Three Ways to Cover a Graph

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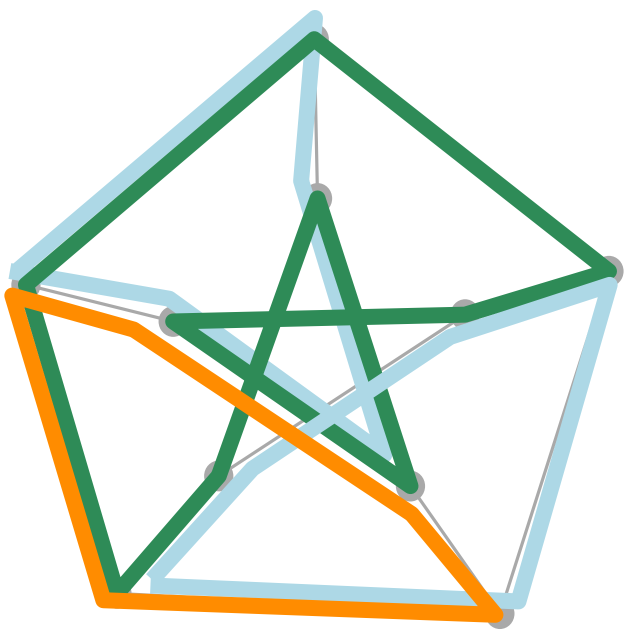
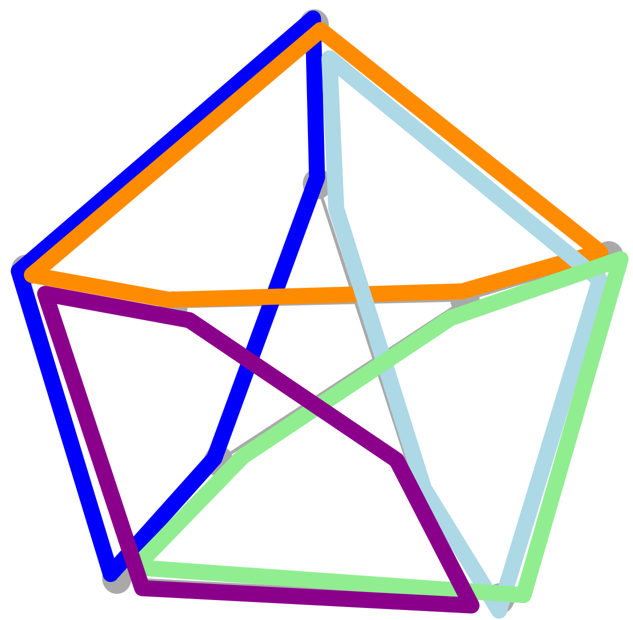
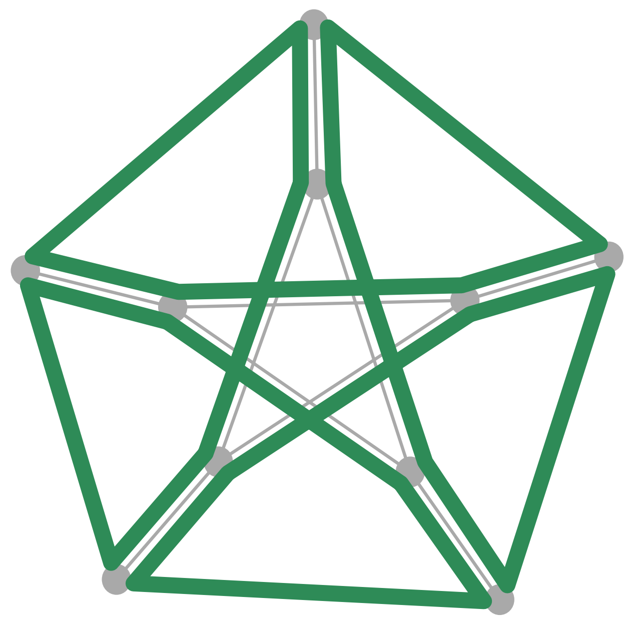






GT Combi du LIX, June 3, 2013

Interval graphs

Intersection graphs of intervals

every v represented by an interval
graph edges $\Leftrightarrow$ interval intersections

- classical graph class
- efficient recognition
- chordal & perfect
- many applications

Intersection graphs of systems of intervals

every v represented by $\leq k$ intervals
graph edges $\Leftrightarrow$ interval intersections

Intersection graphs of systems of intervals

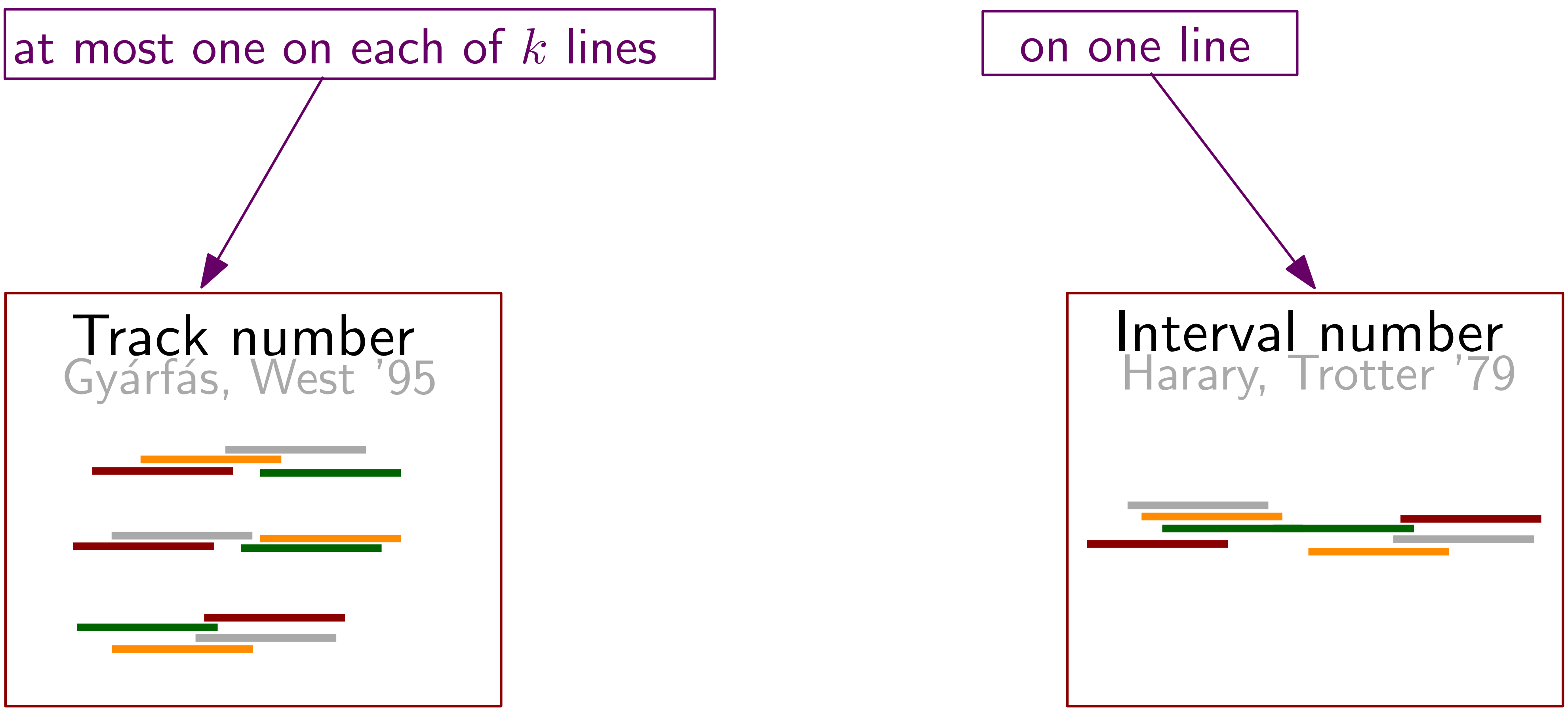
every v represented by $\leq k$ intervals
graph edges $\Leftrightarrow$ interval intersections

on one line

Interval number
Harary, Trotter '79

Intersection graphs of systems of intervals

every v represented by $\leq k$ intervals
graph edges $\Leftrightarrow$ interval intersections



Intersection graphs of systems of intervals

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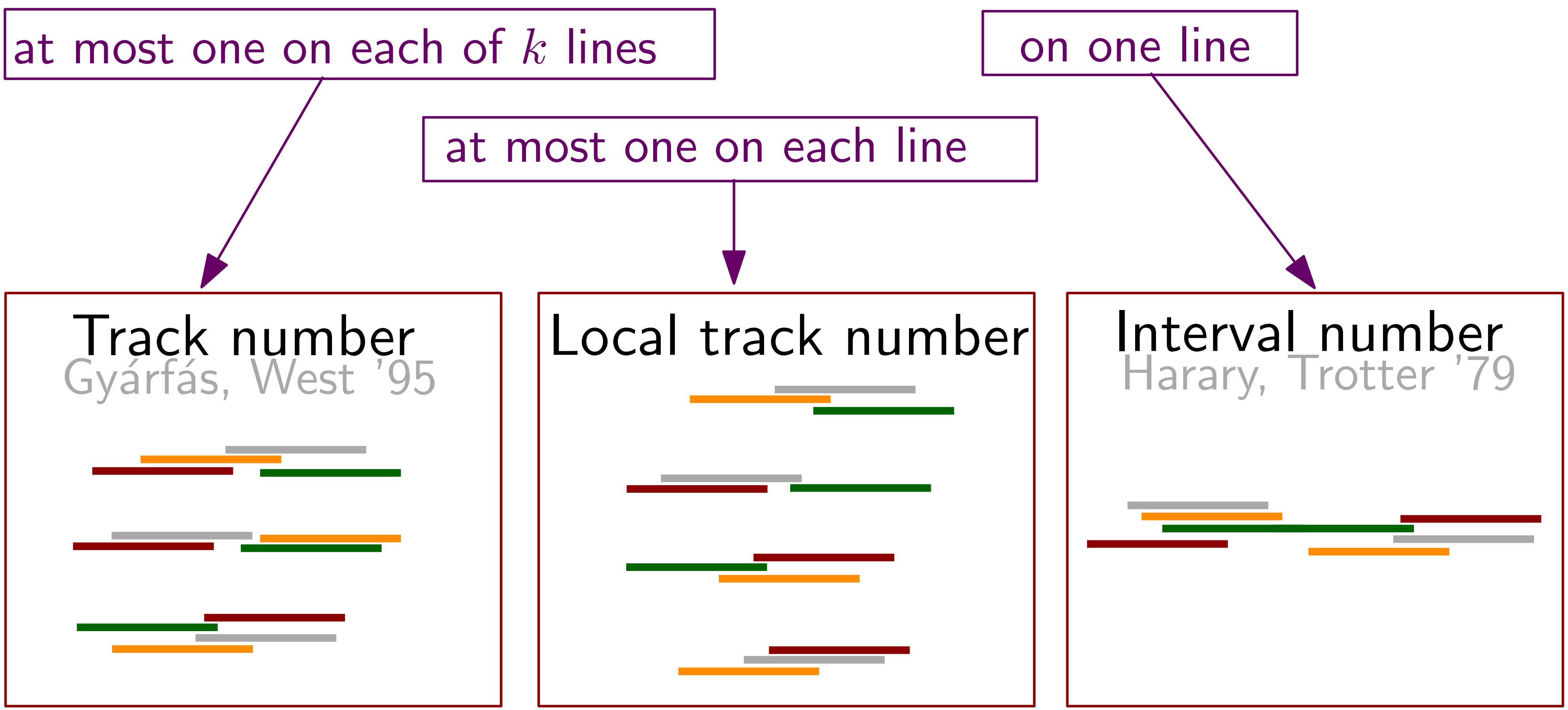


Some Results

	track nr.	local track nr.	interval nr.
outerplanar	2	2	2
bip. planar	4	3	3
planar	4	$?$	3
$\text{tw} \leq k$	$k+1$	k	k
$\text{dg} \leq k$	$2k$	$k+1$	$k+1$

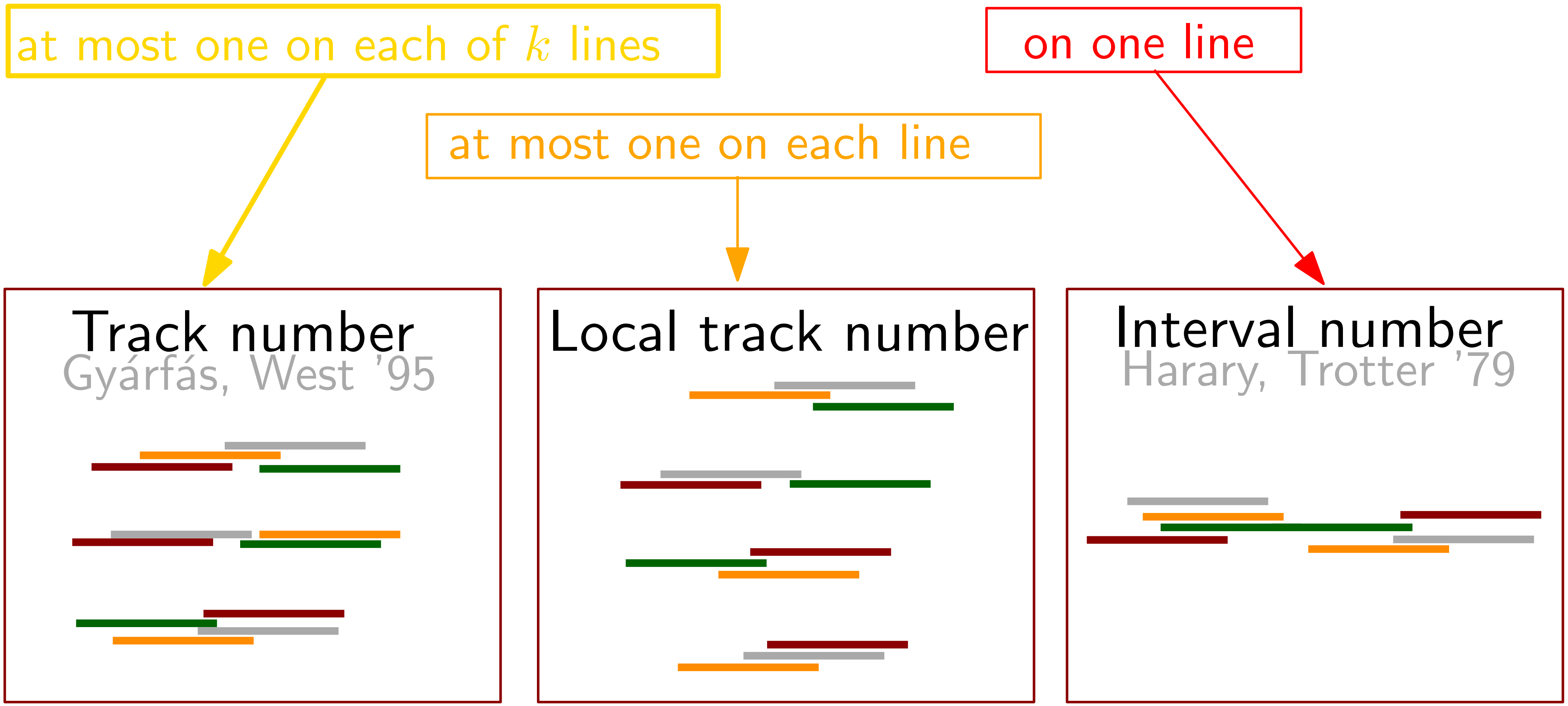
Kostochka, West '99

Scheinermann, West '83

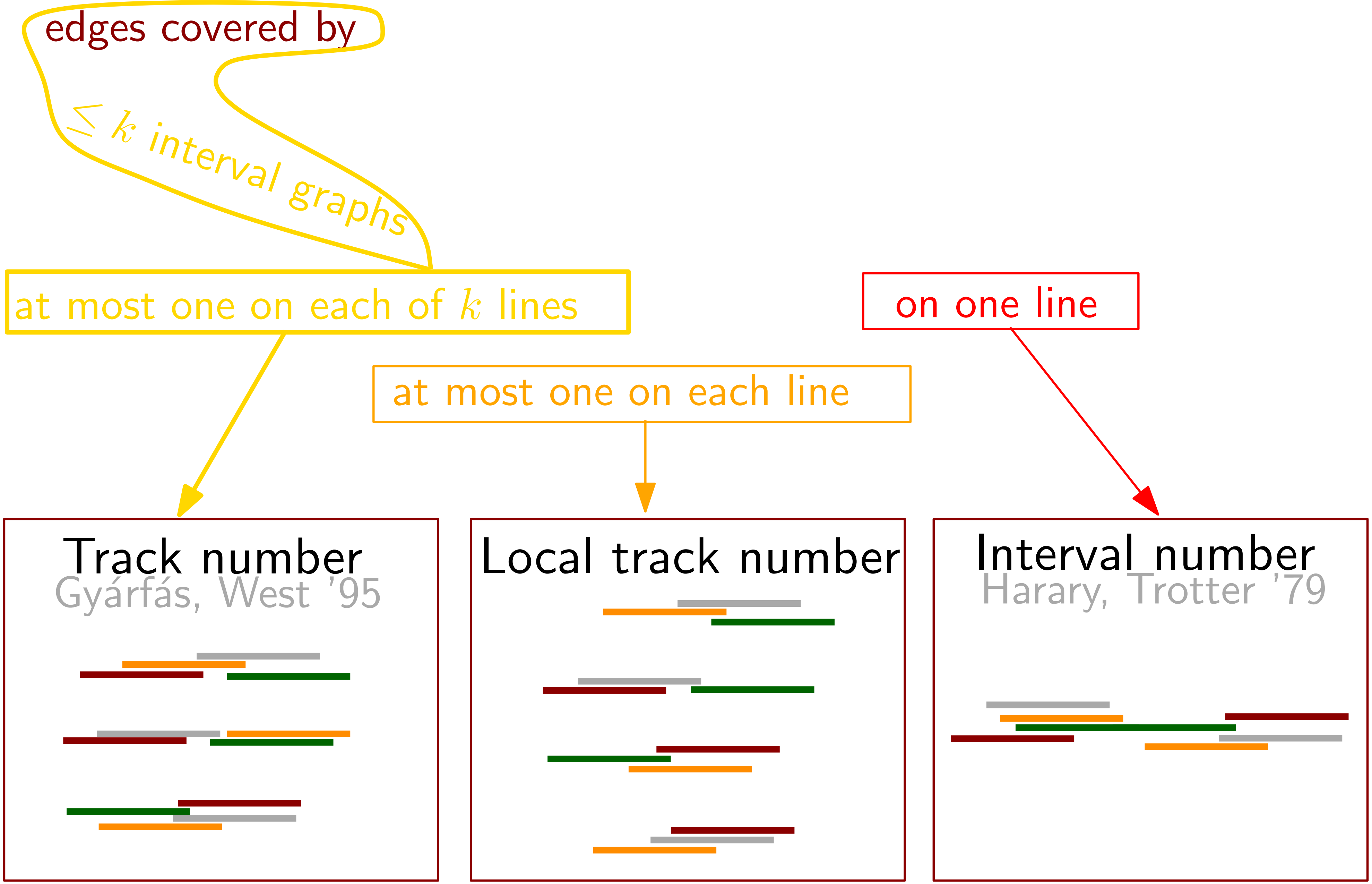
Gonçalves, Ochem '09

KU '12

Intersection graphs of systems of intervals



Intersection graphs of systems of intervals



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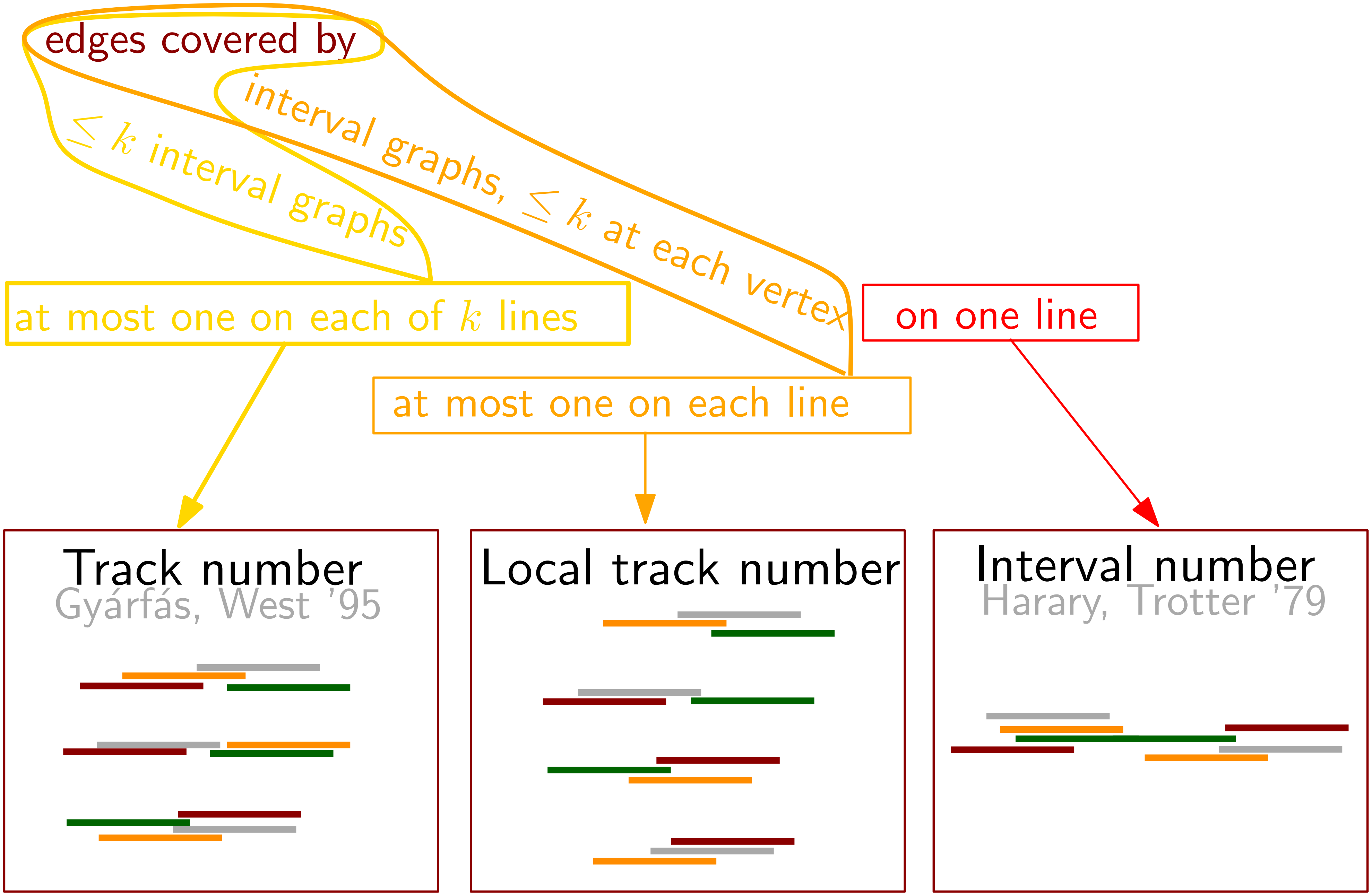
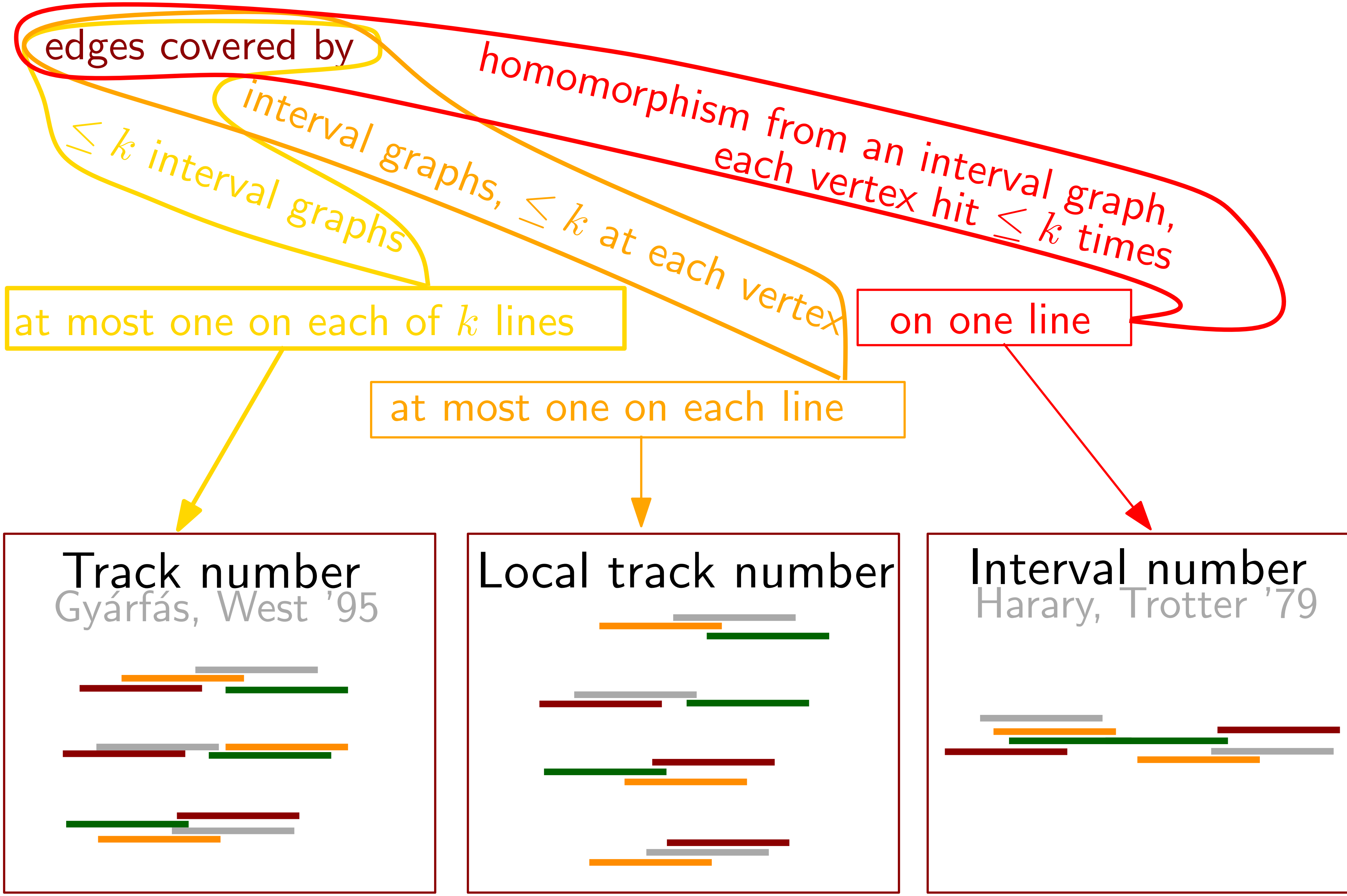


Intersection graphs of systems of intervals



- Global, Local, and Folded Covers
  - Templates = Interval Graphs
- Formal Definitions
- Local and Folded Linear Arboricity
  - Templates = Collections of Paths
- Interrelations
  - Templates = Forests, Pseudo-Forests, Star Forests
- What is known and what is open

More Formally

φ **cover**	$\leftrightsquigarrow$	$\varphi : T_1 \sqcup \cdots \sqcup T_k \to G$ edge-surjective homomorphism
φ **injective**	$\leftrightsquigarrow$	φ restricted to each T_i injective
size of φ	$\leftrightsquigarrow$	$\#$ template graphs in preimage

More Formally

φ cover $\leftrightsquigarrow$ $\varphi : T_1 \sqcup \cdots \sqcup T_k \to G$ edge-surjective homomorphism

φ injective $\leftrightsquigarrow$ φ restricted to each T_i injective

size of φ $\leftrightsquigarrow$ # template graphs in preimage

global

$c_g^{\mathcal{T}}(G) = \min\{\text{size of } \varphi : \varphi \text{ injective cover of } G\}$

local

$c_\ell^{\mathcal{T}}(G) = \min\{\max_{v \in V(G)} |\varphi^{-1}(v)| : \varphi \text{ injective cover of } G\}$

folded

$c_f^{\mathcal{T}}(G) = \min\{\max_{v \in V(G)} |\varphi^{-1}(v)| : \varphi \text{ cover of } G \text{ of size } 1\}$

Basic Properties

We consider template classes that are closed under disjoint union.

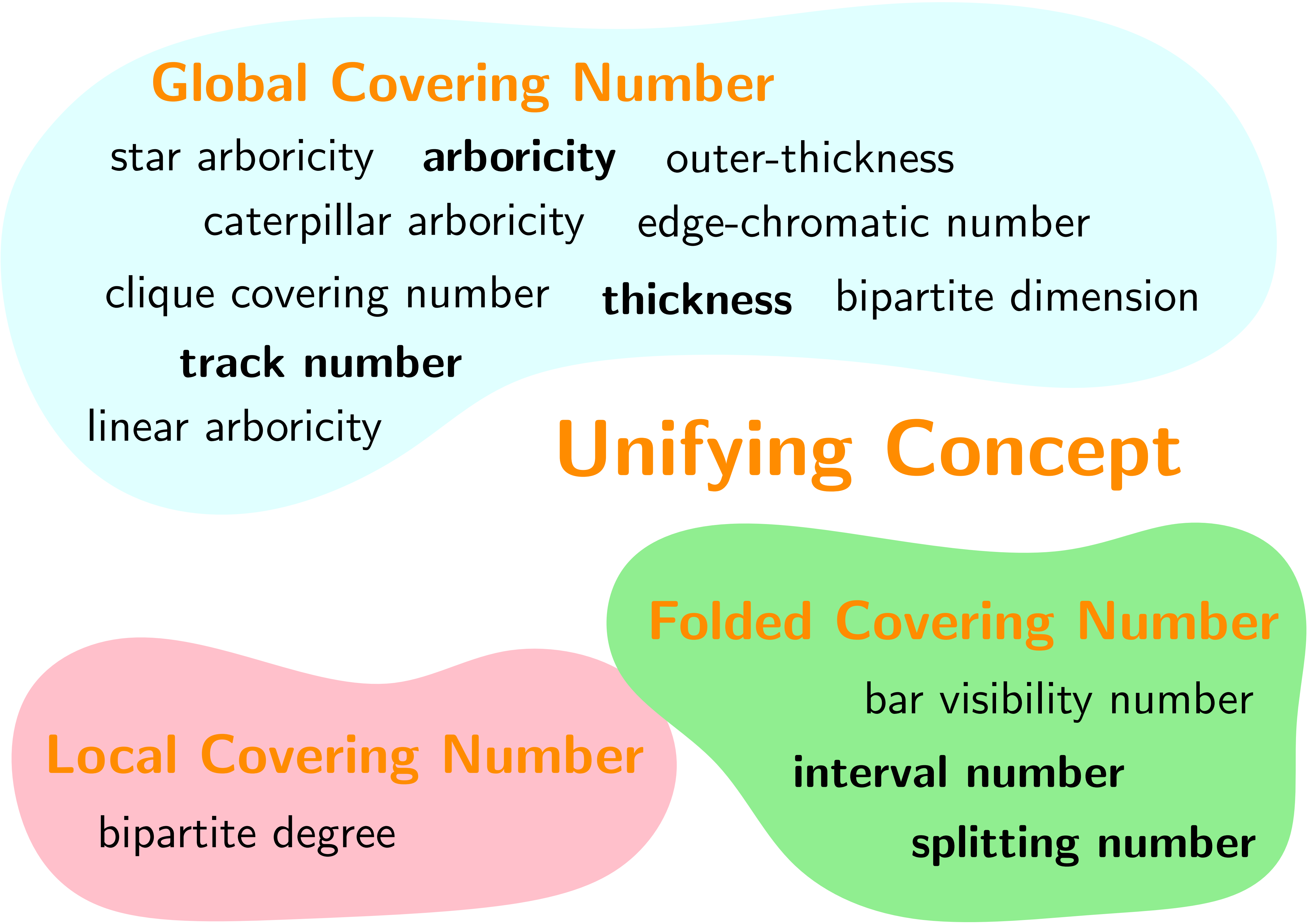
Lemma:

1) $c_g^{\mathcal{T}}(G) \geq c_\ell^{\mathcal{T}}(G) \geq c_f^{\mathcal{T}}(G)$ for every G

define $c_i^{\mathcal{T}}(\mathcal{G}) := \sup\{c_i^{\mathcal{T}}(G) : G \in \mathcal{G}\}$ ($\mathcal{G}$ graph class)

2) $c_i^{\mathcal{T}}(\mathcal{G}) \leq c_i^{\mathcal{T}}(\mathcal{G}')$ $\mathcal{G} \subseteq \mathcal{G}'$

3) $c_i^{\mathcal{T}}(\mathcal{G}) \geq c_i^{\mathcal{T}'}(\mathcal{G})$ $\mathcal{T} \subseteq \mathcal{T}'$


Global Covering Number
star arboricity
arboricity
outer-thickness
caterpillar arboricity
edge-chromatic number
clique covering number
thickness
bipartite dimension
track number
linear arboricity
Unifying Concept
Folded Covering Number
bar visibility number
interval number
splitting number
Local Covering Number
bipartite degree

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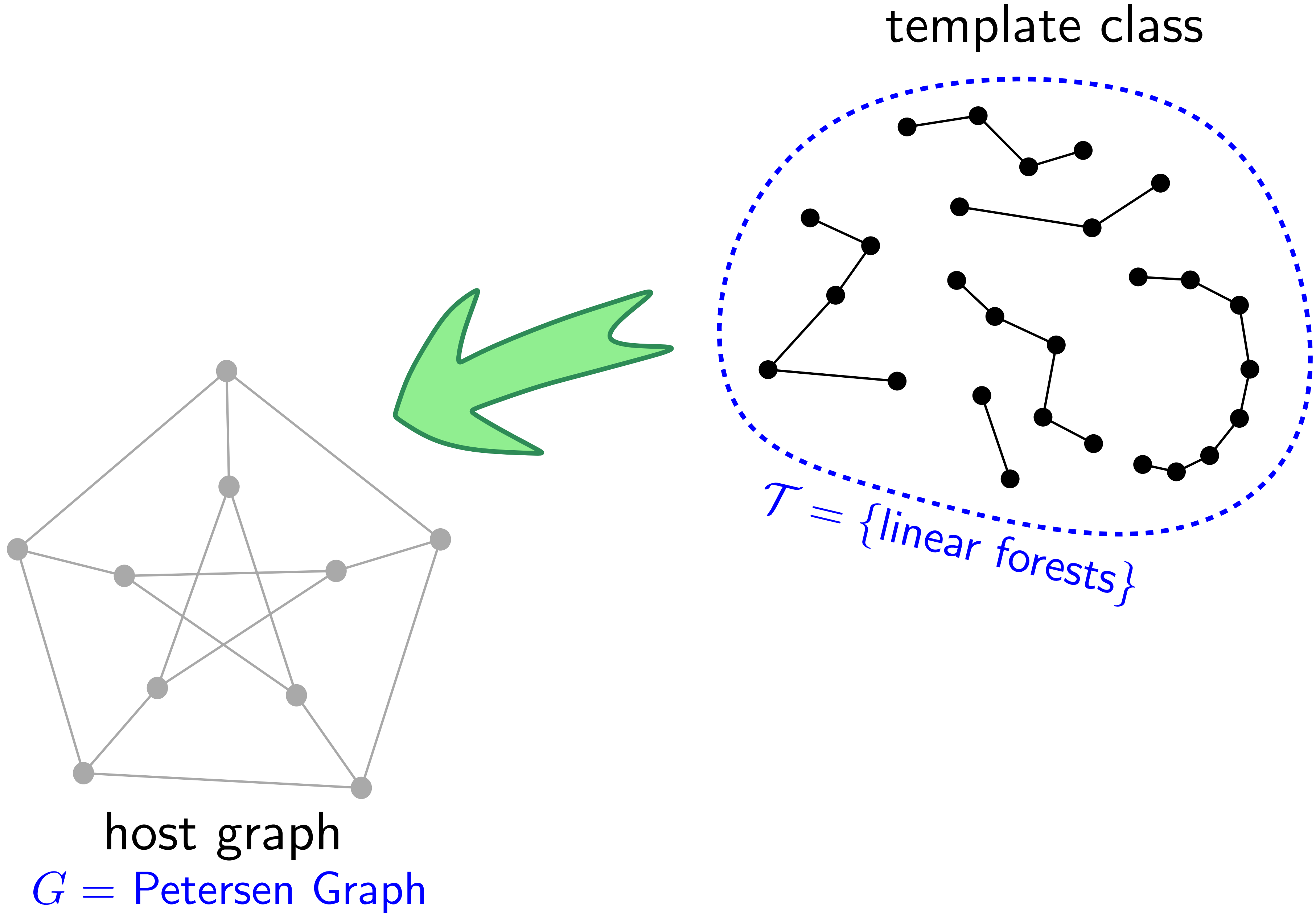
Global and Local Linear Arboricity



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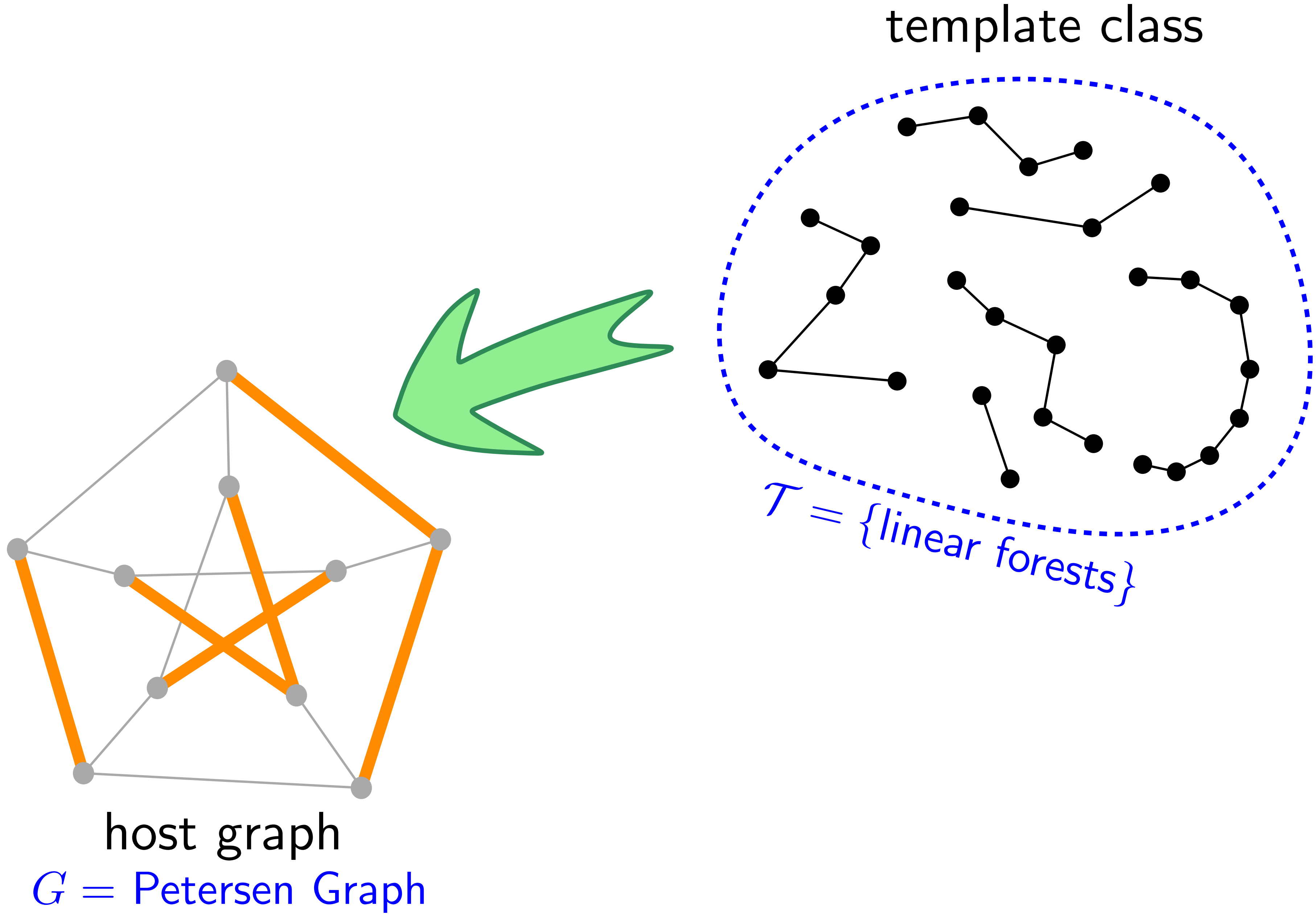
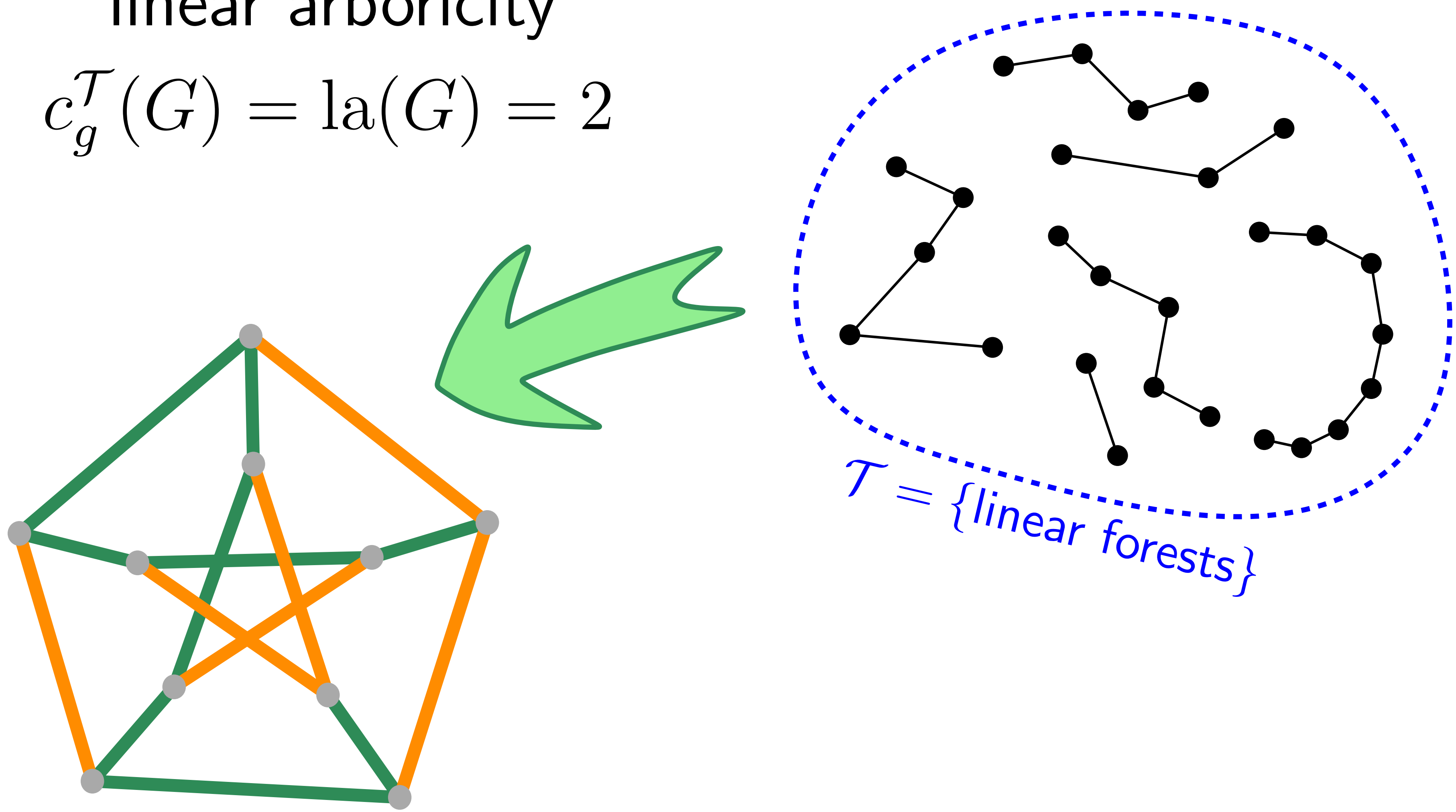


Global and Local Linear Arboricity

template class

linear arboricity

$$c_g^{\mathcal{T}}(G) = \mathrm{la}(G) = 2$$



host graph

G = Petersen Graph

Global and Local Linear Arboricity

template class

linear arboricity

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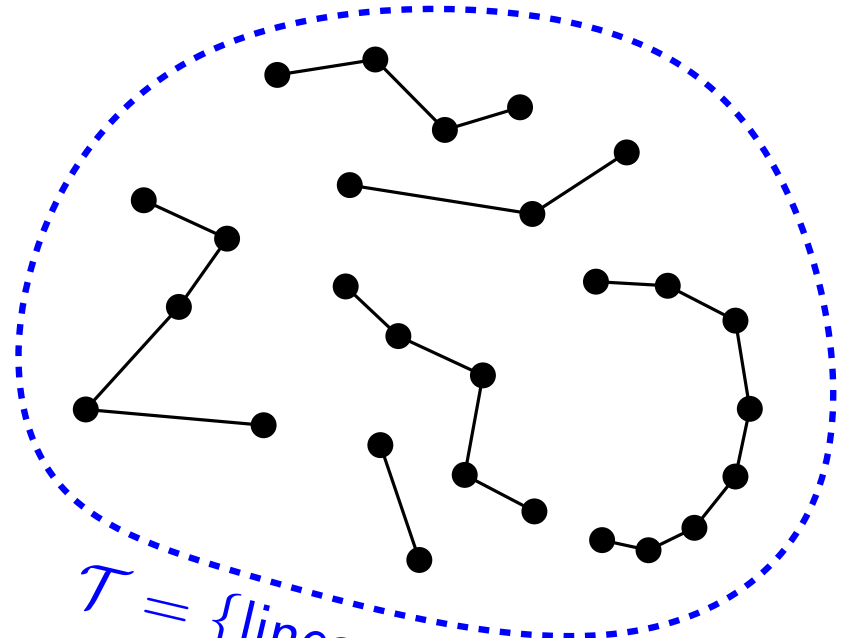
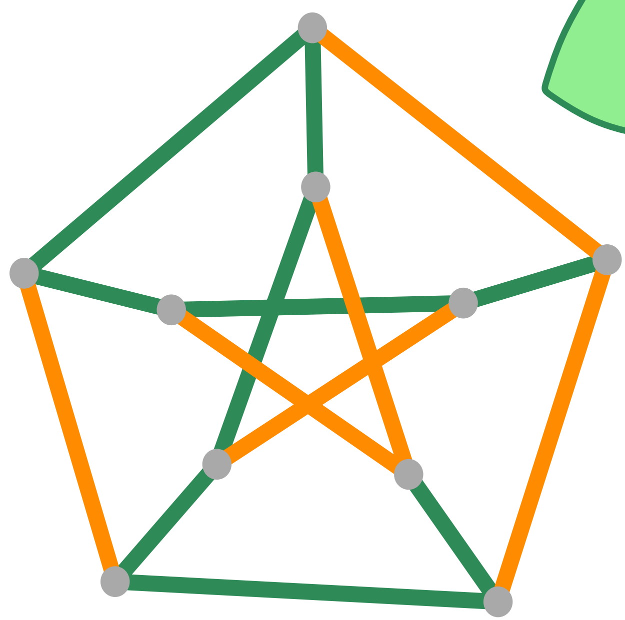


$\mathcal{T} = \{$linear forests$\}$



host graph

G = Petersen Graph

Akiyama et. al. '80

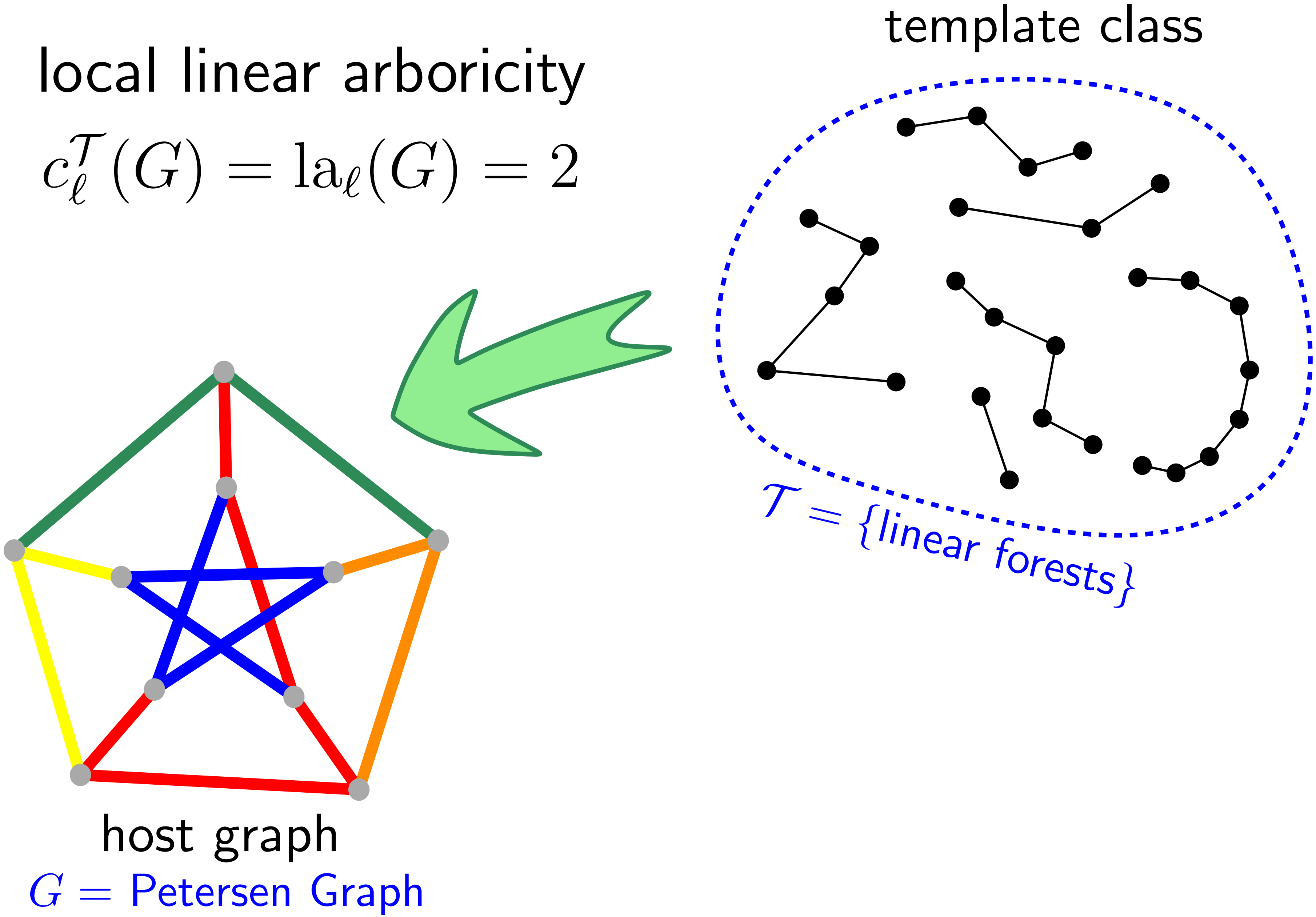
Linear Arboricity Conjecture

$$\mathrm{la}(G) \leq \lceil \frac{\Delta+1}{2} \rceil$$

Global and Local Linear Arboricity

local linear arboricity

$$c_\ell^{\mathcal{T}}(G) = \mathrm{la}_\ell(G) = 2$$



Global and Local Linear Arboricity

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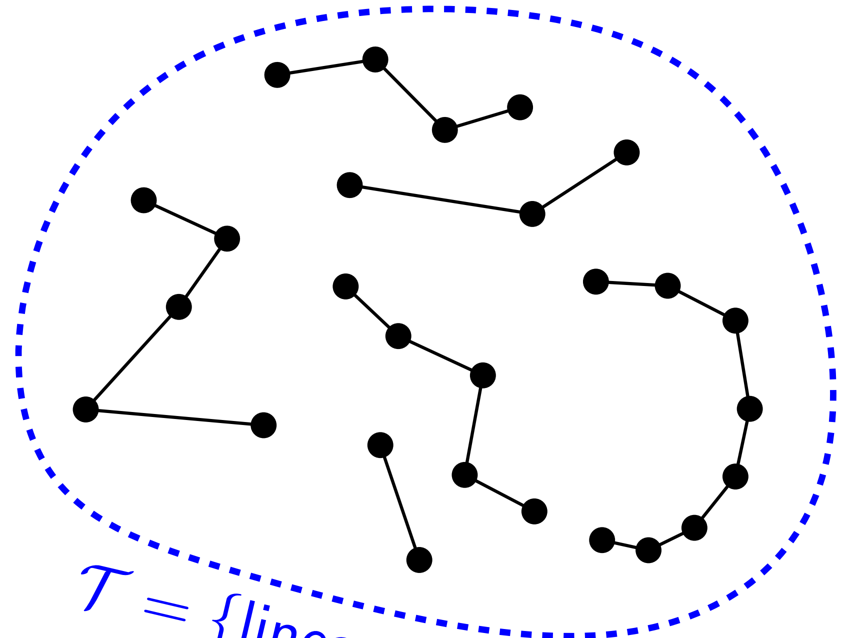

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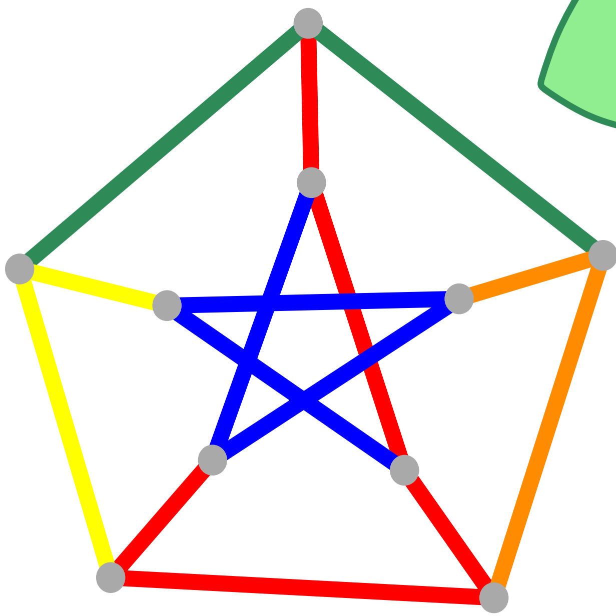


host graph

G = Petersen Graph

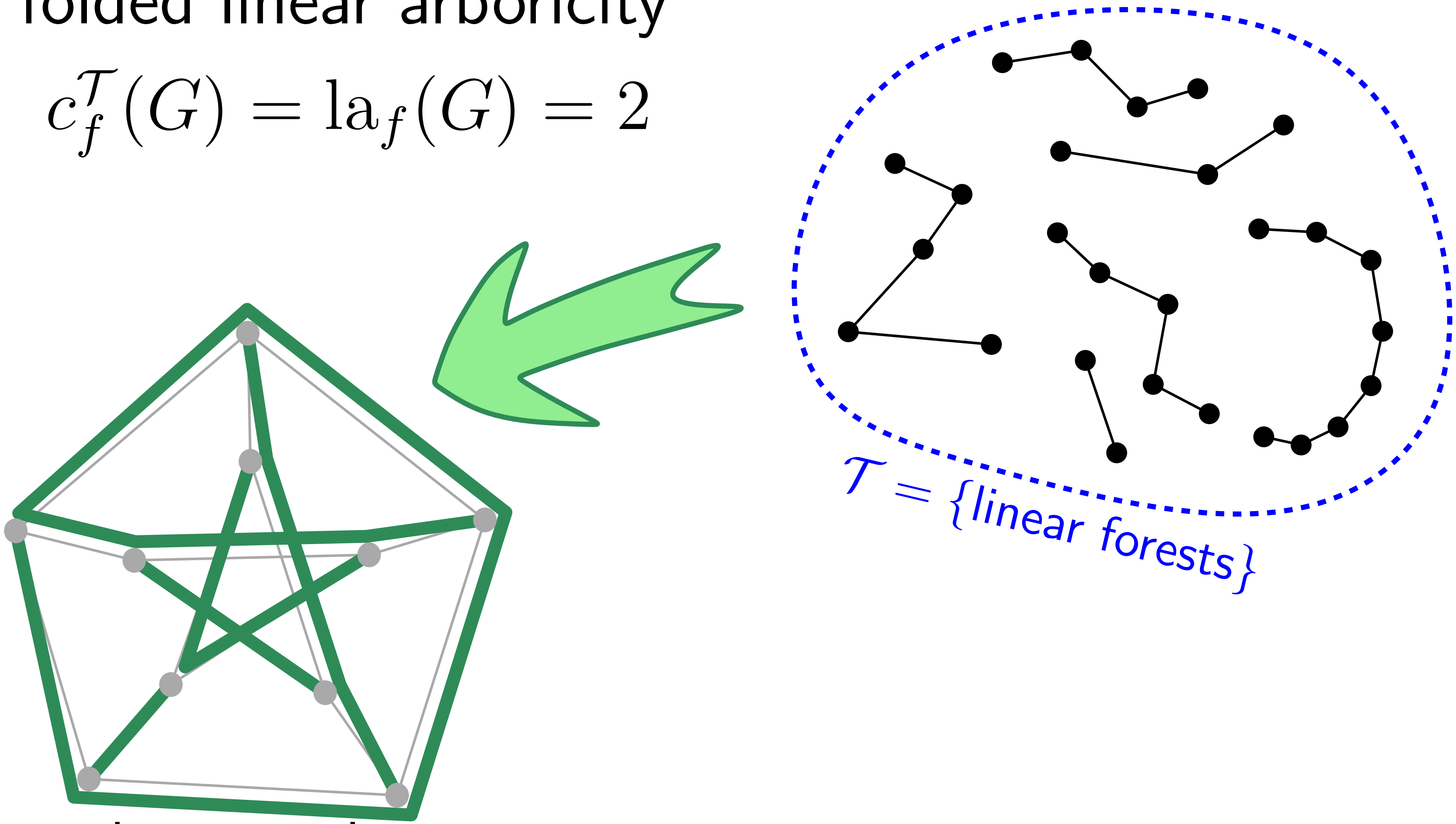
Local Linear Arboricity Conjecture

$$\mathrm{la}_\ell(G) \leq \lceil \frac{\Delta+1}{2} \rceil$$

Folded Linear Arboricity

folded linear arboricity

$$c_f^{\mathcal{T}}(G) = \mathrm{la}_f(G) = 2$$



Folded Linear Arboricity

folded linear arboricity

$$c_f^{\mathcal{T}}(G) = \mathrm{la}_f(G) = 2$$

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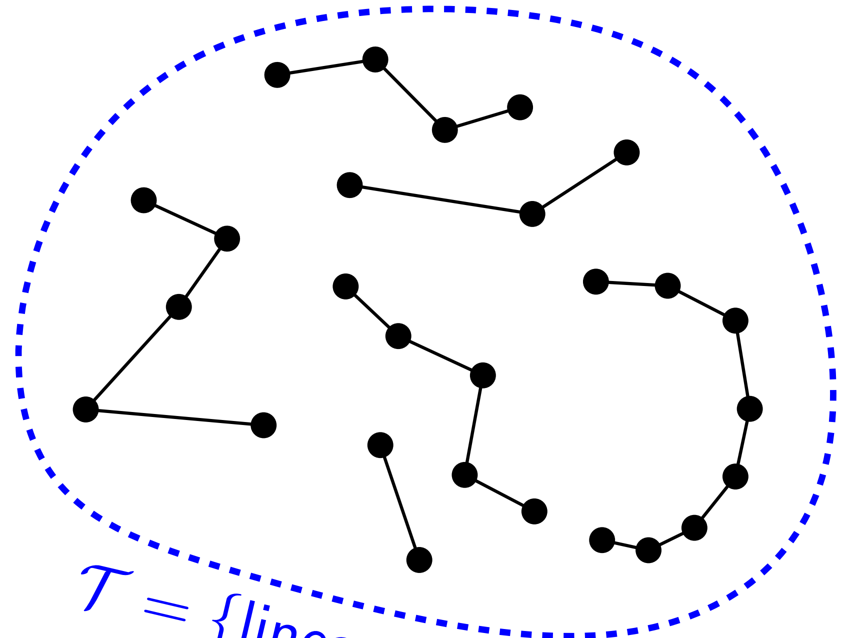
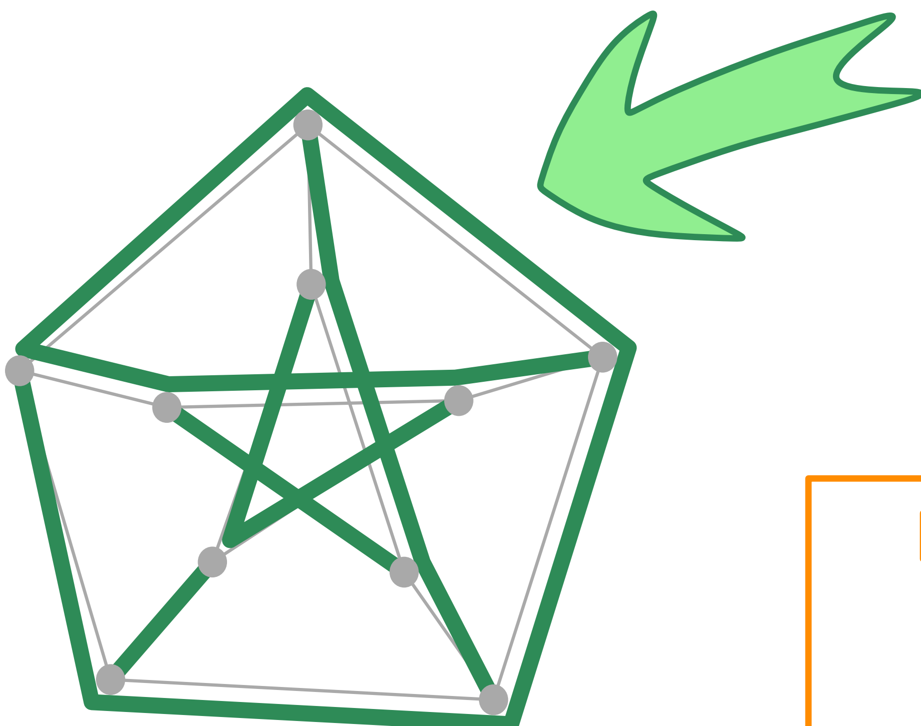


$\mathcal{T} = \{\text{linear forests}\}$



host graph

G = Petersen Graph

Folded Linear Arboricity Theorem[KU]

$$\mathrm{la}_f(G) \leq \lceil \frac{\Delta+1}{2} \rceil$$

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Folded Linear Arboricity Theorem[KU]

$$\mathrm{la}_f(G) \leq \lceil \frac{\Delta+1}{2} \rceil$$

Proof: *(easy)*

Δ **even**:

- add vertices and edges to obtain Eulerian
- take Eulertour
- all visited $\leq \frac{\Delta}{2}$ times
- start-vertex once more
- $1 + \frac{\Delta}{2} = \lceil \frac{\Delta+1}{2} \rceil$

Folded Linear Arboricity Theorem[KU]

$$\mathrm{la}_f(G) \leq \lceil \frac{\Delta+1}{2} \rceil$$

Proof: *(easy)*

Δ **even**:

- add vertices and edges to obtain Eulerian
- take Eulertour
- all visited $\leq \frac{\Delta}{2}$ times
- start-vertex once more
- $1 + \frac{\Delta}{2} = \lceil \frac{\Delta+1}{2} \rceil$

Δ **odd**:

- add vertices and edges to obtain Eulerian
- take Eulertour
- all visited $\leq \frac{\Delta+1}{2}$ times
- start-vertex once more
- start on added vertex
- $\lceil \frac{\Delta+1}{2} \rceil$

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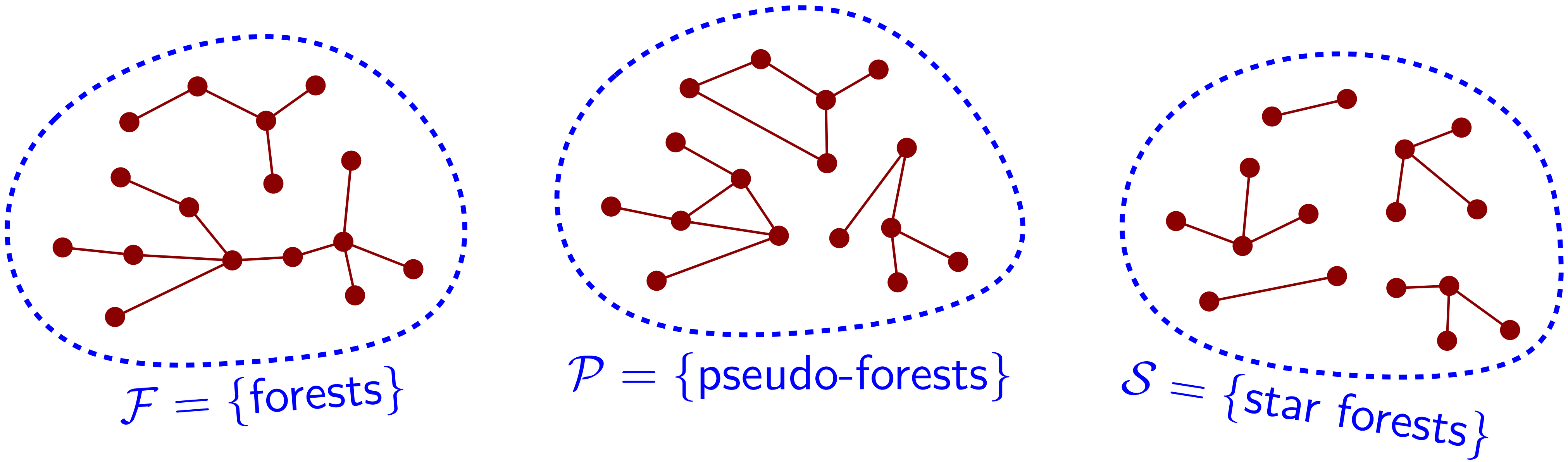
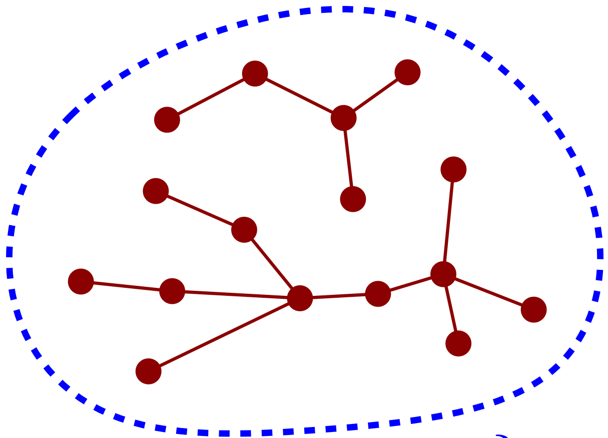


$\mathcal{F}$ = {forests}
$\mathcal{P}$ = {pseudo-forests}
$\mathcal{S}$ = {star forests}



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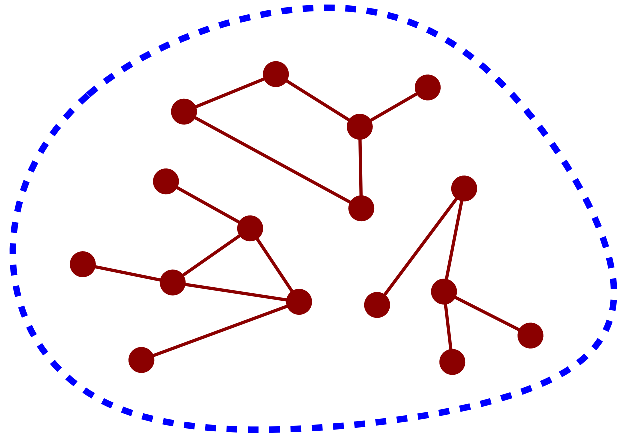

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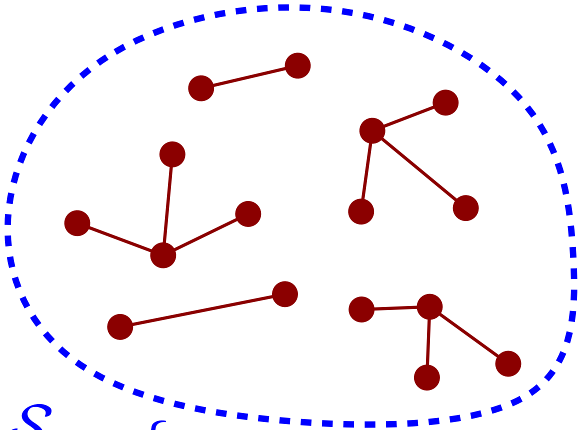


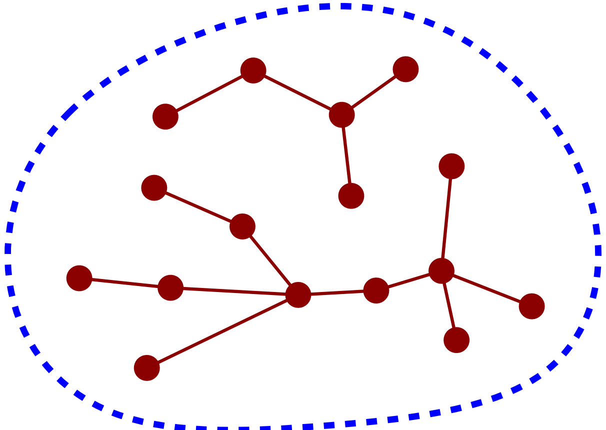
$\mathcal{S} = \{\text{star forests}\}$

Arboricity

$$c_g^{\mathcal{F}}(G) = a(G)$$

[Nash-Williams '64]

$$a(G) = \max_{S \subseteq V(G)} \left\lceil \frac{|E[S]|}{|S| - 1} \right\rceil$$



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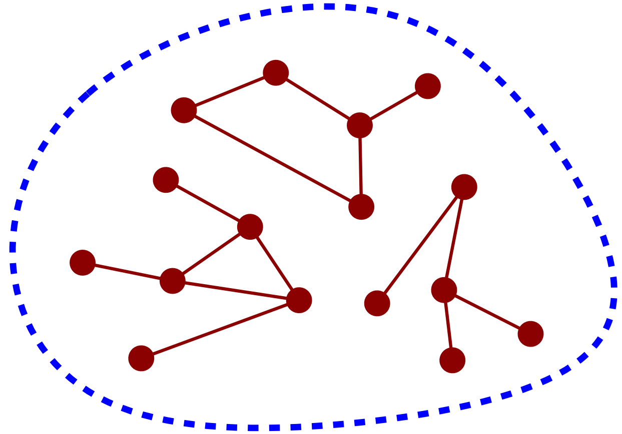

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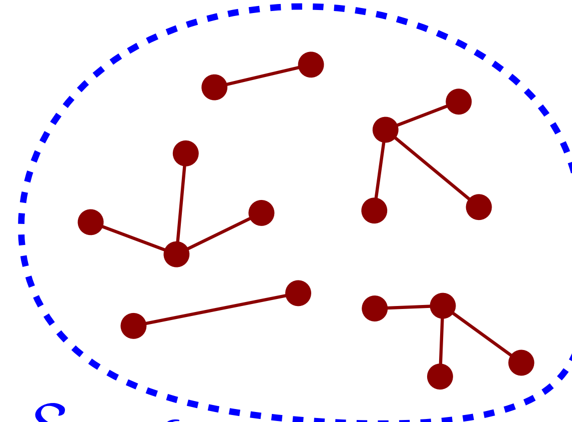

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Arboricity

$c_g^{\mathcal{F}}(G) = a(G)$

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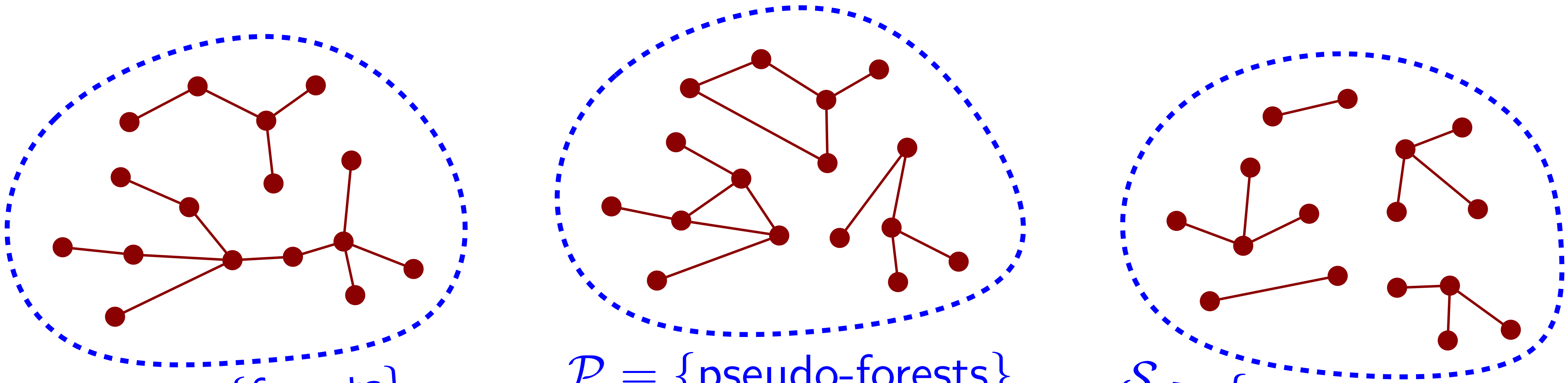
Pseudo-Arboricity

$c_g^{\mathcal{P}}(G) = p(G)$

[Picard et al. '82]

$$p(G) = \max_{S \subseteq V(G)} \left\lceil \frac{|E[S]|}{|S|} \right\rceil$$



$\mathcal{F} = \{\text{forests}\}$ $\mathcal{P} = \{\text{pseudo-forests}\}$ $\mathcal{S} = \{\text{star forests}\}$

Arboricity

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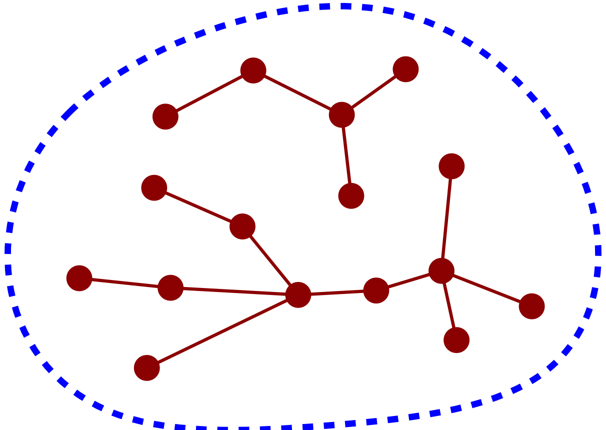
$$p(G) \leq a(G) \leq p(G) + 1$$



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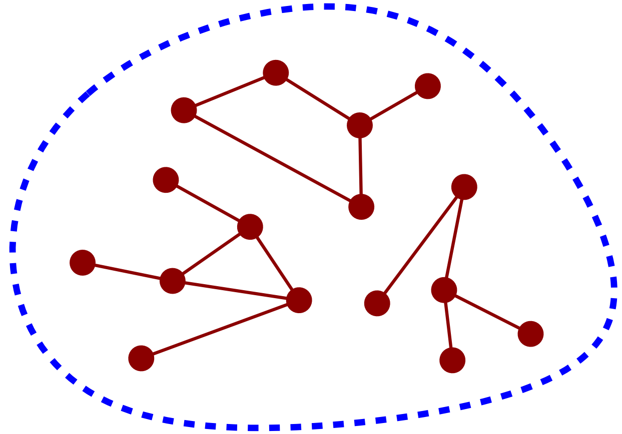

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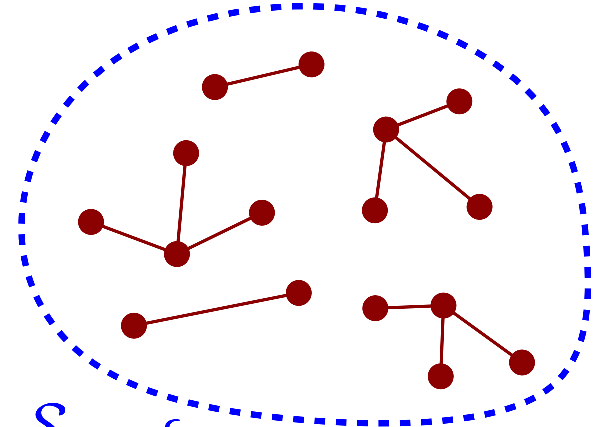

$\mathcal{S} = \{\text{star forests}\}$

Arboricity

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Pseudo-Arboricity

$c_g^{\mathcal{P}}(G) = p(G)$

Star Arboricity

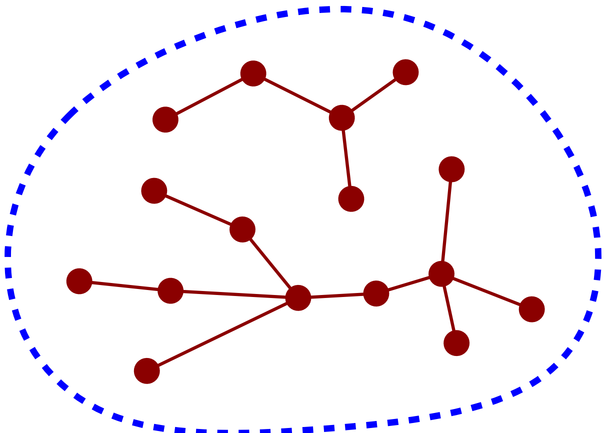
$c_g^{\mathcal{S}}(G) = \mathrm{sa}(G)$

[Nash-Williams '64]

[Picard et al. '82]

$$a(G) = \max_{S \subseteq V(G)} \left\lceil \frac{|E[S]|}{|S| - 1} \right\rceil \quad p(G) = \max_{S \subseteq V(G)} \left\lceil \frac{|E[S]|}{|S|} \right\rceil$$

$$p(G) \leq a(G) \leq p(G) + 1$$



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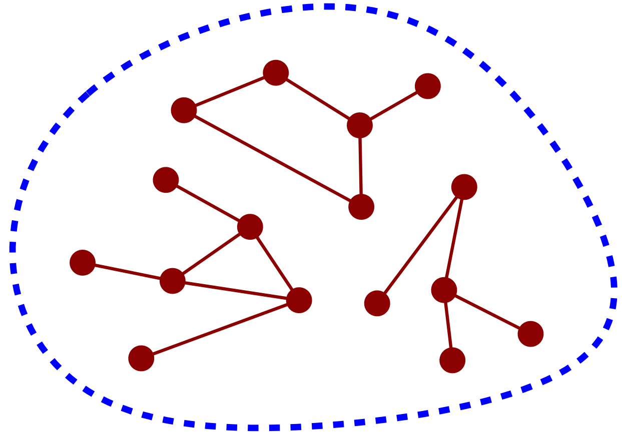

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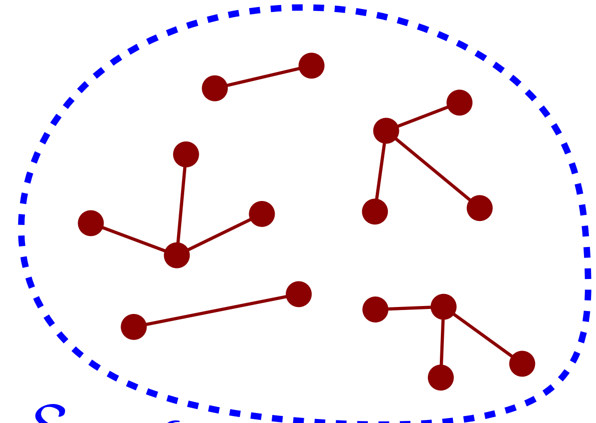

$\mathcal{S} = \{\text{star forests}\}$

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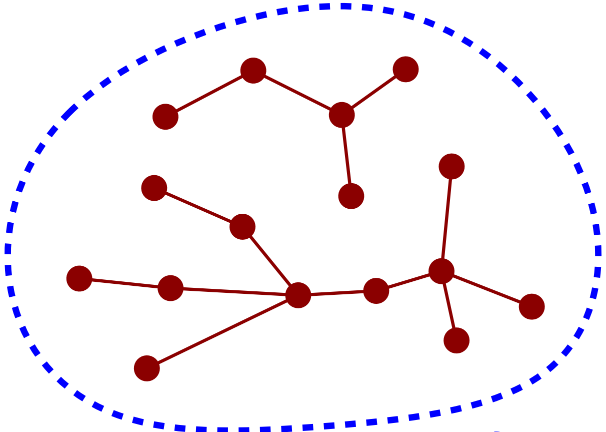
[Picard et al. '82]

$$p(G) = \max_{S \subseteq V(G)} \left\lceil \frac{|E[S]|}{|S|} \right\rceil$$

Local Star Arboricity

$c_\ell^{\mathcal{S}}(G) = \mathrm{sa}_\ell(G)$

$$p(G) \leq a(G) \leq \mathrm{sa}_\ell(G) \leq p(G) + 1$$



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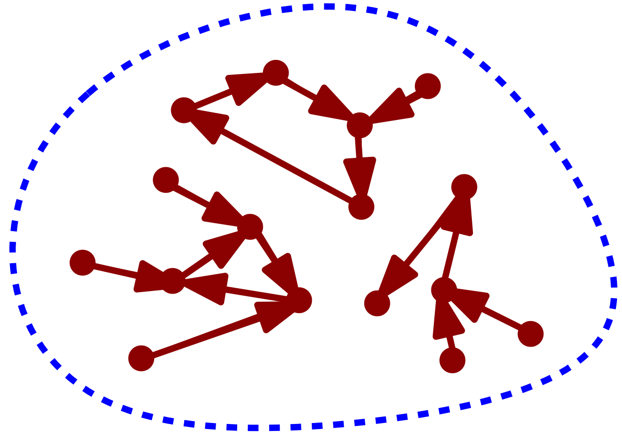
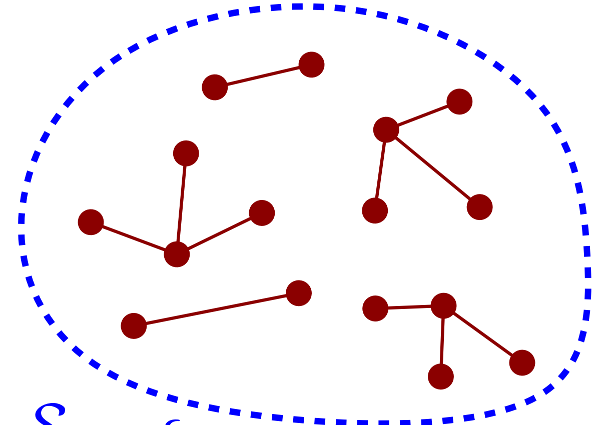


$\mathcal{P} = \{\text{pseudo-forests}\}$



$\mathcal{S} = \{\text{star forests}\}$

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Pseudo-Arboricity

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Star Arboricity

$$c_g^{\mathcal{S}}(G) = \mathrm{sa}(G)$$

Local Star Arboricity

$$c_\ell^{\mathcal{S}}(G) = \mathrm{sa}_\ell(G)$$

$$p(G) \le a(G) \le \mathrm{sa}_\ell(G) \le p(G) + 1$$

Thm.: We have $p(G) \leq a(G) \leq \mathrm{sa}_\ell(G) \leq p(G) + 1.$

(where any of these inequalites can be strict)

Moreover, $p(G) = \mathrm{sa}_\ell(G)$ iff G has an orientation with:

- $\mathrm{outdeg}(v) \leq p(G)$ for every $v \in V(G)$
- $\mathrm{outdeg}(v) = p(G)$ only if $\deg(v) = p(G)$

Thm.: We have $p(G) \leq a(G) \leq \mathrm{sa}_\ell(G) \leq p(G) + 1.$

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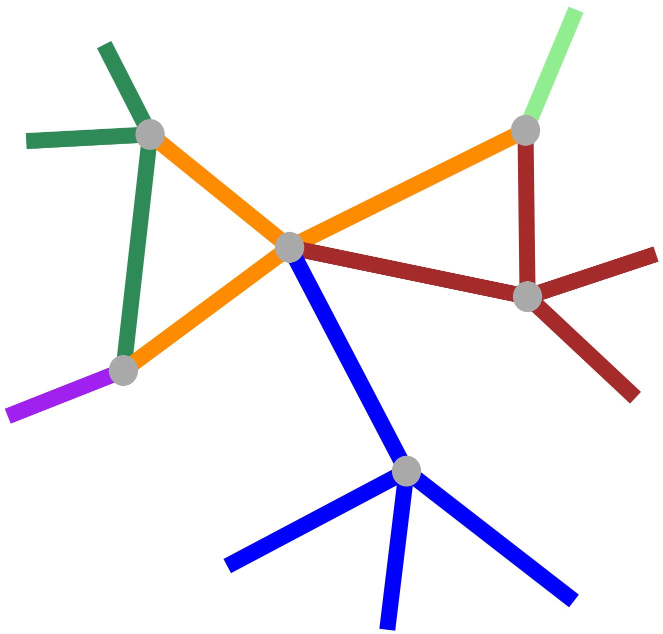
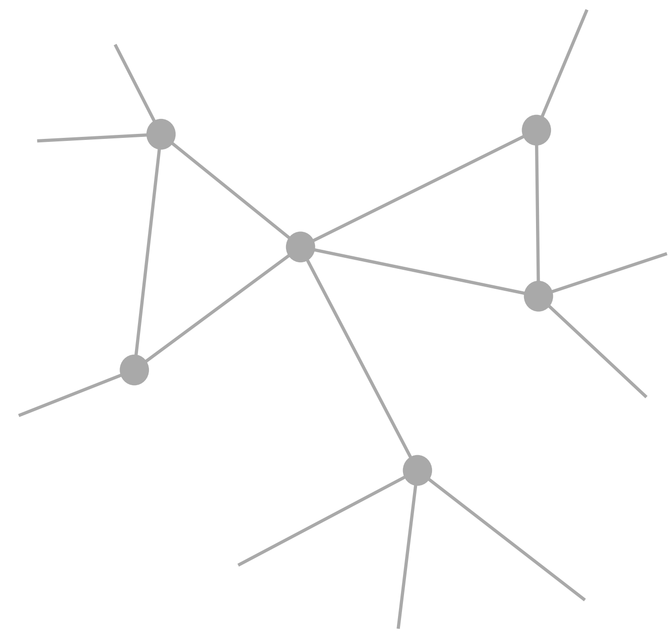
Proofsketch:





Thm.: We have $\boxed{p(G) \leq a(G) \leq \mathrm{sa}_\ell(G) \leq p(G) + 1.}$

(where any of these inequalites can be strict)

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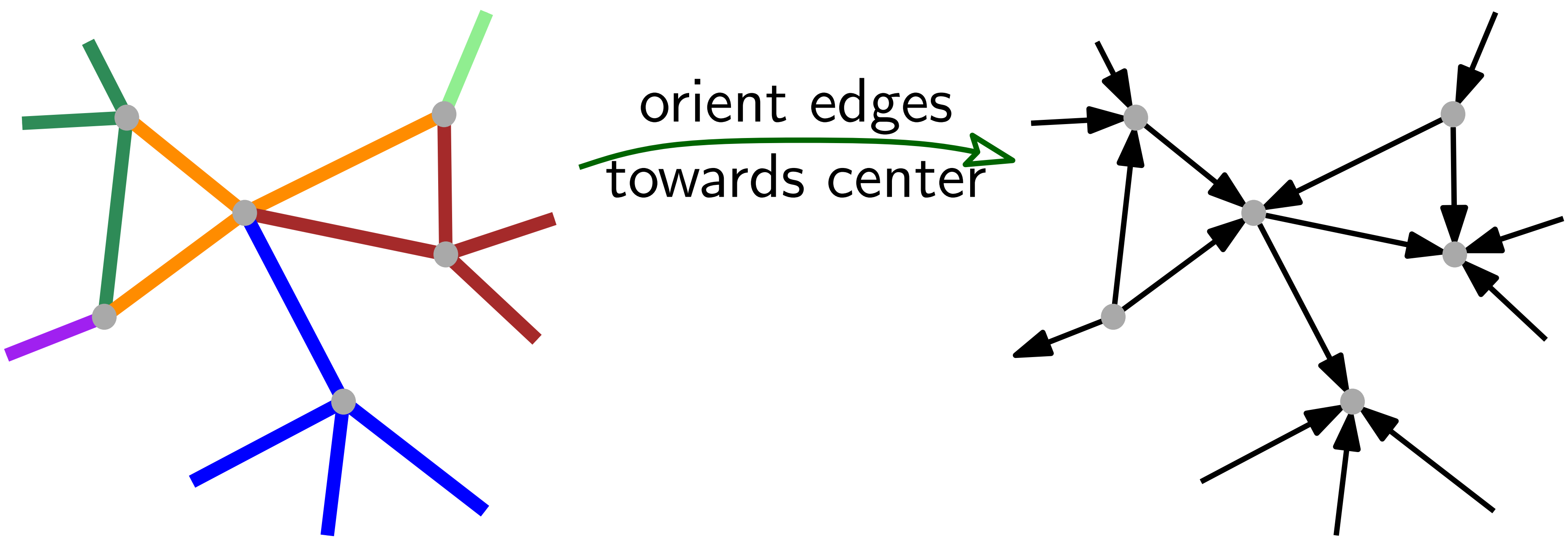

$\mathrm{outdeg}(v) \leq \mathrm{sa}_\ell(G)$

$\boxed{p(G) \leq \mathrm{sa}_\ell(G)}$

Thm.: We have $p(G) \leq a(G) \leq \mathrm{sa}_\ell(G) \leq p(G) + 1.$

(where any of these inequalites can be strict)

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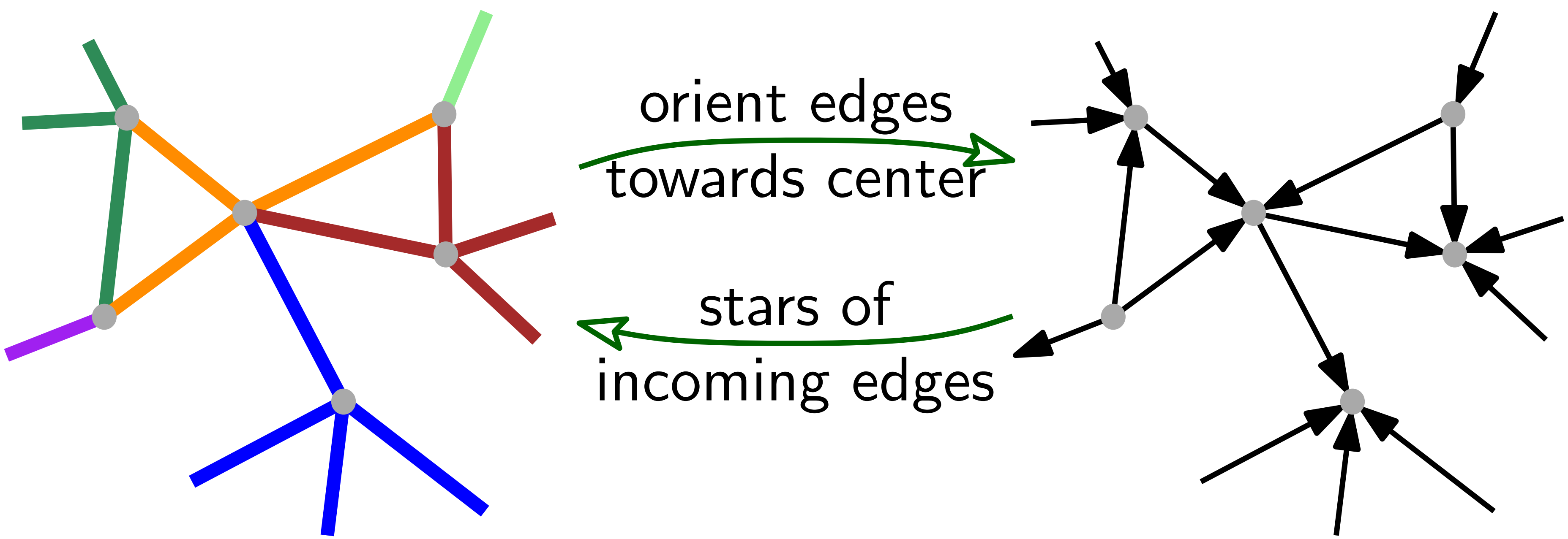

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Remains to show $a(G) \leq \mathrm{sa}_\ell(G)$:

- W.l.o.g. $p(G) = \mathrm{sa}_\ell(G)$
- Orientation with max outdeg $p(G)$ attained only at degree-$p(G)$ vertices
- Remove degree-$p(G)$ vertices
- $p(G') \leq p(G) - 1$, thus $a(G') \leq p(G)$
- Reinsert degree-$p(G)$ vertices
- $a(G) \leq p(G) = \mathrm{sa}_\ell(G)$

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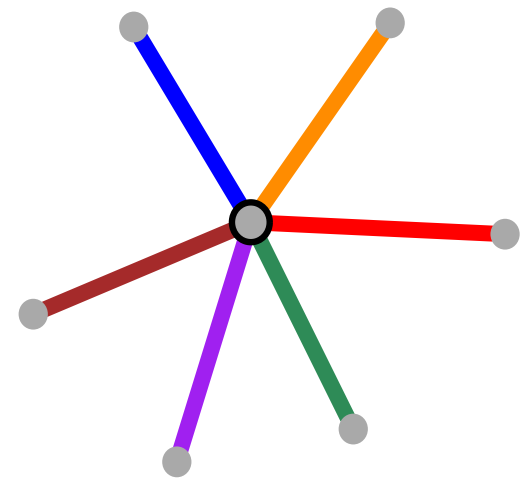

every edge into a different forest

Conclusions (concerning local star arboricity)

Theorem

We have $p(G) \leq a(G) \leq \mathrm{sa}_\ell(G) \leq p(G) + 1$.

Corollary

Local star arboricity can be computed in polynomial time.

[Hakimi, Mitchem, Schmeichel '96]

Deciding $\mathrm{sa}(G) \leq 2$ is NP-complete.

[Alon, McDiarmid, Reed '92]

$\mathrm{sa}(G) \leq 2a(G)$ and this is best possible.

- Global, Local, and Folded Covers
  - Templates = Interval Graphs
- Formal Definitions
- Local and Folded Linear Arboricity
  - Templates = Collections of Paths
- Interrelations
  - Templates = Forests, Pseudo-Forests, Star Forests
- What is known and what is open

What else is known

	Star Forests		Caterpillar Forests		
	g	$\ell = f$	g	ℓ	f
outerplanar	3	3	3	3	3
bip. planar	4	3	4	3	3
planar	5	4	4	4	4
$\mathrm{tw} \leq k$	$k+1$	$k+1$	$k+1$	$k+1$	$k+1$
$\mathrm{dg} \leq k$	$2k$	$k+1$	$2k$	$k+1$	$k+1$

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KU '12

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Kostochka, West '99

Scheinermann, West '83

Algor, Alon '89

Alon et. al. '92

Ding et. al. '98

Gonçalves '07

KU '12

Hakimi et. al. '96

What is open

Local
Linear Arboricity Conjecture

$$\mathrm{la}_\ell(G) \leq \lceil \frac{\Delta+1}{2} \rceil$$

Local track number of planars

$$3 \leq t_\ell \leq 4$$

How much can $c_\ell^{\mathcal{T}}(G)$ and $c_f^{\mathcal{T}}(G)$ differ?

Are there $\mathcal{T}$ and k, where $c_g^{\mathcal{T}}(G) \leq k$ is poly,
but $c_\ell^{\mathcal{T}}(G) \leq k$ or $c_f^{\mathcal{T}}(G) \leq k$ NP-hard?

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...three ways to pack a graph