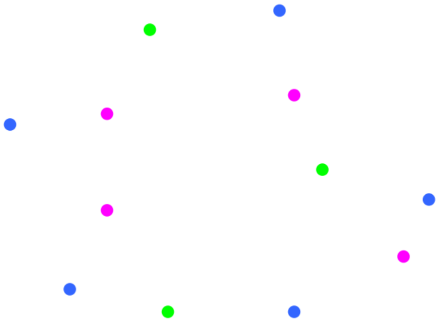
A combinatorial setting for the colourful simplicial depth

Frédéric Meunier – Ecole des Ponts
April 8th, 2012

Talk based on join works with Antoine Deza (McMaster University, Hamilton) and Pauline Sarrabezolles (Ecole des Ponts).

The colourful Carathéodory Theorem in dimension two



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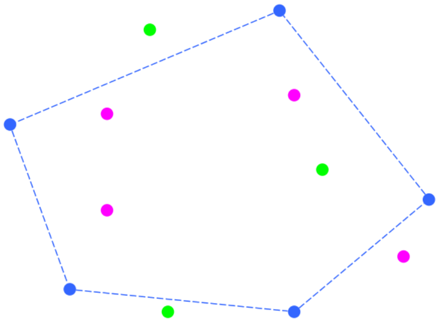

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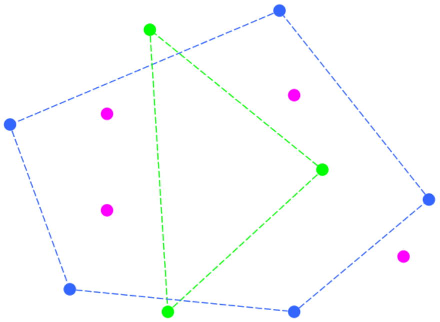

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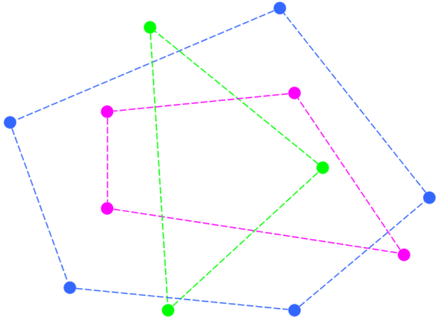

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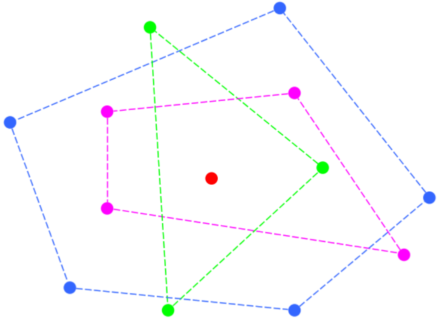

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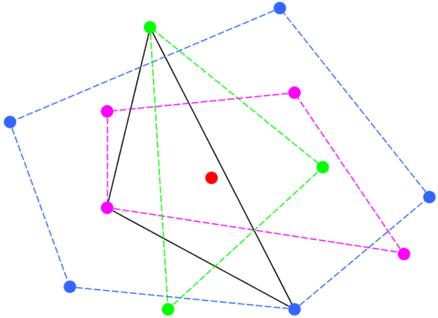


The colourful Carathéodory Theorem

[Bárány 1982]
Let $S_1, \ldots, S_{d+1}$ be sets of points in $\mathbb{R}^d$. If a point $p \in \bigcap_{i=1}^{d+1} \text{conv}(S_i)$, then there is a $T \subseteq \bigcup_{i=1}^{d+1} S_i$ such that $|T \cap S_i| \leq 1$ for $i = 1, \ldots, d+1$ and $p \in \text{conv}(T)$.

$T \subseteq \bigcup_{i=1}^{d+1} S_i$ such that $|T \cap S_i| \leq 1$ for $i = 1, \ldots, d+1$ is *colourful*.

Questions

- Algorithmic questions.
- Counting questions.

Algorithmic questions.

Colourful linear programming

Colourful linear programming, defined by Bárány and Onn in 1997.

Input. $S_1, \ldots, S_k$ sets of points in $\mathbb{R}^d$ and an additional point p

Question. Is there a colourful T such that $p \in \text{conv}(T)$?

Complexity status: **NP**-complete (Bárány and Onn, 1997).

If $S_1 = \ldots = S_k$: usual linear programming.

Colourful linear programming, special **TFNP** case

Input. $S_1, \ldots, S_{d+1}$ sets of points in $\mathbb{R}^d$ and an additional point p such that $p \in \bigcap_{i=1}^{d+1} \text{conv}(S_i)$.

Task. Find a colourful T such that $p \in \text{conv}(T)$.

Complexity status: unknown.

Colourful linear programming, special **PPAD** case

[Deza and M., 2012]
If $S_1, \ldots, S_{d+1}$ are sets of points in $\mathbb{R}^d$ such that $|S_i| = 2$, then there is an even number of colourful T such that $p \in \text{conv}(T)$.

Input. $S_1, \ldots, S_{d+1}$ sets of points in $\mathbb{R}^d$ such that $|S_i| = 2$, and a colourful T such that $p \in \text{conv}(T)$.

Task. Find a colourful $T' \neq T$ such that $p \in \text{conv}(T')$.

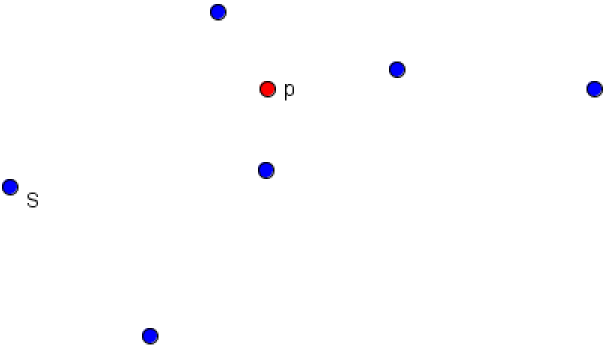
Complexity status: unknown.

Counting questions.

Original motivation: simplicial depth

Let S be a set of points in $\mathbb{R}^d$.

Simplicial depth of a point p = number of d-simplices generated by S and containing p.



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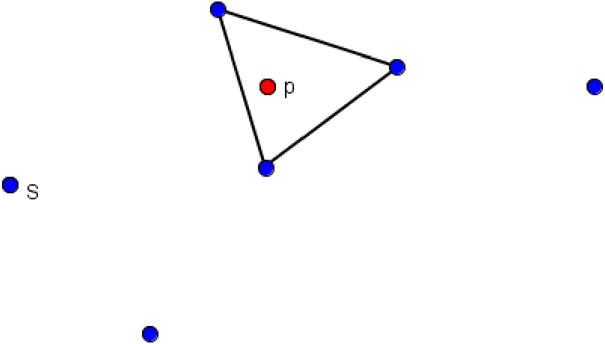

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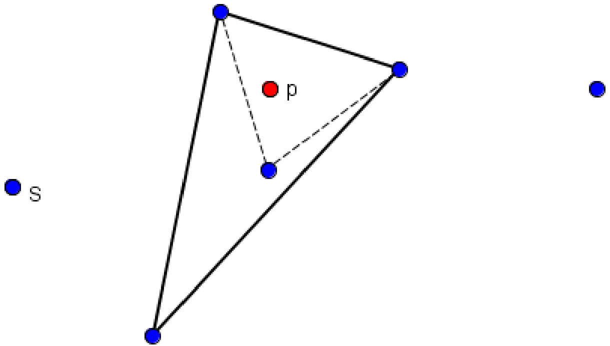

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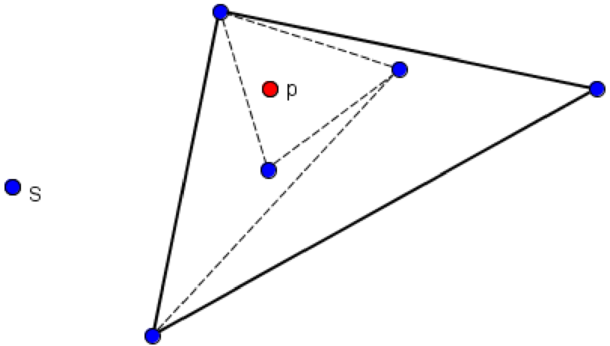

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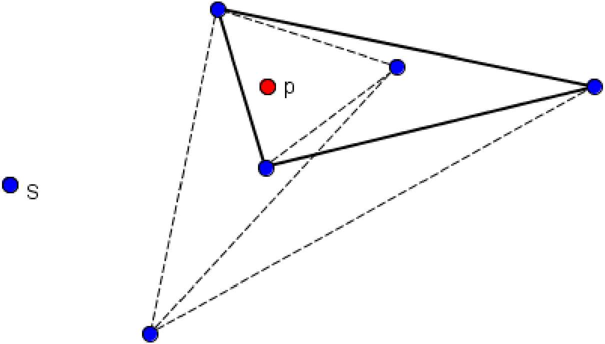

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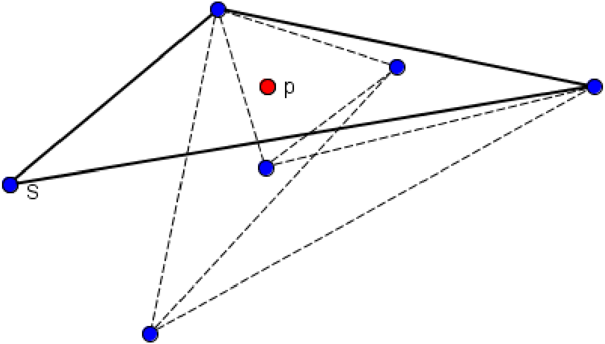

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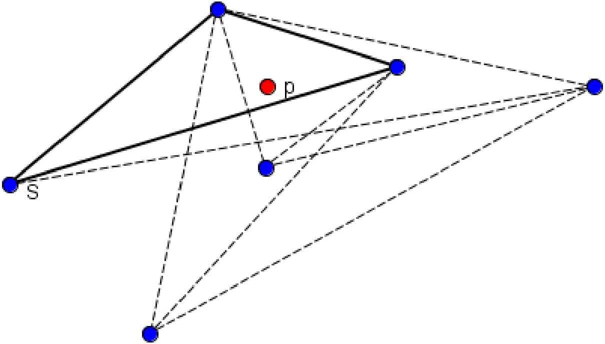
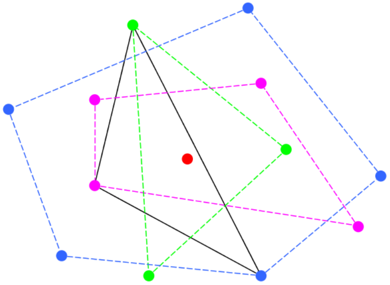


Original motivation: simplicial depth

Let $S_1, \ldots, S_{d+1}$ be $(d+1)$ sets of points in $\mathbb{R}^d$.

Colourful simplicial depth of a point p is:
depth$_{S_1,\ldots,S_{d+1}}(p)$ = number of colourful d-simplices generated by $\bigcup_{i=1}^{d+1} S_i$ and containing p.



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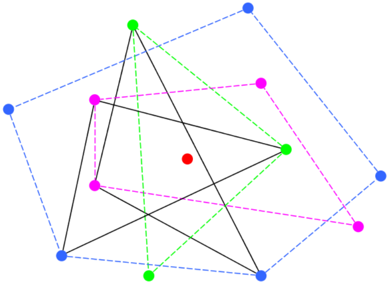

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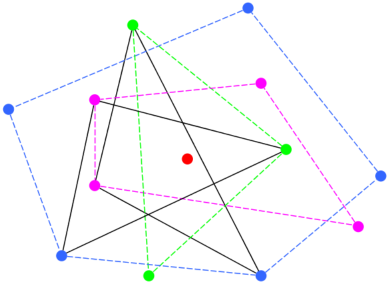

$$\mu(d) = \min_{S_1,\ldots,S_{d+1},p} \textbf{depth}_{S_1,\ldots,S_{d+1}}(p).$$

A lower bound on simplicial depth

For $S \cup \{p\}$ in general position

[Bárány1982]

$$\max_{p} \operatorname{depth}_S(p) \geq \frac{1}{(d+1)^{d+1}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

Proof combines the Tverberg theorem and the colourful Carathéodory theorem.

A lower bound on simplicial depth

For $S \cup \{p\}$ in general position

[Bárány1982]

$$\max_{p} \operatorname{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

Proof combines the Tverberg theorem and the colourful Carathéodory theorem.

A new lower bound for simplicial depth

$$\mu(d) = \min_{\substack{S_1,\ldots,S_{d+1} \\ p \in \bigcap_{i=1}^{d+1} \mathrm{conv}(S_i)}} \#\{T : T \text{ colourful and } p \in \mathrm{conv}(T)\}.$$

Strong version of Colourful Carathéodory Theorem: each point in $\bigcup_{i=1}^{d+1} S_i$ is part of a colourful simplex containing the p.

$$\max_p \mathrm{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{(d+1)}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

What is the exact value of $\mu(d)$?

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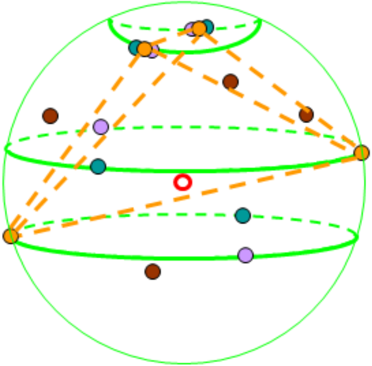
What is the exact value of $\mu(d)$?

Upper bound on the colourful simplicial depth

Unfortunately,
[Deza et al., 2006]

$$\mu(d) \leq d^2 + 1.$$



Gromov's bound

$$\max_{p} \operatorname{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{(d+1)}} \binom{n}{d+1} \quad \text{with } n = |S|,$$

with $\mu(d) = d^2 + 1$ at best.

[Gromov, 2010]

$$\max_{p} \operatorname{depth}_S(p) \geq \frac{2d}{(d+1)!(d+1)} \binom{n}{d+1} \quad \text{with } n = |S|.$$

(simplification by Karasev, 2012).

The conjecture

Conjecture.

$$\mu(d) = d^2 + 1.$$

The successive improvements

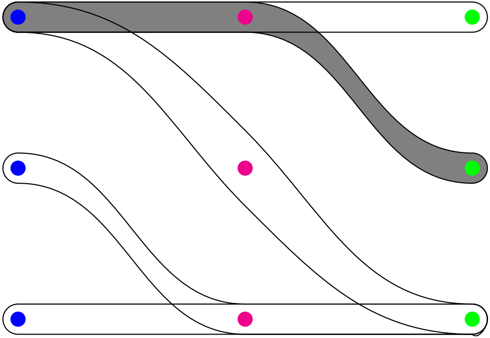
	Lower bound for $\mu(d)$	Conjecture true for d up to
Bárány, 1982	$d+1$	1
Deza et al., 2006	$2d$	2
Bárány and Matoušek, 2007	$\max(3d, \frac{1}{5}d^2 + \frac{1}{5}d)$	3
Stephen and Thomas, 2008	$\frac{1}{4}d^2 + d + 1$	$\emptyset$
Deza, Stephen, and Xie, 2011	$\frac{1}{2}d^2 + d + \frac{1}{2}$	$\emptyset$
Deza, Meunier, and S., 2012	$\frac{1}{2}d^2 + \frac{7}{2}d - 8$	4

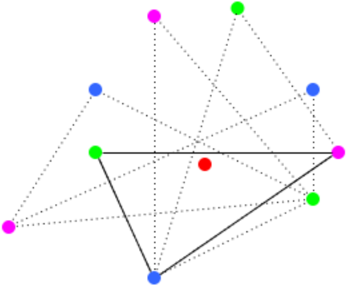
A combinatorial counterpart: octahedral systems

An *octahedral system* Ω in an n-partite hypergraph $(V_1, \ldots, V_n, E)$ satisfying *parity condition:* for any $X \subseteq \bigcup_{i=1}^{n} V_i$ such that $|X \cap V_i| = 2$ for all i, the number of edges of Ω induced by X is even.

Octahedral systems *without isolated vertex* generalize colourful configurations.

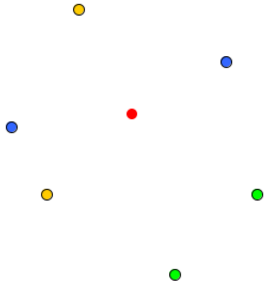
An octahedral system





Two main properties for the geometrical approach

Octahedral Lemma



$X \subseteq S$, $|X \cap S_i| = 2$ for all i $\longrightarrow$ an even number of colourful simplices.

Strong colourful Carathéodory Theorem

If $p \in \text{conv}(S_i)$ for all i, each point is part of some colourful simplices containing p.

Two main properties for the geometrical approach

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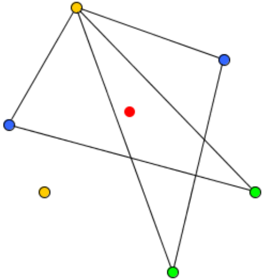

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Strong colourful Carathéodory Theorem

If $p \in \text{conv}(S_i)$ for all i, each point is part of some colourful simplices containing p.

Combinatorial approach

Vertex set: $V = V_1 \cup \cdots \cup V_{d+1}$.

Edge set: E.

Parity condition: The number of edges induced by X, with $|X \cap V_i| = 2$ for all i, is even.

Octahedral systems without isolated vertex: Every point in $\bigcup_{i=1}^{d+1} V_i$ is in at least one edge.

Geometrical approach

A colourful configuration $S = S_1 \cup \cdots \cup S_{d+1}$.

Colourful simplices containing p.

Octahedral Lemma: The number of colourful simplices containing p generated by points in X, with $|X \cap S_i| = 2$ for all i, is even.

Strong Colourful Carathéodory Theorem: Every point in $\bigcup_{i=1}^{d+1} S$ is part of some colourful simplex containing p.

If Ω realizes a colourful configuration, **the number of edges $|E|$ is the number of colourful simplices containing p.**

Definition (ν)

$\nu(d)$ is the minimal number of edges of an octahedral system without isolated vertex with $|V_i| = d+1$ for $i = 1, \ldots, d+1$.

$$\nu(d) \leq \mu(d)$$

Lower bounds

Theorem (Deza, Meunier, S.)

$$\nu(d) \geq \frac{1}{2}d^2 + \frac{7}{2}d - 8$$

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$$\mu(d) \geq \frac{1}{2}d^2 + \frac{7}{2}d - 8$$

Idea of the proof: induction

Inductive approach.

Given an octahedral system $\Omega = (V, E)$ without isolated vertex and one of its vertices v, use the bound for $\Omega' = (V', E') = \Omega \setminus \{v\}$:

$$|E| = |E'| + \deg_\Omega(v).$$

For any such Ω', parity condition automatically satisfied.

We would like to ensure that Ω' is again without isolated vertex.

Main Idea. Delete the vertices one after another until reaching an octahedral system whose number of edges can be estimated.

Idea of the proof: graph $D(\Omega)$

Octahedral system $\Omega = (V_1, \ldots, V_n, E)$.

Directed graph $D(\Omega) = (V, A)$ with $V = \bigcup_{i=1}^{n} V_i$ and arc $(u, v) \in A$ if every edge in E containing v contains u as well.

Idea. If u is removed from Ω, the vertex v becomes isolated.

Transitivity of $D(\Omega)$

$D(\Omega)$ is *transitive*:

Arc $(u, v) \in A$: every edge in E containing v contains u as well.

Arc $(v, w) \in A$: every edge in E containing w contains v as well.

$$\Rightarrow$$

Arc $(u, w) \in A$: every edge in E containing w contains u as well.

Looking for complete subgraphs without outneighbour

In a transitive directed graph, there is always a complete subgraph without outneighbour.

Let Ω be an octahedral system without isolated vertex. If X induces a complete subgraph without outneighbour in $D(\Omega)$, then $\Omega' = \Omega \setminus X$ is an octahedral system without isolated vertex.

If $|X| = 1$

$$\text{lower bound for } \Omega \geq \text{lower bound for } \Omega \setminus X + \deg_{\Omega}(X).$$

If $|X| \geq 2$

$$\text{lower bound for } \Omega \gtrsim \min_{i=1,\dots,n} |V_i|^2.$$

Small instances

An octahedral system with $n = 5$*,* $|V_1| = \ldots = |V_5| = 5$ *and without isolated vertex has at least* 17 *edges.*

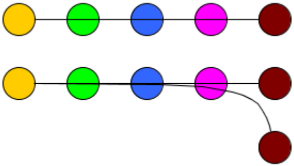
Proposition

$$\mu(4) = 17.$$

Computational approach “branch-and-bound” $\mu(4) \geq 14$, (Deza, Stephen, and Xie, 2012).

Small instances

$$n = 5,\ |V_1| = \ldots = |V_5| = 5 \implies |E| \geq 17.$$



$|E| \geq 3$

Small instances

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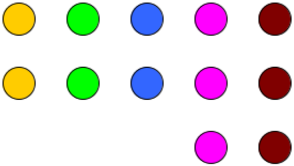

$|E| \geq 4$

Small instances

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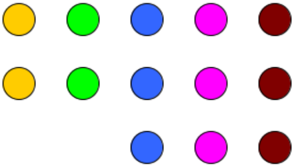

$|E| \geq 5$

Small instances

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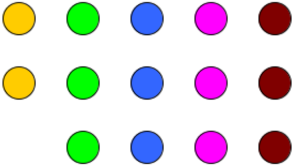

$|E| \geq 6$

Small instances

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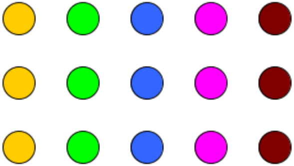

$|E| \geq 7$

Small instances

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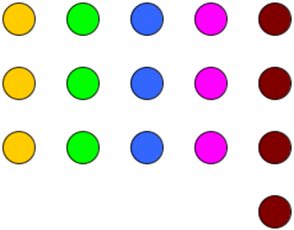

$|E| \geq 8$

Small instances

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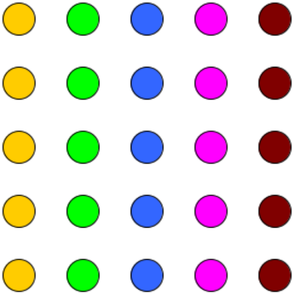


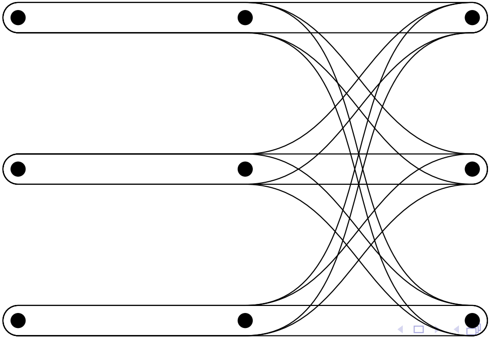
$|E| \geq 17$

Realisability

Is any octahedral system Ω with $|V_i| = d+1$ for $i = 1, \ldots, d+1$ and without isolated vertex the combinatorial counterpart of sets of points $S_1, \ldots, S_{d+1}$ in $\mathbb{R}^d$?

No.

Counterexample.



Counting the number of distinct octahedral systems

Given n disjoint finite sets $V_1, \ldots, V_n$, we have

number of octahedral systems on $V_1, \ldots, V_n = 2^{\prod_{i=1}^{n}|V_i| - \prod_{i=1}^{n}(|V_i|-1)}$.

Idea of the proof

Identification

n-partite hypergraph $\cong$ subspace of $\mathcal{H} = \mathbb{F}_2^{V_1} \otimes \ldots \otimes \mathbb{F}_2^{V_n}$.

Define $\mathcal{X} = F_1 \otimes \ldots \otimes F_n$ where F_i =vectors of $\mathbb{F}_2^{V_i}$ with an even number of 1.

$$\begin{array}{rccc} \psi : & \mathcal{H} & \to & \mathcal{X}^* \\ & H & \mapsto & \langle H, \cdot \rangle \end{array}$$

ψ is surjective, $\ker \psi$ is identified with the set of all octahedral systems

$$\dim \ker \psi = \dim \mathcal{H} - \dim \mathcal{X}.$$

Counting the number of distinct octahedral systems

$\mathcal{H} = \mathbb{F}_2^{V_1} \otimes \ldots \otimes \mathbb{F}_2^{V_n}$ thus

$$\dim \mathcal{H} = \Pi_{i=1}^{n} |V_i|.$$

$\mathcal{X} = F_1 \otimes \ldots \otimes F_n$ where $F_i =$vectors of $\mathbb{F}_2^{V_i}$ with an even number of 1 thus

$$\dim \mathcal{X} = \Pi_{i=1}^{n}(|V_i| - 1).$$

$$\Rightarrow$$

Given n disjoint finite sets $V_1, \ldots, V_n$, we have

number of octahedral systems on $V_1, \ldots, V_n = 2^{\Pi_{i=1}^{n}|V_i| - \Pi_{i=1}^{n}(|V_i|-1)}$.

Open questions

- Complexity status of colourful linear programming under Bárány's conditions.
- Complexity status of colourful linear programming, **PPAD** version.
- $\mu(d) \stackrel{?}{=} d^2 + 1$.
- Non-realisable octahedral systems for $d \geq 3$?
- Number of non-isomorphic octahedral systems (using Polya's theory?).
- Monotony of $\nu(m_1, \ldots, m_n)$.

Thank you.