THE INTERSECTION OF A MATROID AND AN ORIENTED MATROID

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Overview

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- Combinatorics; definitions
  - Matroids
  - Oriented matroids

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  - Bárány's 1982 theorem
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  - Bárány's 1982 theorem
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- Topology; proof
  - Meshulam's condition for the existence of colorful simplices
  - Folkman-Lawrence topological representation theorem
  - The nerve theorem

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INDEPENDENCE AXIOMS

A matroid $\mathcal{M}$ on a finite set E is a non-empty collection $\mathcal{I}$ of subsets of E called *independent sets* satisfying,

1. $A \subset B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$.
2. $A, B \in \mathcal{I}$ and $|A| < |B| \Rightarrow \exists b \in B$ such that $A \cup \{b\} \in \mathcal{I}$.

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$$\rho(S) := \max_{A \subset S}\{|A| \ : \ A \in \mathcal{I}\} \qquad \text{(rank function)}$$

$$rk(\mathcal{M}) := \rho(E) \qquad \text{(rank of } \mathcal{M}\text{)}$$

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The independent sets are the linearly independent subsets of E.

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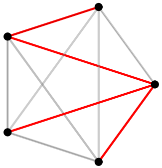
Special case: The *cycle free* subgraphs of a graph.
The maximum independent sets are the spanning forests.



The cycles of the graph are the *circuits* of the matroid,
that is, the minimal dependent sets.

Example 2. Partition a finite set E into non-empty parts

$$E_1 \cup E_2 \cup \cdots \cup E_k$$

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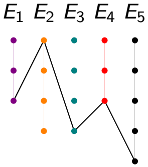
Declare $S \subset E$ to be independent if and only if $|S \cap E_i| \leq 1$ for every i. This gives us a *partition matroid*.



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Denote: $\underline{X} = X^+ \cup X^-$, $-X = (X^-, X^+)$. Assume: $\underline{X} \neq \emptyset$.

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CIRCUIT AXIOMS

An oriented matroid $\mathcal{M}$ on a finite set E is a non-empty collection $\mathcal{C}$ of signed subsets of E called *circuits* satisfying,

1. $X \in \mathcal{C} \Rightarrow -X \in \mathcal{C}$.
2. $X, Y \in \mathcal{C}$ and $\underline{X} \subset \underline{Y} \Rightarrow X = \pm Y$.
3. $X \neq \pm Y \in \mathcal{C}$ and $z \in X^+ \cap Y^- \Rightarrow \exists Z \in \mathcal{C}$ such that $Z^+ \subset X^+ \cup Y^+ \setminus \{z\}$ and $Z^- \subset X^- \cup Y^- \setminus \{z\}$.

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$\{\underline{X}\}_{X \in \mathcal{C}}$ are the circuits of the *underlying matroid*, $\underline{\mathcal{M}}$,

$rk(\mathcal{M}) := rk(\underline{\mathcal{M}})$, $X = (X^+, \emptyset) \in \mathcal{C}$ is called a **positive circuit**.

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The edges are the ground set E.
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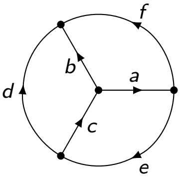

$(\{a, f\}, \{b\})$
$(\{b, c\}, \{d\})$
$(\{a, e, c\}, \emptyset)$
$(\{a, f, c\}, \{d\})$
$(\{b\}, \{a, d, e\})$
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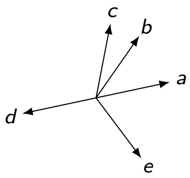

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The positive circuits are the minimal sets which contain the origin in their *convex hull*.

Carathéodory's theorem states that given a set $P \subset \mathbb{R}^d$ and a point $x \in \operatorname{conv} P$, there exists a subset $Q \subset P$ such that $|Q| \leq d+1$ and $x \in \operatorname{conv} Q$.

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In 1982 Bárány gave the following generalization.

COLORFUL CARATHÉODORY THEOREM

Let $P_1, \ldots, P_{d+1}$ be point sets in $\mathbb{R}^d$. If $x \in \bigcap_{i=1}^{d+1} \operatorname{conv} P_i$, then there exists $p_1 \in P_1$, ..., $p_{d+1} \in P_{d+1}$ such that $x \in \operatorname{conv}\{p_1, \ldots, p_{d+1}\}$.

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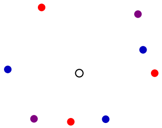

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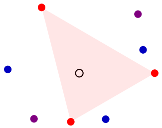

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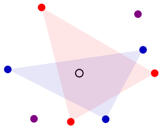

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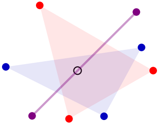

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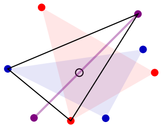

Around 2007 several people discovered a weakening of the hypothesis in Bárány's theorem, resulting in the following.

STRONG COLORFUL CARATHÉODORY THEOREM

Let $P_1, \ldots, P_{d+1}$ be point sets in $\mathbb{R}^d$. If $x \in \bigcap_{1 \le i < j}^{d+1} \operatorname{conv}(P_i \cup P_j)$, then there exists $p_1 \in P_1$, ..., $p_{d+1} \in P_{d+1}$ such that $x \in \operatorname{conv}\{p_1, \ldots, p_{d+1}\}$.

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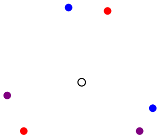

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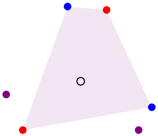

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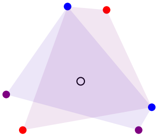

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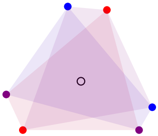

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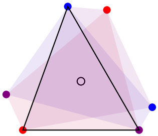

THEOREM

Let $\mathcal{M}$ be an oriented matroid, $\mathcal{N}$ a matroid with rank function ρ, both on the same ground set E and satisfying $rk(\mathcal{M}) < rk(\mathcal{N})$.
If every $A \subset E$ with $\rho(E - A) < rk(\mathcal{M})$ contains a positive circuit of $\mathcal{M}$, then $\mathcal{M}$ contains a positive circuit which is independent in $\mathcal{N}$.

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A special case is the "strong colorful Carathéodory theorem": Take $\mathcal{M}$ to be the oriented matroid of the vector configuration $V = P_1 \cup \cdots \cup P_{d+1}$ (rank d) and $\mathcal{N}$ is the partition matroid induced by the color classes (rank $d + 1$).

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Another special case is when the edges of K_m, with $m \geq 4$, are identified with points in $\mathbb{R}^{m-2}$ and the complement of every clique of size $m-2$ contains the origin. Then there exists a spanning tree corresponding to a simplex which contains the origin.

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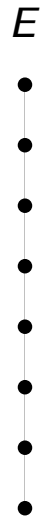
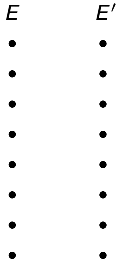
Dual support complex

$$X^{\star}_{\mathcal{M}} = \{U \subset E \;:\; U \text{ disjoint from some positive circuit of } \mathcal{M}\}$$


E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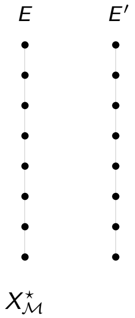
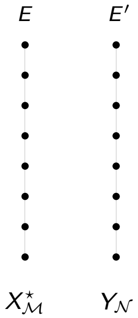

E
E'
$X^{\star}_{\mathcal{M}}$


E
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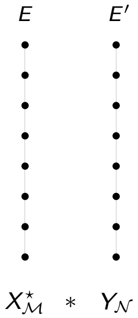
E
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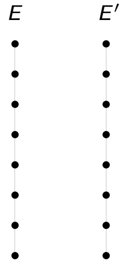
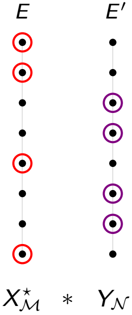
$X^{\star}_{\mathcal{M}} * Y_{\mathcal{N}}$



$$X^{\star}_{\mathcal{M}} \quad * \quad Y_{\mathcal{N}}$$

Goal: Find a *spanning simplex* in the join



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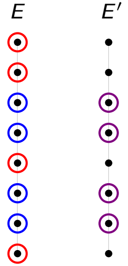

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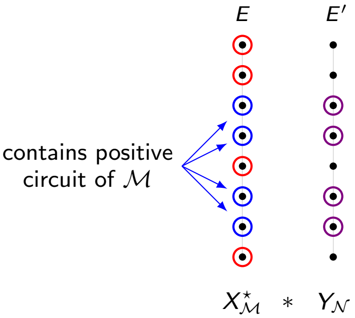

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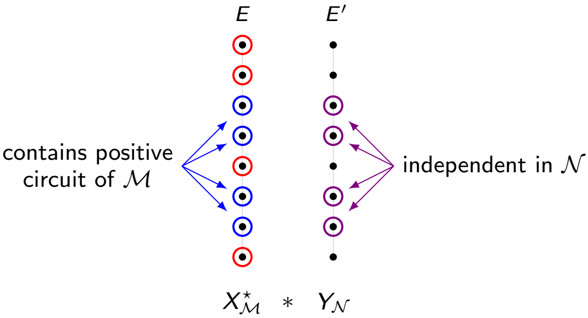

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Let K be a simplicial complex on vertex set $V = V_1 \cup \cdots \cup V_m$
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MESHULAM'S CONDITION

If for every $1 \leq k \leq m$ *and* $1 \leq i_1 < \cdots < i_k \leq m$,

$$\eta(K\,[V_{i_1} \cup \cdots \cup V_{i_k}]) \geq k,$$

then K *contains a colorful simplex.*

$K[V']$: induced subcomplex on $V' \subset V$

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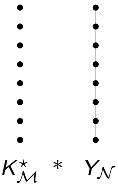

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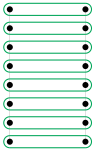

$K^\star_{\mathcal{M}} \; * \; Y_{\mathcal{N}}$

For every $\emptyset \neq S \subset E$, we must show

$$\eta(X^{\star}_{\mathcal{M}}[S]) + \eta(Y_{\mathcal{N}}[S]) \geq |S|$$

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If $S \notin X^{\star}_{\mathcal{M}}$, then $T = E \setminus S \in X_{\mathcal{M}} \Rightarrow \widetilde{H}_i(X^{\star}_{\mathcal{M}}[S]) \cong \widetilde{H}_{|S|-i-3}(lk(T, X_{\mathcal{M}}))$

(Last implication uses the fact that $E \notin X_{\mathcal{M}}$ and Alexander duality.)

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New goal: Compute $\widetilde{H}_j(lk(T, X_{\mathcal{M}}))$

Fact: If $rk(\mathcal{M}) = r$, then

$$\widetilde{H}_j(X_{\mathcal{M}}) = 0 \text{ for all } j \geq r$$

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Follows from the Folkman-Lawrence representation theorem and the Nerve theorem.

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$$\eta(X^\star_{\mathcal{M}}[S]) + \eta(Y_{\mathcal{N}}[S]) \geq |S| - r + \rho(S) \geq |S|$$

$$(S \notin X^\star_{\mathcal{M}} \Rightarrow E \setminus S \in X_{\mathcal{M}} \Rightarrow \rho(S) \geq r)$$

Thank you for your attention!