On the rank of configurations on graphs and a kind of Riemann Roch formula

Robert Cori, joint work with Yvan Le Borgne

LIX, december 3, 2012

Graphs

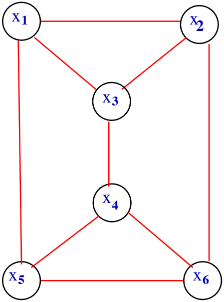
Definition

A graph is given by a set $X = \{x_1, x_2, \dots, x_n\}$ of vertices and a set E of edges, consisting of pairs of vertices.

Definition

The adjacency matrix of a graph denoted also by E is given by: $e_{i,j}$ is equal to the number of edges between the vertices x_i and x_j.

An example



$$E = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Graphs, conventions

- There are no loops, multiple edges may be allowed
- All graphs are supposed connected
- The degree of a vertex is the number of edges incident to it

$$d_i = \sum_{j=1}^{n} e_{i,j}$$

- The number of edges of the graph is denoted m.

Configurations on graphs

- A **configuration** is an assignment of integers (positive or negative) to vertices.
  Two different writings will be used:

  $$u = (u_1, u_2, \cdots, u_n) \text{ or } \quad u = \sum_{i=1}^{n} u_i x_i$$

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$$deg(u) = \sum_{i=1}^{n} u_i$$

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- $$v = u + (e_{i,1}, e_{i,2}, \ldots, e_{i,i-1}, -d_i, e_{i,i+1}, \ldots e_{i,n})$$

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- Corresponds to the rows of the Laplacian matrix

$$\Delta = D - E$$

  where D is the diagonal matrix such that $D_{i,i} = d_i$ and E is the adjacency matrix.

Laplacian equivalence

- If $u - v$ is a linear combination of the Laplacian configurations $\Delta^{(i)}$ we will write

$$u \sim_{L_G} v$$

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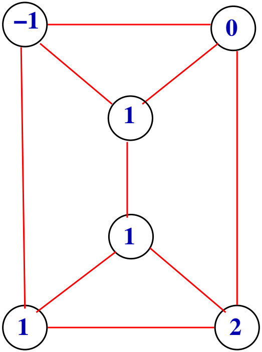
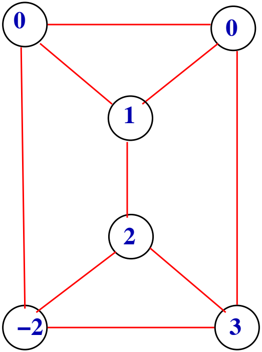
- **Algebraic translation:** Does the exist a linear combination v of the $\Delta^{(i)}$ such that $u + v$ is positive.
- A configuration u for which such a v exists will be called here **effective**

An effective configuration



$$u = -x_1 + x_3 + x_4 + x_5 + 2x_6$$

An effective configuration



$$u \sim_{L_G} x_3 + 2x_4 - 2x_5 + 3x_6$$

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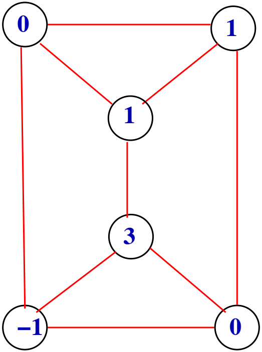

$$u \sim_{L_G} x_2 + x_3 + 3x_4 - x_5$$

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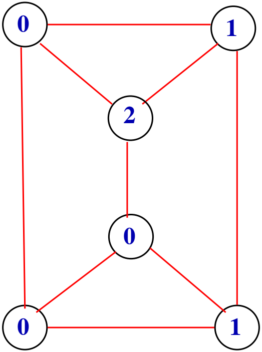

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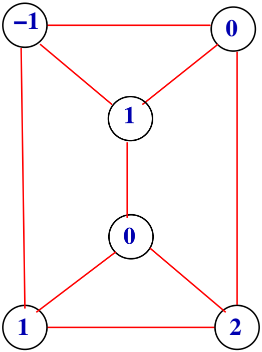
A non effective configuration



Similar games

- A. Björner , L. Lovász , P. W. Shor. *Chip-firing games on graphs (1991)* European J. Combin, 1991, Vol 12.

Consider only positive configurations, determine if there exists an infinite sequence of topplings, beginning with a given configuration and remaining positive.

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- Deepak Dhar *Self Organised Critical State of Sandpile Automaton Models* Phys. Rev. Lett. 64, No.14,(1990)

The rules are:

1. Add 1 to a vertex different from the sink (x_n)
2. Perform topplings in vertices different from the sink;

Configurations associated to acyclic orientations

- **Orientation of a graph:** For each edge $\{x_i, x_j\}$ of the graph determine which of x_i, x_j is the tail and which is the head. The orientation is **acyclic** if there is no circuit.
- For the cycle C_n, the number of acyclic orientations is **The configuration associated** to $\overrightarrow{G}$ is denoted $u_{\overrightarrow{G}}$ and given by:

$$(u_{\overrightarrow{G}})_i = d_i^- - 1$$

Where d_i^- is the number of edges which have head x_i.

- **Remark** The degree of $u_{\overrightarrow{G}}$ is $m - n$, where m is the number of edges of G.

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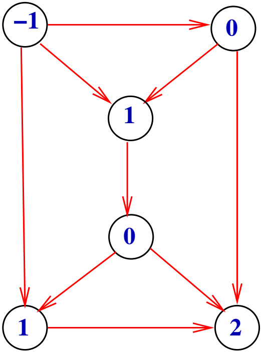
An Example



An acyclic orientation and the configuration associated to it

Non effectiveness

Proposition

For any acyclic orientation $\overrightarrow{G}$ *of* G *the configuration* $u_{\overrightarrow{G}}$ *associated to it is non effective.*

Non effectiveness

Proof.

□

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- Consider a configuration v such that $v \sim_{L_G} u_{\vec{G}}$ then

$$v = u_{\vec{G}} + \sum_{i=1}^{n} a_i \Delta^{(i)}$$

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- Let j be such that $-a_j$ is maximal among the $-a_i$, then $v_j \geq 0$ implies that there exists an edge x_k, x_j oriented from x_k to x_j such that $a_k = a_j$;

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- Let j be such that $-a_j$ is maximal among the $-a_i$, then $v_j \geq 0$ implies that there exists an edge x_k, x_j oriented from x_k to x_j such that $a_k = a_j$;
- Repeating this remark implies that $\overrightarrow{G}$ has a circuit, contradicting acyclicity.

□

Parking configurations

- Definition

  A parking configuration is a sandpile configuration u such that for any subset Y of $\{x_1, x_2, \ldots, x_{n-1}\}$ there exists a vertex x_i in Y such that u_i is less than the number of edges with have as end points x_i and an x_j not in Y. More precisely:

$$\exists x_i \in Y, s.t: \quad u_i < \sum_{x_j \notin Y} e_{i,j}$$

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- Remark

  In the case of the complete graph:

$$\sum_{x_j \notin Y} e_{i,j} = n - |Y|$$

so that the condition correspond to the usual parking functions.

Parking and recurrent configurations

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  A configuration u is a parking configuration if and only if $\overline{u}$ given by: $\overline{u}_i = d_i - 1 - u_i$ is a recurrent configuration.

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  A configuration u *is a parking configuration if and only if* $\overline{u}$ *given by:* $\overline{u}_i = d_i - 1 - u_i$ *is a recurrent configuration.*

- Corollary

  For each configuration u *there is a (unique) parking configuration* v *such that* $u \sim_{L_G} v$

Parking configurations and acyclic orientations

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Proof.

$$Y = \{x_1, x_2, \ldots, x_{n-1}\}$$

While $Y \neq \emptyset$ do:

- Find $x_k \in Y$ such that $u_k < \sum_{x_j \notin Y} e_{k,j}$
- Orient the edges joining $x_j \notin Y$ to x_k from x_j t(tail) to x_k (head)
- Remove x_k from Y.

Deciding effectiveness

Corollary

u is effective if and only if the parking configuration w such that $w \sim_{L_G} u$ satisfies $w_n \geq 0$

Main Theorem

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  For any configuration u one of the following assertions holds:
  (I) u is effective
  (II) There exists an acyclic orientation $\overrightarrow{G}$ of G such that $u_{\overrightarrow{G}} - u$ is effective.

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- Remark
  Assertions (I) and (II) cannot hold simultaneously since:

$$u + (u_{\overrightarrow{G}} - u) = u_{\overrightarrow{G}}$$

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- $(u_{\overrightarrow{G}})_i = d_i^- - 1 \geq w_i$
- $d_n^- - 1 - w_n \geq 0$, since $d_n^- \geq 0$ and $w_n < 0$

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Algorithm for testing effectiveness of u

Find the parking configuration w such that $u \sim_{L_G} w$ then check if $w_n \geq 0$

Rank of a configuration

Definition

The rank $\rho(u)$ of a configuration u is given by

$$\rho(u) = Min(deg(f)) - 1$$

where the minimum is taken among all the positive configurations f such that $u - f$ is not effective.

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Remark

The rank of a non effective configuration is -1.

Example of calculation

The configuration $u = (0, 2, 2, 2)$ on K_4

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Subtract a positive configuration of degree 4 and prove non effectiveness

Subtraction of $(0, 3, 1, 0)$ gives $(0, -1, 1, 2)$ which is associated to an acyclic orientation

Rank of a configuration of high degree

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$$\rho(u) = deg(u) - m + n - 1$$

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Proof.

Let $r = deg(u) - m + n - 1$. If $deg(f) = r$ then $deg(u - f) = m - n + 1$ hence it is effective and $\rho(u) \geq r$

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Let $\overrightarrow{G}$ be an acyclic orientation of G,
$deg(u - u_{\overrightarrow{G}}) > 2(m - n) - m + n$ hence $u - u_{\overrightarrow{G}}$ is effective

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Let $\overrightarrow{G}$ be an acyclic orientation of G, $deg(u - u_{\overrightarrow{G}}) > 2(m - n) - m + n$ hence $u - u_{\overrightarrow{G}}$ is effective

Let f be such that $f \sim_{L_G} u - u_{\overrightarrow{G}}$ then $u - f \sim_{L_G} u_{\overrightarrow{G}}$ is not effective. Since $deg(f) = deg(u) - m + n = r + 1$, we have $\rho(u) < r + 1$. □

Riemann Roch Theorem for graphs

M. Baker, S. Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph (2007)* Advances in Maths, **215**, 766-788.

Theorem

Let K be the configuration such that $K_i = d_i - 2$, so that $deg(K) = 2(m-n)$. Any configuration u satisfies:

$$\rho(u) = deg(u) - m + n + \rho(K-u)$$

Riemann Roch Theorem for graphs

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Showing that $K - u - g$ is not effective hence

$$\rho(K-u) < deg(g) = deg(u_{\overrightarrow{G}} - (u-f)) = m - n - deg(u) + \rho(u) + 1$$

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Since $\rho(K-u)$ is an integer we have:

$$\rho(u) + m - n - deg(u) = \rho(K-u)$$

A greedy algorithm computing the rank on K_n.

A greedy algorithm to compute the rank on K_n

Lemma. u effective if and only if $parking(u)$ is positive.

```
u ← parking(u)
rank ← −1
while s ≥ 0 do
    substract 1 to a u_i = 0 and i < n
    u ← parking(u)
    rank ← rank + 1
end while
Return rank
```

$$\rho(u) = \min_{f \geq 0 \text{ and } u - f \text{ non-effective}} deg(f) - 1$$

Lemma. For a positive configuration u where $u_i = 0$, there exists in

$$\{f | \; u - f \text{ non-effective}, \; f \geq 0 \text{ and } degree(f) = \rho(u) + 1\}$$

a configuration g such that $\boxed{g_i > 0}$.

Proof.

$$\begin{array}{rccc}
 & u & & \\
-(f - a\epsilon^{(j)}) & \downarrow & & \\
\{v_i = 0 = v_j\} & v & \overset{(ij).}{\longleftrightarrow} & (ij).v \\
-a\epsilon^{(j)} & \downarrow & & \downarrow -a\epsilon^{(i)} \; \{= (ij).[-a\epsilon^{(j)}]\} \\
\{ \text{ non-effective } \} & u - f & \overset{(ij).}{\longleftrightarrow} & (ij).[u - f]
\end{array}$$

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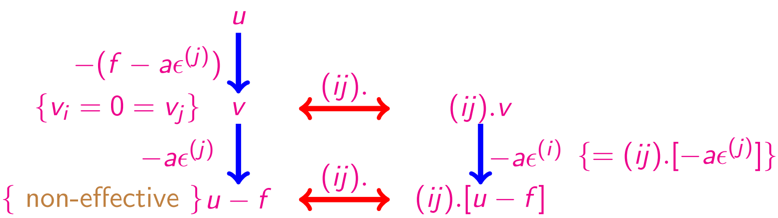


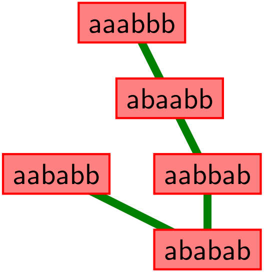
Let $\epsilon^{(i)}$ the configuration where $\epsilon_i^{(i)} = 1$ and $\epsilon_j^{(i)} = 0$ for $j \neq i$.
Let $f \geq 0$ be an optimal configuration so $u - f$ non-effective.
Assume $f_i = 0$, let $j \neq i$ such that $u_j - f_j = -a < 0$.

Algorithm in terms of Dyck words

A non-increasing parking configuration $u = (u_1, \ldots, u_{n-1}, s)$ is decomposed into

$$(d(u)\text{: Dyck word of (semi-)length } n-1, s{:}\mathbb{Z}).$$

$(d, s) \leftarrow (d(u), s(u))$
$rank \leftarrow -1$
while $s \geq 0$ **do**
  match d **with** $ad'bd''$
  $d \leftarrow d''abd'$
  $rank \leftarrow rank + 1$
  $s \leftarrow s - |ad'b|_a$
end while
Return $rank$



The *prerank* of a Dyck word d is the number of loop iterations leading to the fixpoint belonging to $(ab)^*$.

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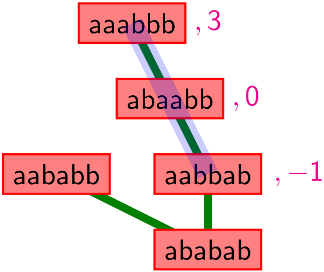
```
(d, s) ← (d(u), s(u))
rank ← −1
while s ≥ 0 do
    match d with a d′ b d″
    d ← d″ a b d′
    rank ← rank + 1
    s ← s − |a d′ b|_a
end while
Return rank
```

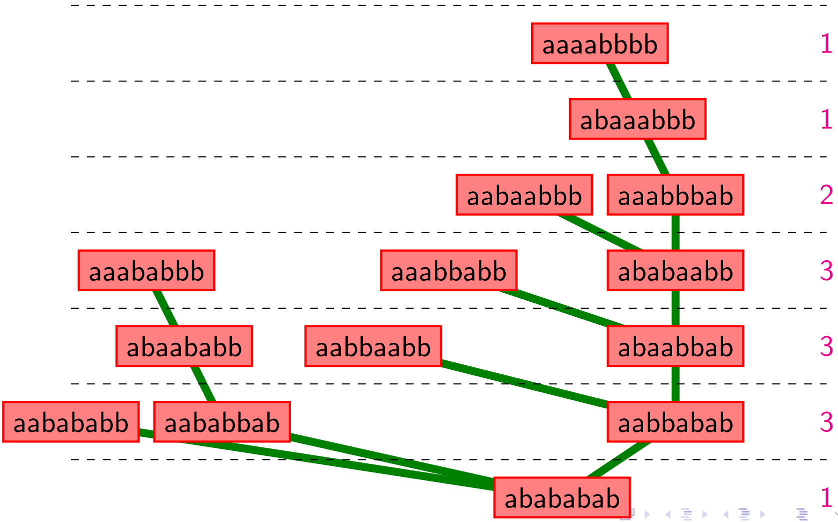
$u = (0, 0, 0, 3)$



$\rho(u) = -1 + 2 = 1$

The *prerank* of a Dyck word d is the number of loop iterations leading to the fixpoint belonging to $(ab)^*$.

beginframe


aaaabbbb
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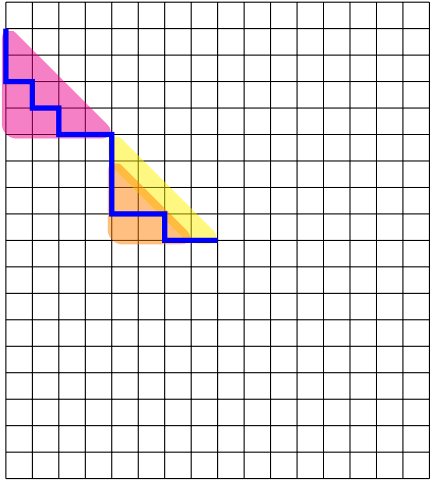
1

Symmetry of *area*, *prerank* distribution.



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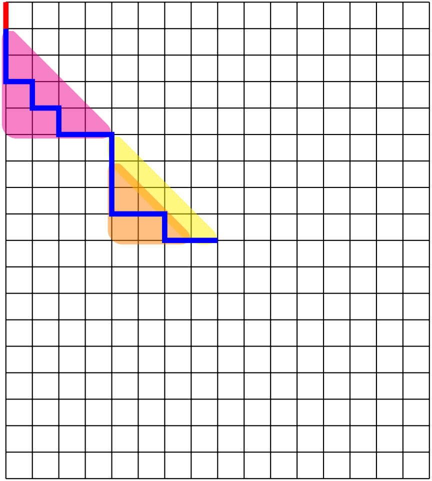

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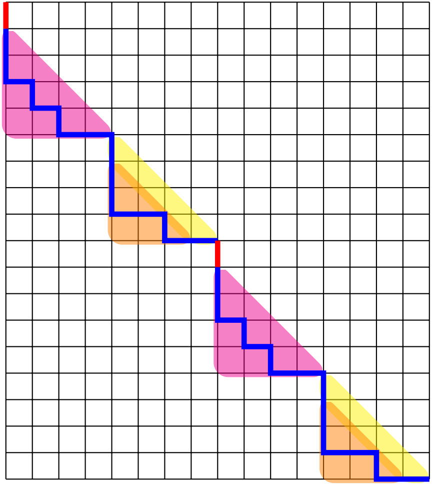

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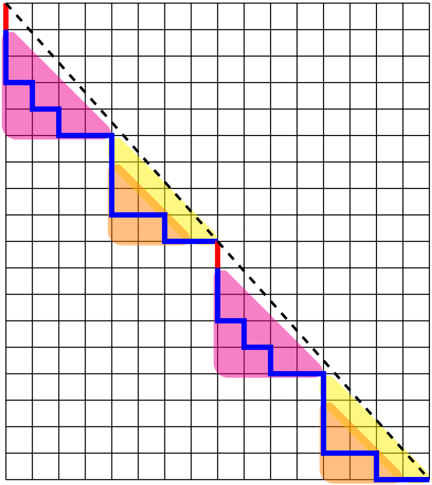

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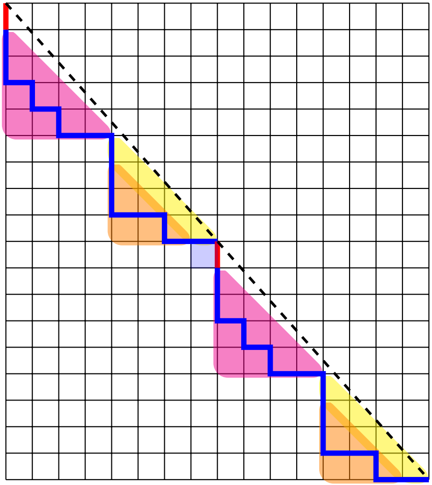

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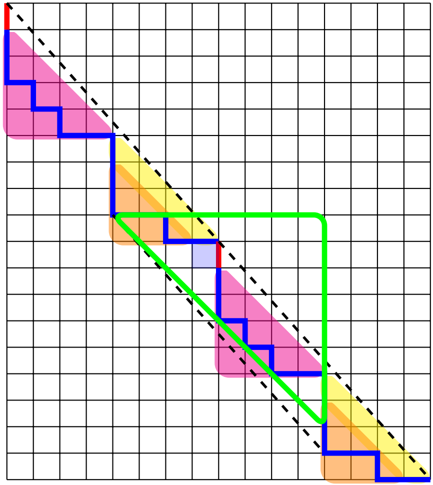

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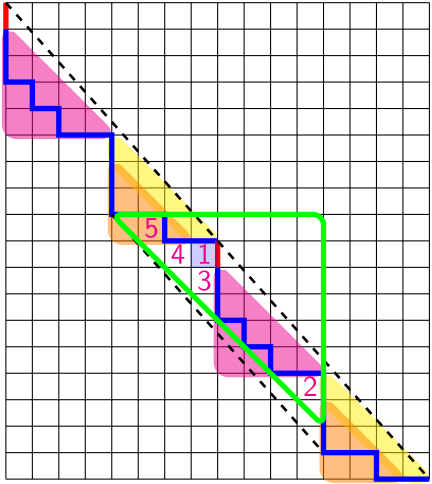

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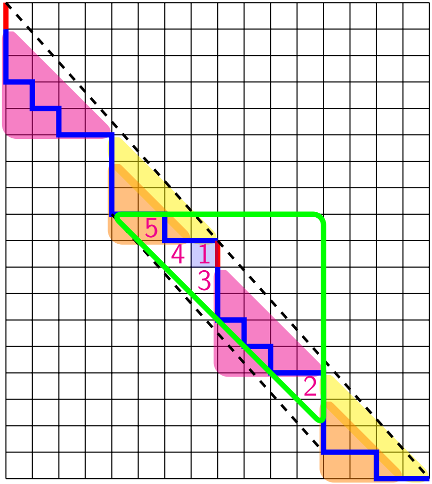
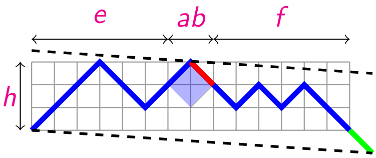


match d with



$\longrightarrow ebaf$

ea cumulant at height h

$\overline{f}$ cumulant at height $h-1$

The loss in sink is related to:

either previous vertex at height h

or last vertex at height $h-1$

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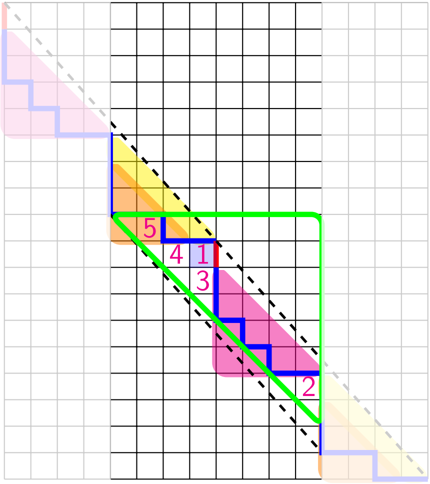
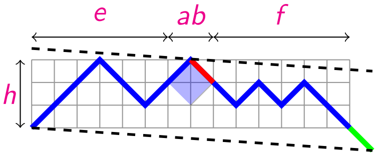


match d with



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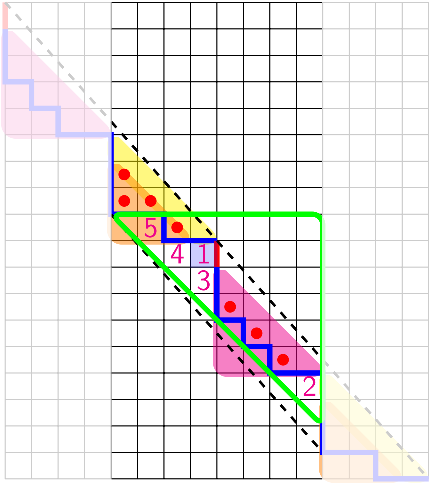
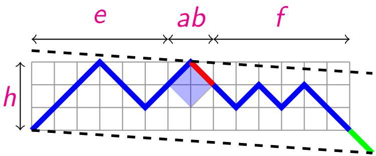


match *d* with



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