Random sampling “à la Rémy” for Motzkin trees

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LIPN, LIPN, LIPN

29 novembre 2012

Sommaire

Remy Algorithm

Remy(n)**:**
Let T be a leaf.
For $i = 0$ to n do:
  Uniformly draw a node
  With probability $1/2$:
    Extend to the right
  Else:
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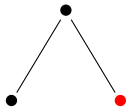

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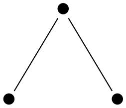

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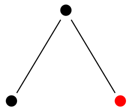

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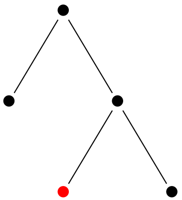

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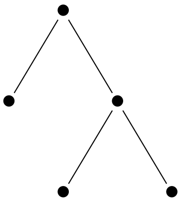

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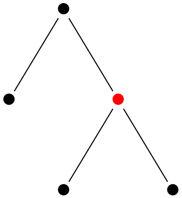

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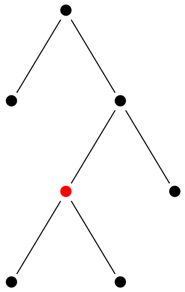

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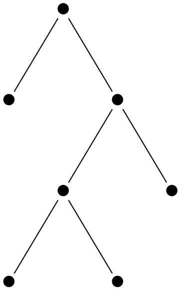

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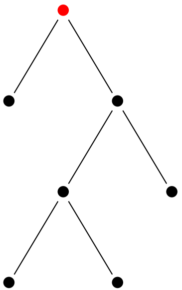

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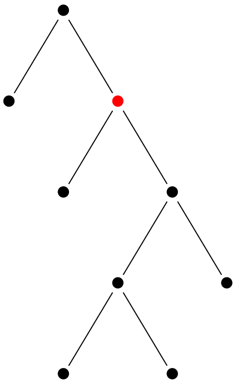

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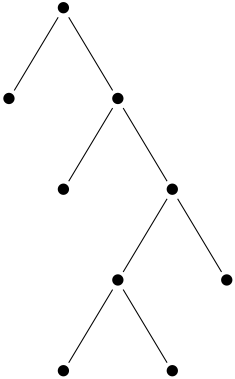

Theorem

Remy(n) draws a tree of size $2n+1$ uniformly among all Catalan trees of size $2n+1$.

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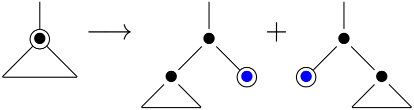
$$\mathcal{C}^{\bullet(leaf)} = \mathcal{Z} + 2\mathcal{Z}^2\mathcal{C}^{\bullet}$$



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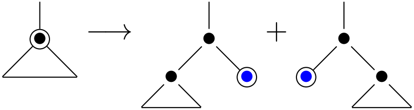

As a tree of size $2i+1$ always have $i+1$ leaves, we can forget the point and keep uniformity at each step of Remy algorithm.

Remy Algorithm for Motzkin trees

1. Find a constructive holonomic specification for Motzkin trees.
2. Use it to build a random sampler
3. Trasnform it into a Remy-like algorithm

An holonomic equation is a linear differential equation with polynomial coefficients.

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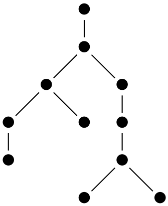
Definition

A unary-binary tree, or Motzkin tree is a planar rooted tree where all internal nodes have 1 or 2 children.
The size of the tree is the total number of nodes (internal + leaves).



$u = \#$unary nodes, $\ell = \#$leaves, $n =$ size

$$2\ell + u = n + 1$$

Specification

Let $\mathcal{M}$ be the class of Motzkin trees.

Theorem

$$\mathcal{M} + \mathcal{M}^{\bullet} = 2\mathcal{Z} + 3\mathcal{Z}^2\mathcal{M}^{\bullet} + \mathcal{Z}\mathcal{M}^{\bullet} + \mathcal{Z}(\mathcal{M} + \mathcal{M}^{\bullet})$$

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$$\mathcal{M}^{\color{blue}{\bullet\text{(leaf)}}} + \mathcal{M}^{\color{red}{\bullet\text{(leaf)}}} + \mathcal{M}^{\color{green}{\bullet\text{(unary)}}} = 2\mathcal{Z} + 3\mathcal{Z}^2\mathcal{M}^{\bullet} + \mathcal{Z}\mathcal{M}^{\bullet} + \mathcal{Z}(\mathcal{M}^{\color{blue}{\bullet\text{(leaf)}}} + \mathcal{M}^{\color{red}{\bullet\text{(leaf)}}} + \mathcal{M}^{\color{green}{\bullet\text{(unary)}}})$$

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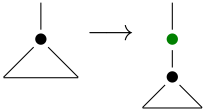
- $\mathcal{Z}\mathcal{M}^{\bullet}$ 

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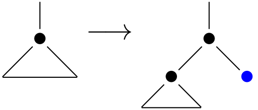
- $\mathcal{Z}^2\mathcal{M}^{\bullet}$



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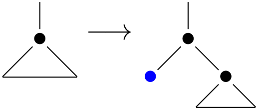

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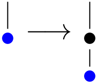

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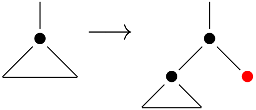
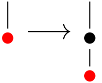
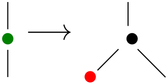
- $\mathcal{Z}^2\mathcal{M}^{\bullet}$ 
- $\mathcal{Z}\mathcal{M}^{\bullet(\text{leaf})}$ 
- $\mathcal{Z}\mathcal{M}^{\bullet(\text{unary})}$ 

Sommaire

Method overview

1. Design a Boltzmann sampler for the holonomic specification
   - Random size of the output
   - Evaluation of the generating function in several points
2. Turn it into a critical Boltzmann sampler
   - Infinite process
   - Iterative form
3. Stop the process when the targeted size is reached
   - Exact size
   - Constant number of rejection in average

Boltzmann sampler

Definition

A Boltzmann sampler $\Gamma_x \mathcal{A}$ *draws an object* α *of* $\mathcal{A}$ *with probability*

$$\frac{x^{|\alpha|}}{A(x)}$$

- x must be in the radius of convergence of $A(z)$
- All objects of the same size have the same probability to be drawn
- The size of the output is a random variable

Boltzmann Algorithms

$\Gamma_x \mathcal{A} + \mathcal{B}$

If Bernouilli$\left(\frac{A(x)}{A(x)+B(x)}\right) = 1$
 return $\Gamma_x \mathcal{A}$
Else
 return $\Gamma_x \mathcal{B}$

$\Gamma_x \mathcal{A} \times \mathcal{B}$

return $(\Gamma_x \mathcal{A}, \Gamma_x \mathcal{B})$

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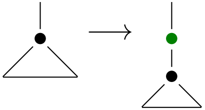
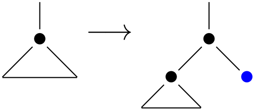
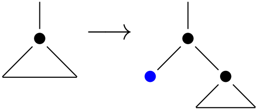
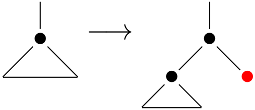
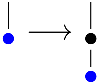
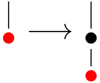
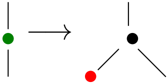
- $\mathcal{Z}\mathcal{M}^{\bullet}$ 
- $\mathcal{Z}^2\mathcal{M}^{\bullet}$ 
- $\mathcal{Z}^2\mathcal{M}^{\bullet}$ 
- $\mathcal{Z}^2\mathcal{M}^{\bullet}$ 
- $\mathcal{Z}\mathcal{M}^{\bullet(\text{leaf})}$ 

$\mathcal{Z}\mathcal{M}^{\bullet(\text{leaf})}$ 

$\mathcal{Z}\mathcal{M}^{\bullet(\text{unary})}$ 

Critical Boltzmann sampler

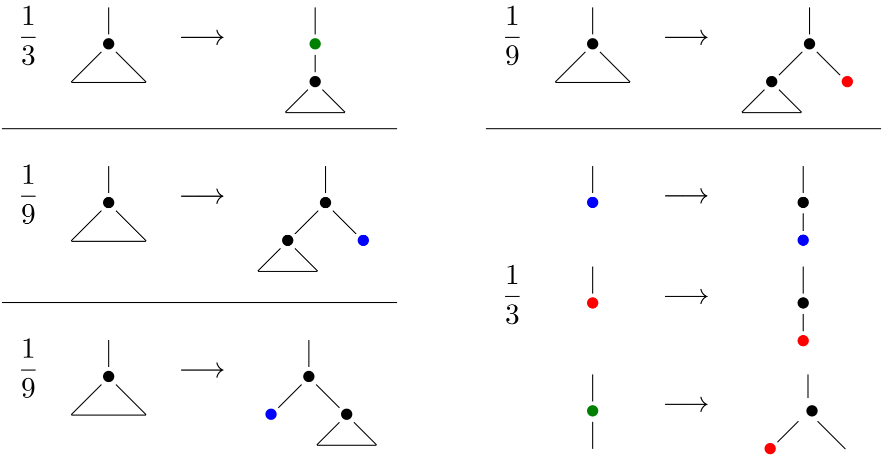
Definition

A critical Boltzmann sampler of $\mathcal{A}$ is the limit process of $\Gamma_x\mathcal{A}$ when $x \to \rho$.

$\tilde{\Gamma}\mathcal{M} + \mathcal{M}^\bullet$ does not terminate:

$$\mathbb{P}(stop) = \frac{2x}{M(x) + M^\bullet(x)} \to 0 \qquad \text{when } x \to \frac{1}{3}$$

Critical Boltzmann sampler



Exact-size sampler

Main Algorithm(n)

$M :=$ • or •
While $|M| < n$
  With Probability $1/3$:
    ⋮
  With Probability $1/9$:
    ⋮
If $|M| = n$ **return** M
Else **return** Main Algorithm(n)

- Uniform
- Constant number of rejection in average
- With size n or $n+1$ accepted, linear in worst case.

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Perspectives

- Weight the unary and binary nodes differently

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- Weight the unary and binary nodes differently
- Extend the method to other classes:
  - k-regular trees, trees with arity 1 to k
  - Classes of walks
  - Maps and graphs
  - ...

  Main difficulty: Need a constructive holonomic form first!