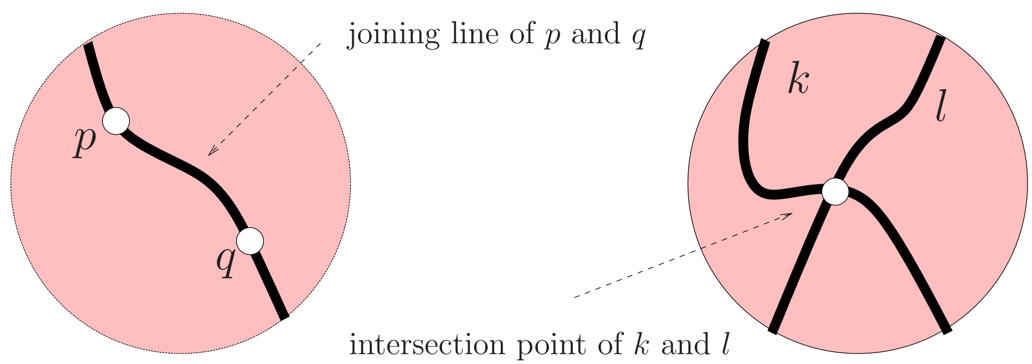
TH 1 (Habert and P. 2006). *Arrangements of DP-ribbons of genus* 1 *are exactly, modulo the addition of topological disks along their boundaries, the arrangements of double pseudolines, i.e., the dual arrangements of finite families of pairwise disjoint convex bodies of (real two-dimensional) projective planes.*

CROSS-SURFACES, PSEUDOLINES AND DOUBLE PSEUDOLINES



PROJECTIVE PLANES

Hilbert 1899, Kolmogoroff 1932, Köthe 1939, Skornjakov 1954, Salzmann 1955, Freudenthal 1957



DF. *A* **(real two-dimensional) projective plane** *is a topological point-line incidence structure* $(\mathcal{P}, \mathcal{L}, \in)$ *whose point space* $\mathcal{P}$ *is a cross-surface and whose line space* $\mathcal{L}$ *is a subspace of the space of pseudolines of* $\mathcal{P}$ *such that (1) any two distinct points are contained in exactly one line which depends continuously on the two points; (2) any two distinct lines intersect in exactly one point which depends continuously on the two lines.*

STANDARD PROJECTIVE PLANE

DF. *The* **standard projective plane** *is the projective plane whose point space is the standard cross surface* $\mathbb{P}^2$ *and whose line space is the image under the canonical projection* $\mathbb{S}^2 \longrightarrow \mathbb{P}^2$ *of the space of great circles of* $\mathbb{S}^2$*.*

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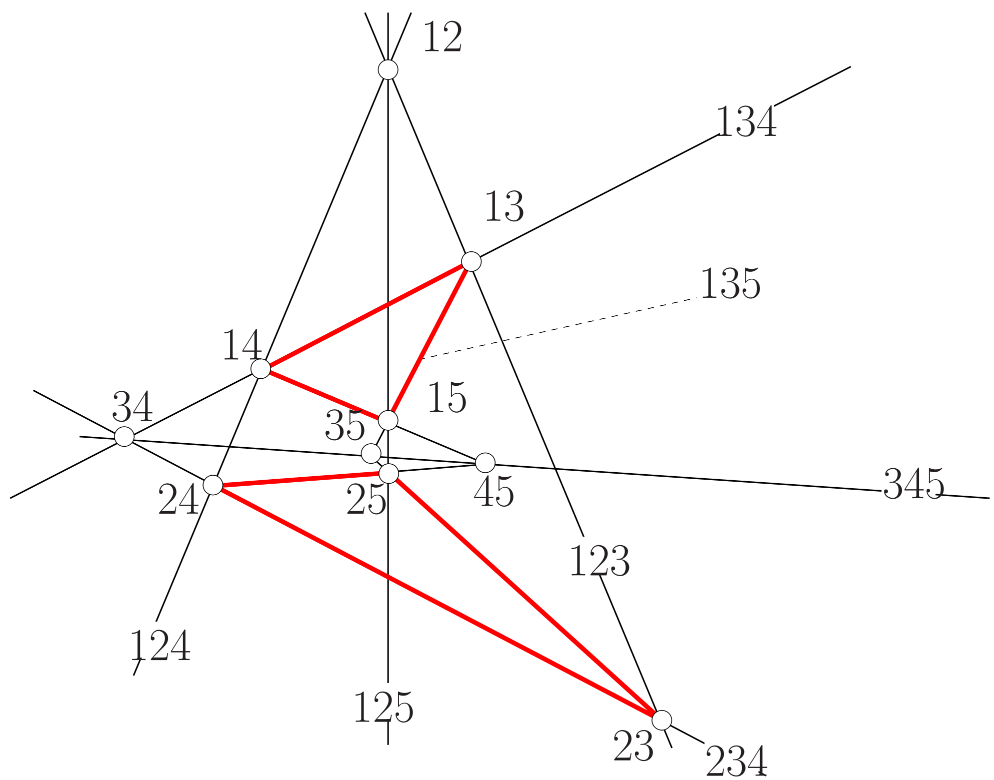
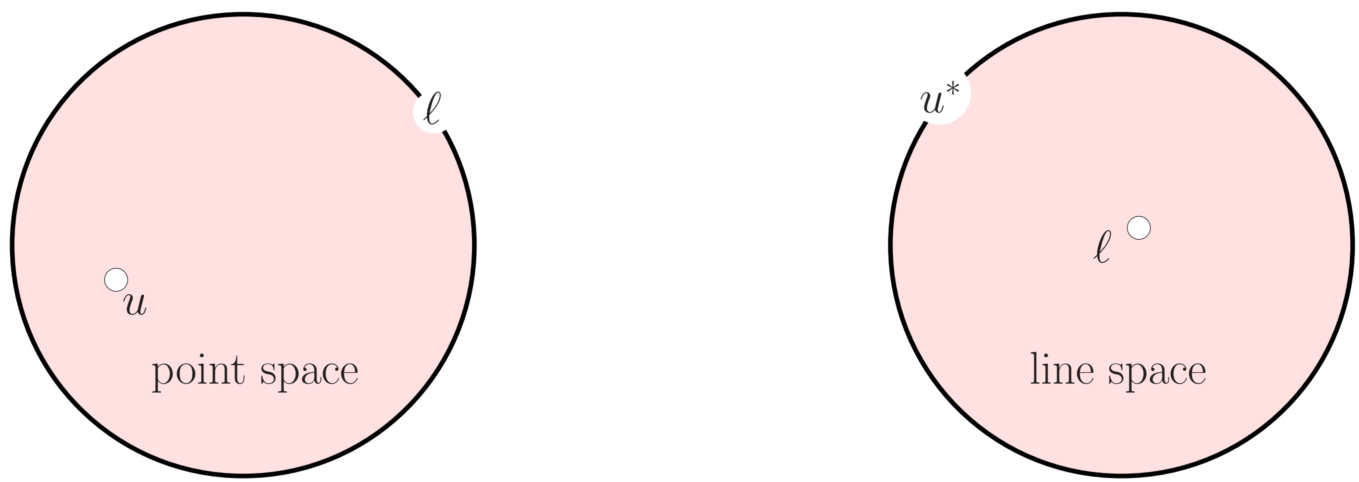
TH (Hilbert, 1899). *The standard projective plane is the unique* **desarguesian** *projective plane.*



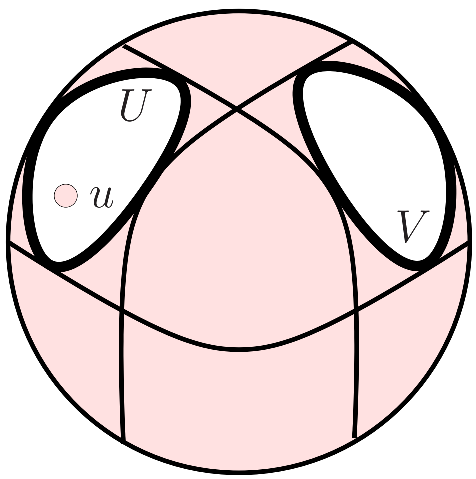
DUALITY IN PROJECTIVE PLANES

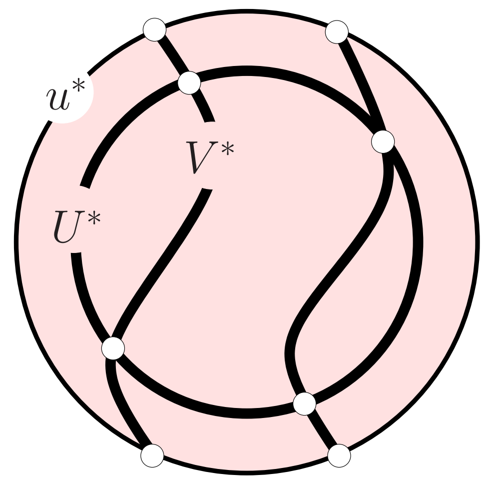


TH. *The line space of a projective plane is a cross-surface and the dual of a point of a projective plane (i.e., the pencil of lines through that point) is a pseudoline of its line space.*

$$(\mathcal{P}, \mathcal{L}) \to (\mathcal{L}, \mathcal{P}^*) \to (\mathcal{P}^*, \mathcal{L}^*) \approx (\mathcal{P}, \mathcal{L})$$

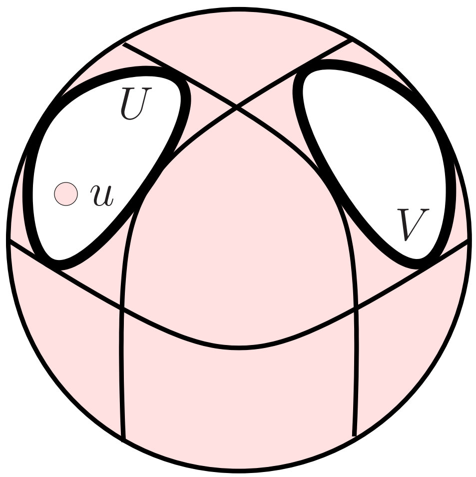
DUAL OF A PAIR OF DISJOINT CONVEX BODIES

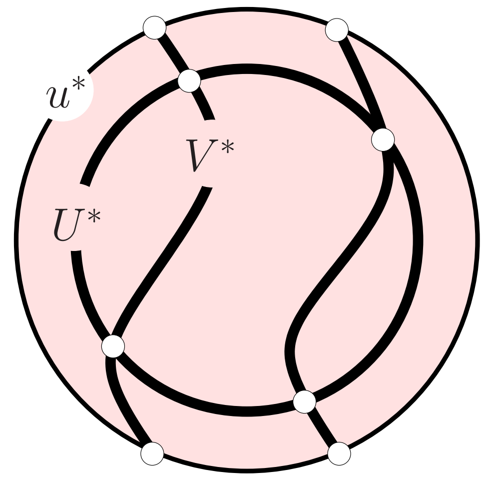




TH (Habert and P. 2006). *Up to homeomorphism, the dual of a pair of disjoint convex bodies is the unique pair of double pseudolines that intersect transversely in four points and induce a cellular decomposition of their underlying cross-surface.*

ARRANGEMENTS OF DOUBLE PSEUDOLINES





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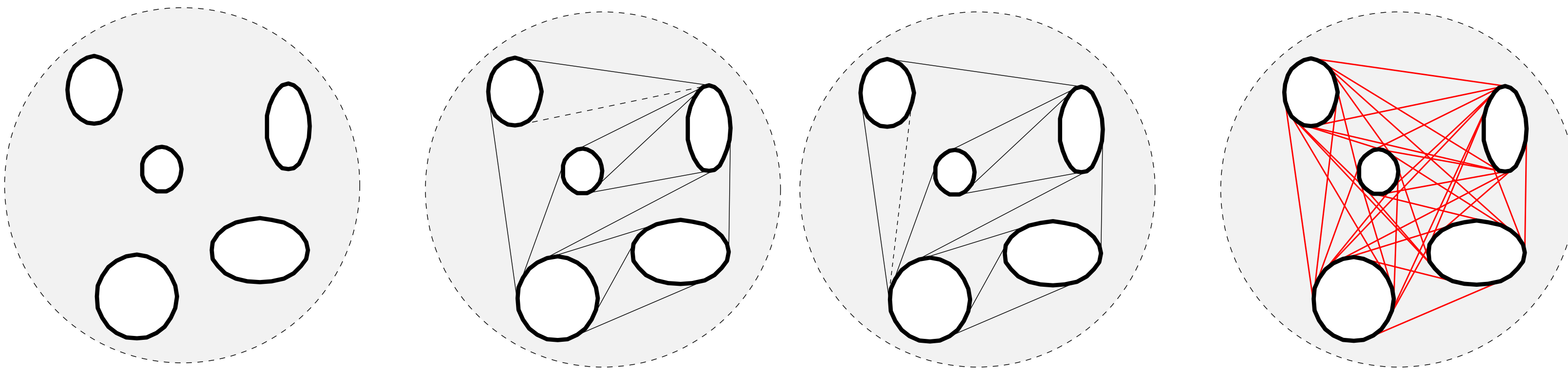
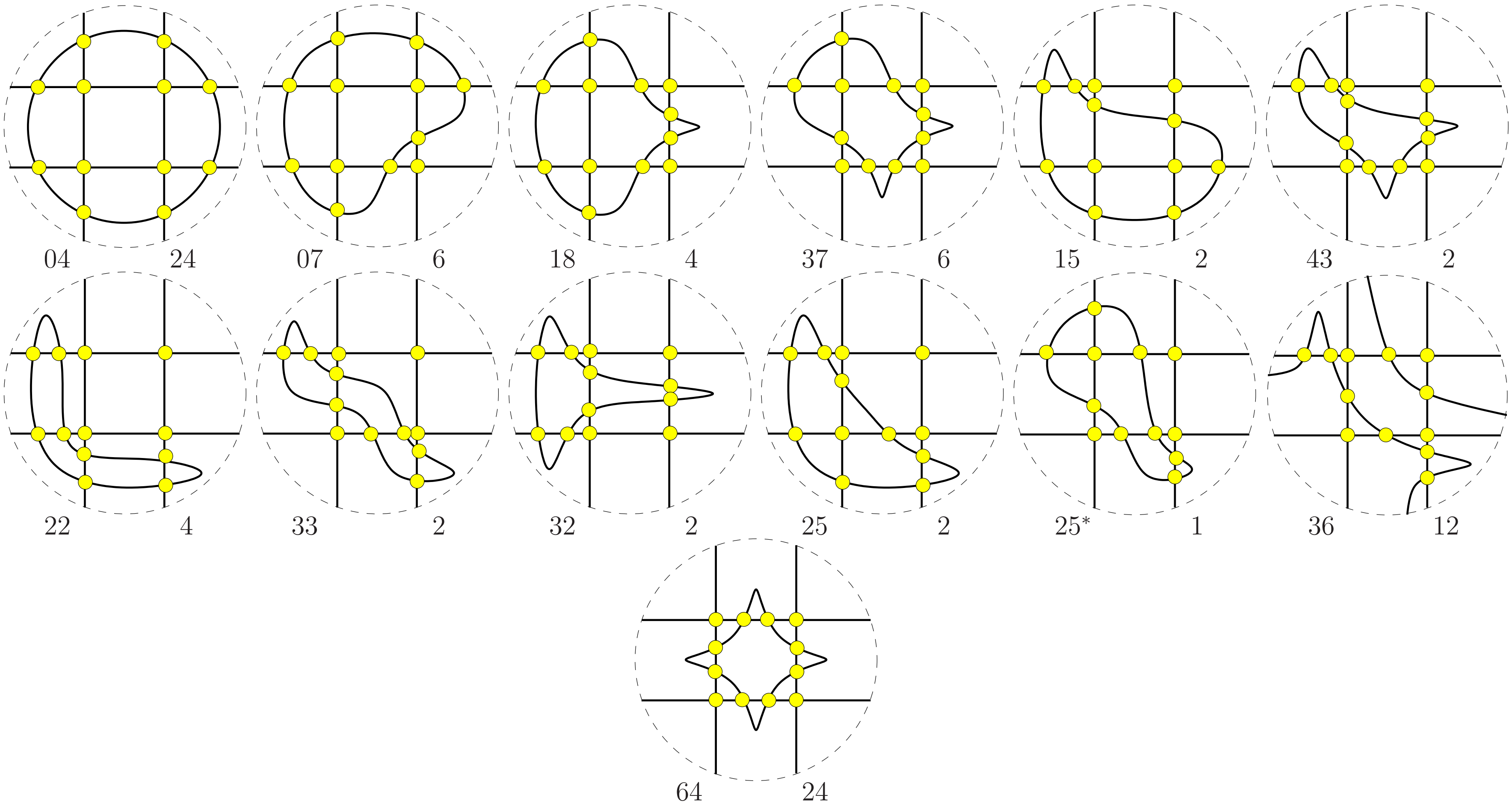
ALGORITHMICS OF VISIBILITY GRAPHS



TH (Habert & P. 12, Angelier & P. 03, P. & Vegter 96). *The k free bitangent line segments of a planar family of n pairwise disjoint convex bodies presented by its chirotope are computable in $O(k + n \log n)$ time and $O(n)$ working space.*

SIMPLE ARRANGEMENTS OF THREE DOUBLE PSEUDOLINES



ENUMERATION OF ARRANGEMENTS OF DOUBLE PSEUDOLINES

$$a_n(1) = \#\text{ simple arrangements of } n \text{ double pseudolines}$$
$$b_n(1) = \#\text{ simple indexed arrangements of } n \text{ oriented double pseudolines}$$

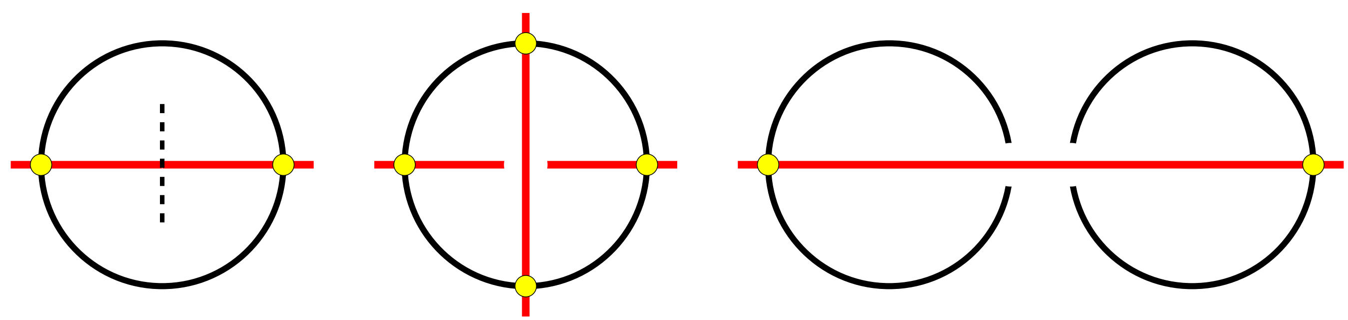
n	2	3	4	5
$a_n(1)$	1	13	6570	181 403 533
$b_n(1)$	1	216	2 415 112	nc

Ferté, Pilaud and P. 2008

LR CHARACTERIZATION

TH 2 (Habert and P. 2006). *An arrangement of DP-ribbons is of genus 1 if and only if its subarrangements of size 3, 4 and 5 are of genus 1.*

- mutation
- separation



(A) (B) (C)

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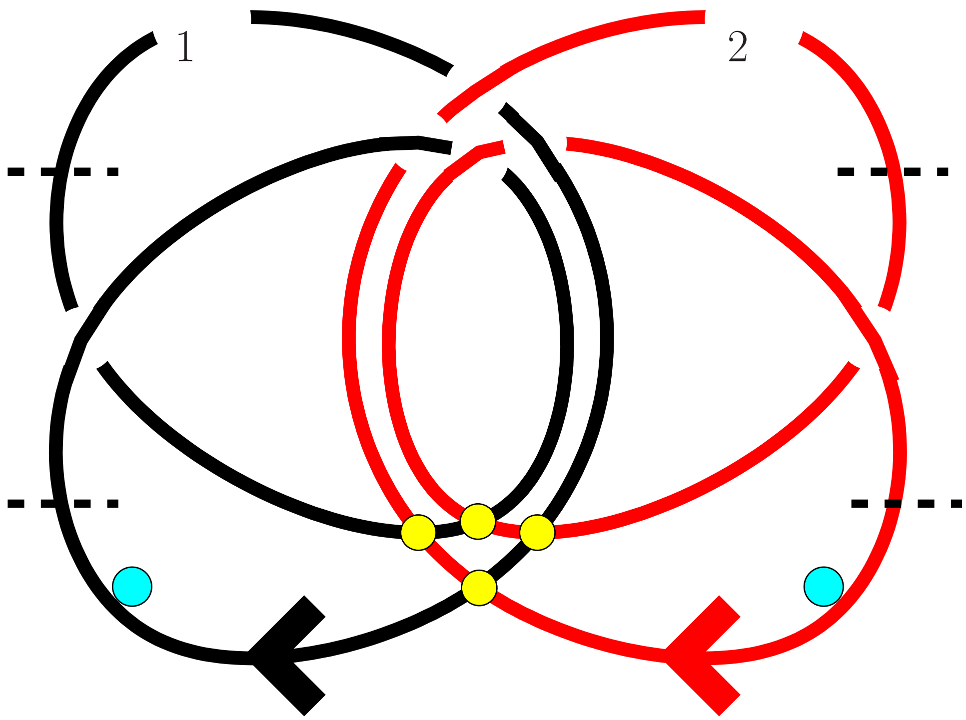
CJ (P. 2010). *An arrangement of 5 DP-ribbons is of genus 1 if and only if its subarrangements of size 4 (hence 3) are of genus 1.*

TH 3 (P. 2013). *An arrangement of 5 DP-ribbons whose subarrangements of size 4 are of genus 1 is of genus* 1 *or its subarrangements of size* 4 *belong to a well-defined family of few tens of arrangements.*

ENUMERATION

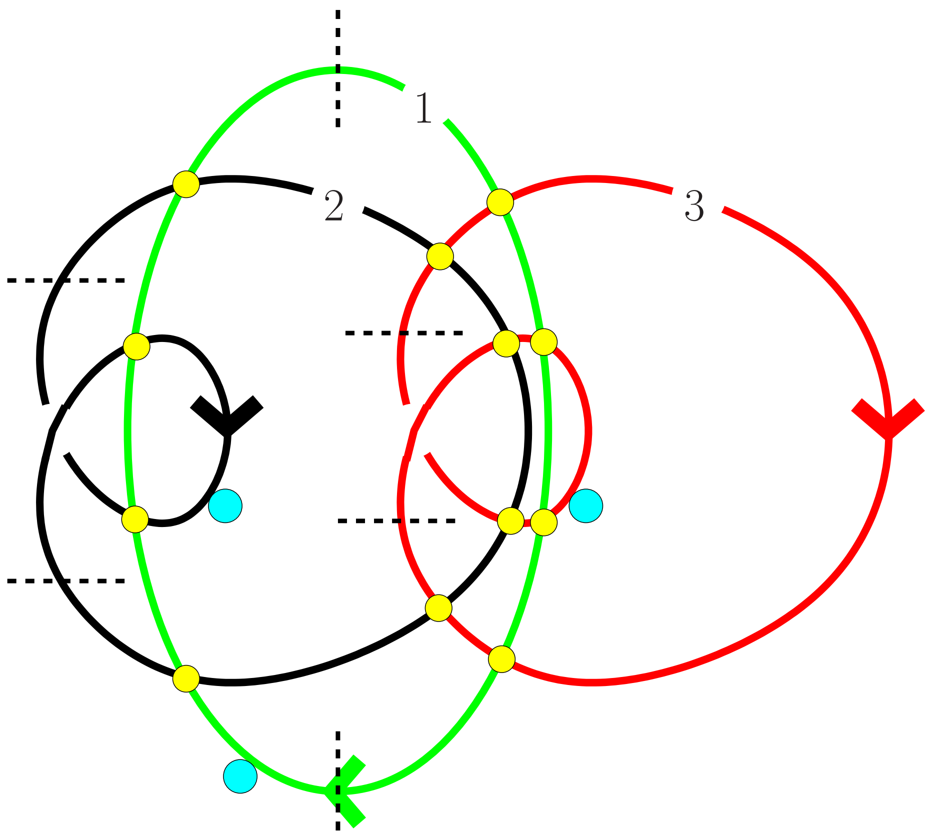


$1: \overline{22}22$

$2: \overline{11}11$

PP 2. *There is a natural correspondence between indexed arrangements of n oriented DP-ribbons and the n-tuples of suffles of the $n-1$ circular sequences $\overline{jj}jj$, $j = 2, 3, \ldots, n$. Furthermore ...*

ENUMERATION (CONTINUED)



$1: \overline{22}2233\overline{33}$

$2: \overline{11}1133\overline{33}$

$3: 1221\overline{1221}$

ENUMERATION (CONTINUED)

PP 2. *There is a natural correspondence between indexed arrangements of n oriented DP-ribbons and the n-tuples of suffles of the $n-1$ circular sequences $\overline{jj}jj$, $j = 2, 3, \ldots, n$. Furthermore the number b_n of indexed arrangements of n oriented DP-ribbons is*

$$\left\{4^{n-2}\binom{4n-5}{3,4,4,\ldots,4}\right\}^n$$

and the number a_n arrangements of n DP-ribbons is bounded from below by

$$b_n/(2^n n!).$$

$$b_3 = \left\{4^1\tbinom{7}{3,4}\right\}^3 = 140^3 = 2\,744\,000 \;\Big\|\; \lceil b_3/(2^3 3!)\rceil = 57167 \;\Big\|\; a_3 = 58042$$

$$b_4 = \left\{4^2\tbinom{11}{3,4,4}\right\}^4 = 184800^4$$

$$b_5 = \left\{4^3\tbinom{15}{3,4,4,4}\right\}^5 = 1009008000^5$$

ENUMERATION (CONTINUED)

$$a_n(g) = \#\text{ arrangements of } n \text{ DP-ribbons of genus } g$$
$$b_n(g) = \#\text{ indexed arrangements of } n \text{ oriented DP-ribbons of genus } g$$

g	1	2	3	4	5	6	7	8	9	10	11	12	13
$a_3(g)$	13	20	77	197	674	1127	2707	5173	10073	11943	13633	9115	3290
$b_3(g)$	216	636	2756	8292	29032	50848	123240	240196	475920	565016	653528	436496	157824

C. Lange and M.P. 2013

ENUMERATION (CONTINUED)

$$a_4^*(g) = \#\text{ arrang. of size 4 and genus } g \text{ whose subarrangements of size 3 are of genus 1}$$
$$b_4^*(g) = \#\text{ indexed and oriented versions}$$

g	1	2	3	4	5	6	7	≥ 8
$a_4^*(g)$	6 570	0	455	0	18	0	1	0
$b_4^*(g)$	2 415 112	0	135 664	0	4 560	0	16	0
$\lceil b_4^*(g)/2^4 4! \rceil$	6 290	0	354	0	12	0	1	0

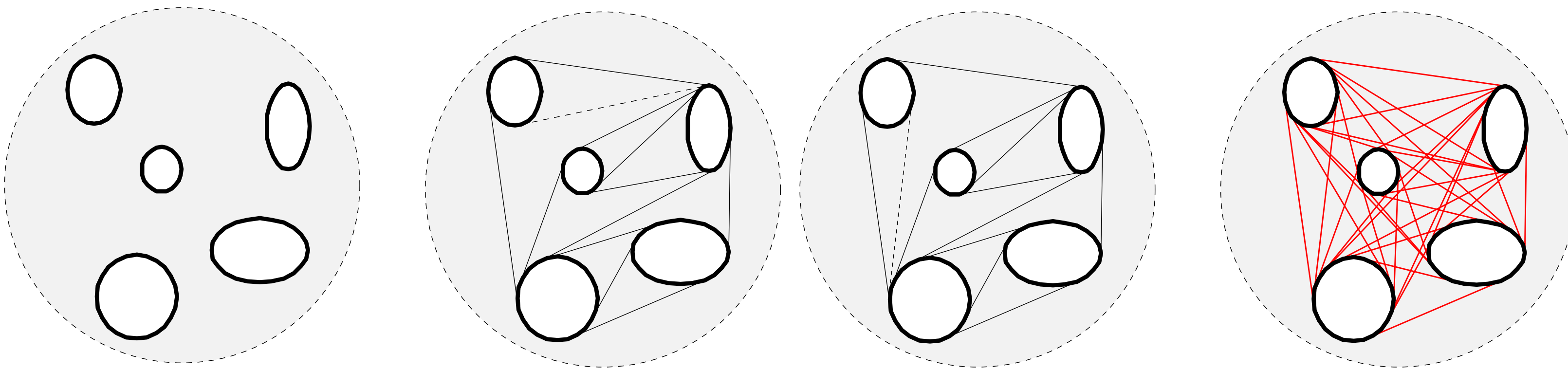
C. Lange and M.P. 08/09/2013

ENUMERATION (END)

$$a_5^*(g) = \#\text{ arrang. of size 5 and genus } g \text{ whose subarrangements of size 4 are of genus 1}$$
$$b_5^*(g) = \#\text{ indexed and oriented versions}$$

g	1	≥ 2
$a_5^*(g)$	180 403 533	??
$b_5^*(g)$	??	

ALGORITHMICS OF VISIBILITY GRAPHS



TH (Habert & P. 12, Angelier & P. 03, P. & Vegter 96). *The k free bitangent line segments of a planar family of n pairwise disjoint convex bodies presented by its chirotope are computable in $O(k + n \log n)$ time and $O(n)$ working space.*

OPEN PROBLEMS

Problem 1. *Devise a quadratic time algorithm to compute an arrangement of n double pseudolines presented by its subarrangements of size 3.*

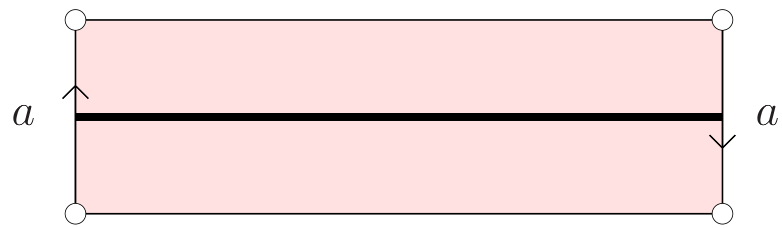
Problem 2. *Give asymptotic formulae for the numbers $b_n(g)$.*

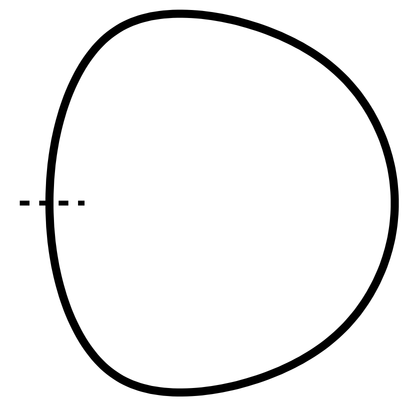
$$\sum_g b_n(g) \;=\; b_n = \left\{4^{n-2}\binom{4n-5}{3,4,4,\ldots,4}\right\}^n$$

$$\ln b_n \;=\; \Theta(n^2 \log n)$$

$$\ln b_n(1) \;=\; \Theta(n^2)$$

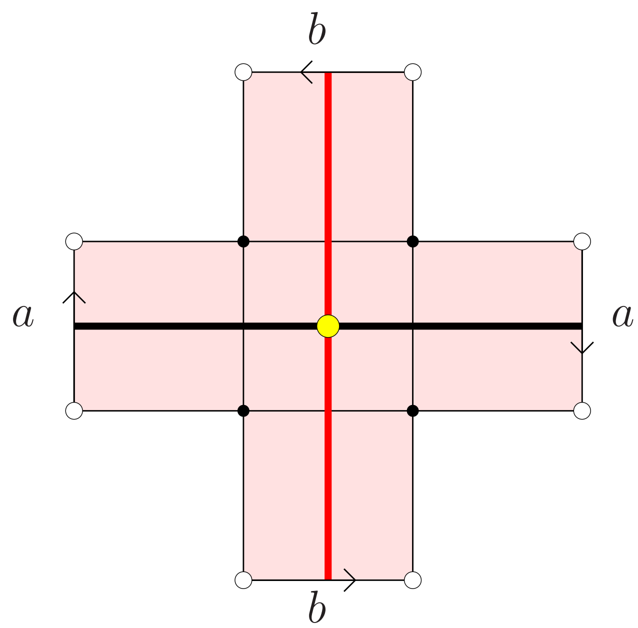
P-RIBBONS

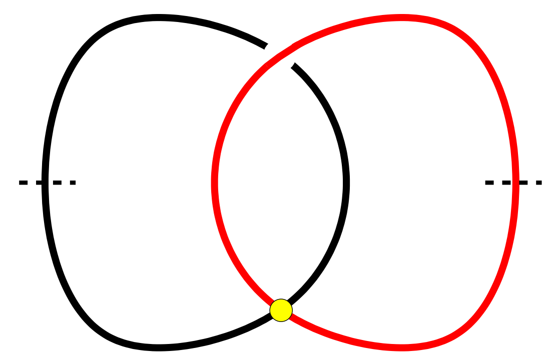




DF 3. *A P-ribbon is a crosscap with a distinguished core circle.*

ARRANGEMENTS OF P-RIBBONS

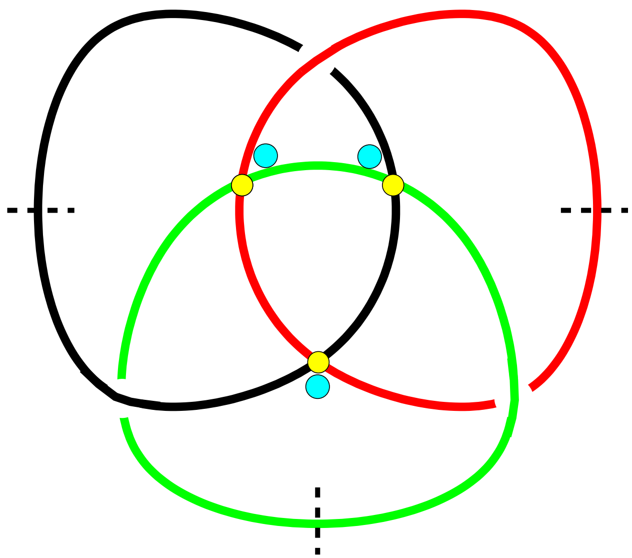




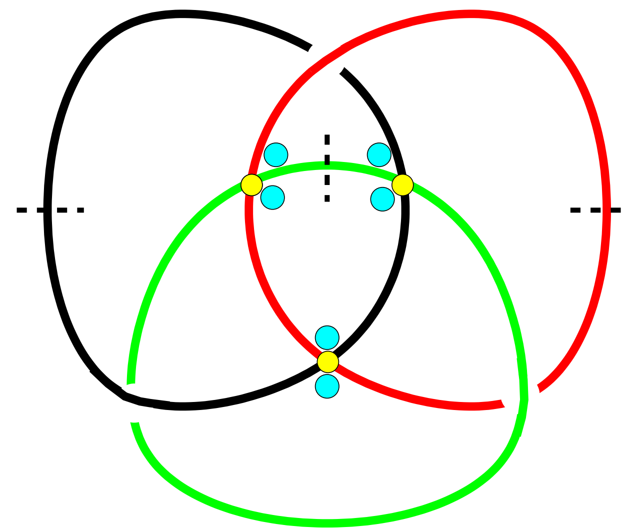
DF 4. *An arrangement of P-ribbons is a finite family of P-ribbons pairwise attached as indicated in the above figure.*

PP 3. *An arrangement of two P-ribbons lives in a sphere with 1 crosscap and 2 boundaries (2 digons).*

ARRANGEMENTS OF 3 P-RIBBONS



4 trigons
genus 1



2 hexagons
genus 3

LR CHARACTERIZATION

TH 4 (Habert and P. 2013). *An arrangement of P-ribbons is of genus 1 if and only if its subarrangements of size 3 are of genus 1.*

TOWARDS ARRANGEMENTS OF DOUBLE PSEUDOHYPERPLANES

TH (HP2006)**.** *The map that assigns to an isomorphism class of indexed arrangements of oriented double pseudolines its chirotope is one-to-one. Furthermore its range is the set of map χ on the set of* 3*-subsets of a finite set I such that for every 3-, 4-, and 5-subset J of I the restriction of χ to the set of* 3*-subsets of J is the chirotope of an indexed arrangement of oriented double pseudolines.*

TOWARDS ARRANGEMENTS OF DOUBLE PSEUDOHYPERPLANES

TH (HP2006)**.** *The map that assigns to an isomorphism class of indexed arrangements of oriented double pseudolines its chirotope is one-to-one. Furthermore its range is the set of map* χ *on the set of* 3*-subsets of a finite set* I *such that for every 3-, 4-, and 5-subset* J *of* I *the restriction of* χ *to the set of* 3*-subsets of* J *is the chirotope of an indexed arrangement of oriented double pseudolines.*

TH (Folkman and Lawrence, 1978). *The map that assigns to an isomorphism class of indexed arrangements of oriented d-dimensional pseudohyperplanes its chirotope is one-to-one. Furthermore its range is the set of map* χ *on the set of d-subsets of a finite set* I *such that for every d-,* $d+1$*-, and* $d+2$*-subset* J *of* I *the restriction of* χ *to the set of d-subsets of* J *is the chirotope of an indexed arrangement of oriented d-dimensional pseudohyperplanes.*

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NOTES