The probability of planarity of a random graph near the critical point

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The material of this talk

1.– **Planarity on the critical window for random graphs**

2.– **Our result. The strategy**

3.– **Generating Functions: algebraic methods**

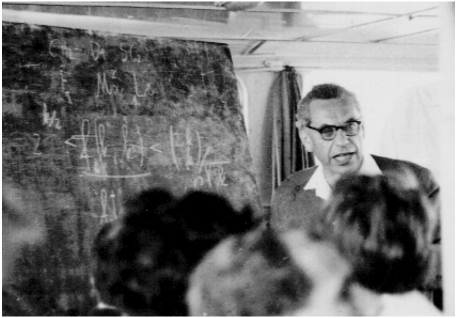
4.– **Cubic planar multigraphs**

5.– **Computing large powers: analytic methods**

6.– **Other applications**

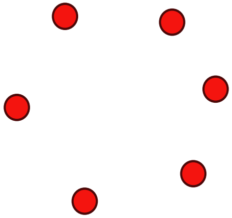
7.– **Further research**

Planarity on the critical window for random graphs



The model $G(n, p)$

Take n labelled vertices and a probability $p = p(n)$:



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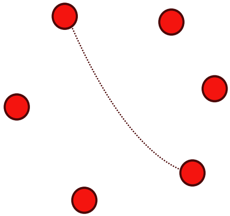

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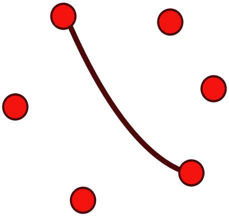

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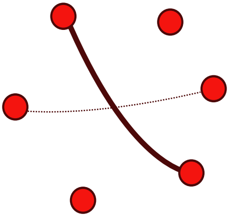

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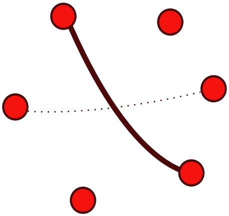

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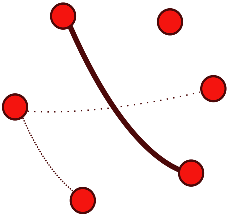

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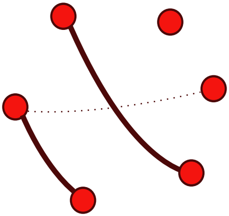

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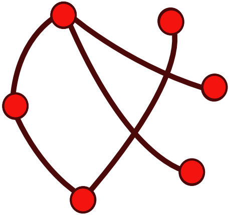


- ♡ Independence in the choice of edges. ✓
- ♣ The expected number of edges is $M = \binom{n}{2}p$. ✓
- ♠ We do not control the number of edges.

The model $G(n, M)$

There are $2^{\binom{n}{2}}$ labelled graphs with n vertices.

A random graph $G(n, M)$ is the probability space with properties:

- Sample space: set of labelled graphs with n vertices and $M = M(n)$ edges.
- Probability: Uniform probability $\left(\binom{\binom{n}{2}}{M}^{-1}\right)$

Properties:

♡ Fixed number of edges ✓

♣ The probability that a fixed edge belongs to the random graph is $p = \binom{n}{2}^{-1} M$. ✓

♠ There is not independence.

EQUIVALENCE: $G(n, p) = G(n, M), (n \to \infty)$ for

$$M = \binom{n}{2} p$$

The Erdős-Rényi phase transition

Random graphs in $G(n, M)$ present a dichotomy for $M = \frac{n}{2}$:

1.– **(Subcritical)** $M = cn$, $c < \frac{1}{2}$: a.a.s. all connected components have size $O(\log n)$, and are either trees or unicyclic graphs.

2.– **(Critical)** $M = \frac{n}{2} + Cn^{2/3}$: a.a.s. the largest connected component has size of order $n^{2/3}$

3.– **(Supercritical)** $M = cn$, $c > \frac{1}{2}$: a.a.s. there is a **unique** component of size of order n.

Double jump in the creation of the ***giant component***.

The problem; what was known

ON THE EVOLUTION OF RANDOM GRAPHS

by

P. ERDŐS and A. RÉNYI

Dedicated to Professor P. Turán at his 50th birthday.

We can show that for $N(n) = \frac{n}{2} + \lambda\sqrt{n}$ with any real λ the probability of $\Gamma_{n,N(n)}$ not being planar has a positive lower limit, but we cannot calculate ts value. It may even be 1, though this seems unlikely.

PROBLEM: Compute

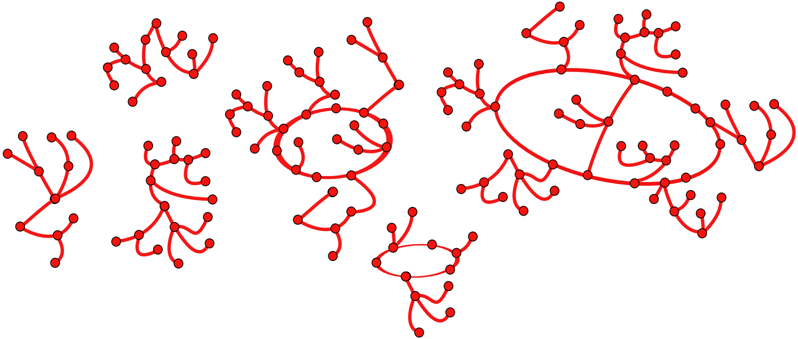
$$p(\lambda) = \lim_{n\to\infty} \Pr\left\{G\left(n, \tfrac{n}{2}(1+\lambda n^{-1/3})\right) \text{ is planar}\right\}$$

What was known:

- Janson, Łuczak, Knuth, Pittel (94): $0{,}9870 < p(0) < 0{,}9997$
- Łuczak, Pittel, Wierman (93): $0 < p(\lambda) < 1$

Our contribution: **the whole description of $p(\lambda)$**

Our result. The strategy



The main theorem

Theorem (Noy, Ravelomanana, R.) Let $g_r (2r)!$ be the number of cubic planar weighted multigraphs with $2r$ vertices. Write

$$A(y,\lambda) = \frac{e^{-\lambda^3/6}}{3^{(y+1)/3}} \sum_{k\geq 0} \frac{\left(\frac{1}{2}3^{2/3}\lambda\right)^k}{k!\,\Gamma\big((y+1-2k)/3\big)}.$$

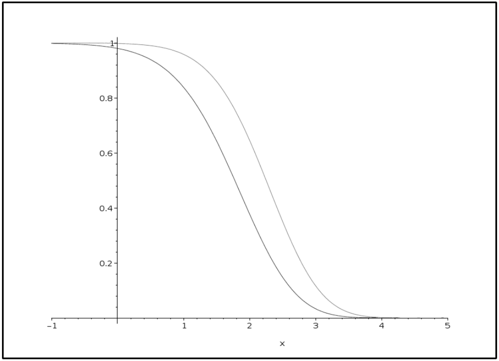
Then the limiting probability that the random graph $G\left(n, \frac{n}{2}(1+\lambda n^{-1/3})\right)$ is planar is

$$p(\lambda) = \sum_{r\geq 0} \sqrt{2\pi}\, g_r A\left(3r+\frac{1}{2}, \lambda\right).$$

In particular, the limiting probability that $G\left(n, \frac{n}{2}\right)$ is planar is

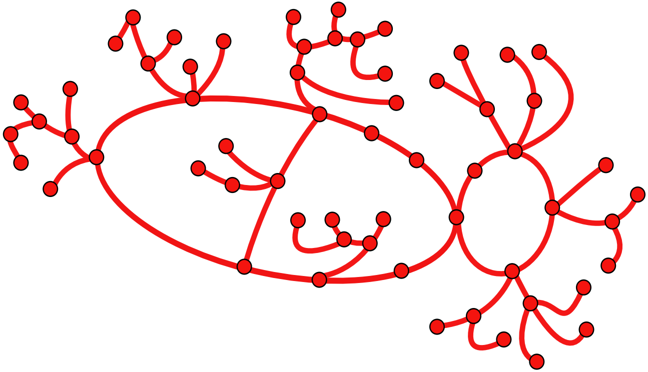
$$p(0) = \sum_{r\geq 0} \sqrt{\frac{2}{3}} \left(\frac{4}{3}\right)^r g_r \frac{r!}{(2r)!} \approx 0{,}99780.$$

A plot



Probability curve for planar graphs and SP-graphs
(top and bottom, respectively)

The strategy (I): pruning a graph



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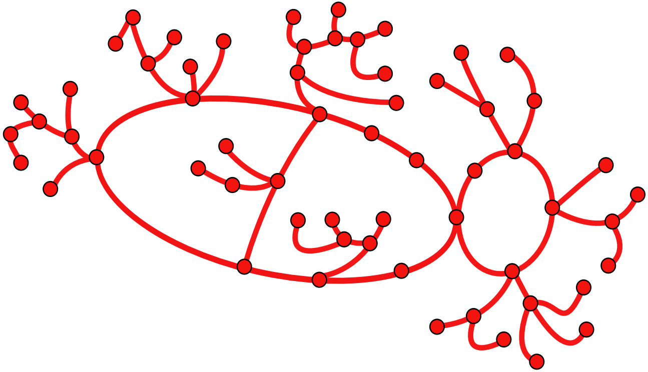

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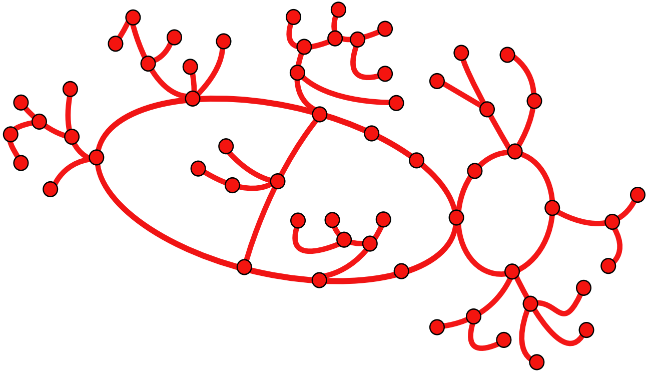

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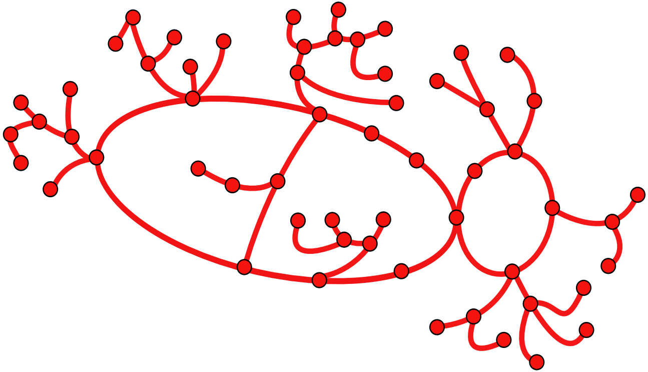

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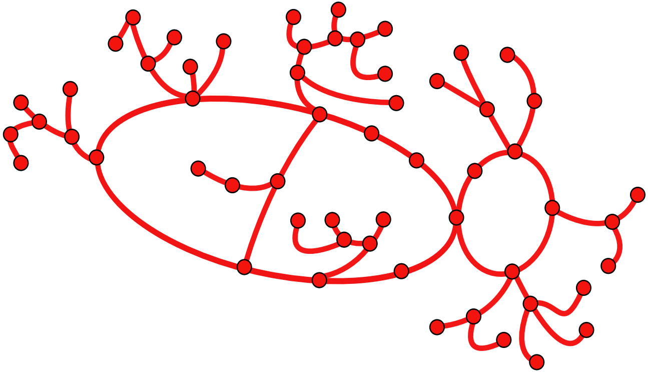

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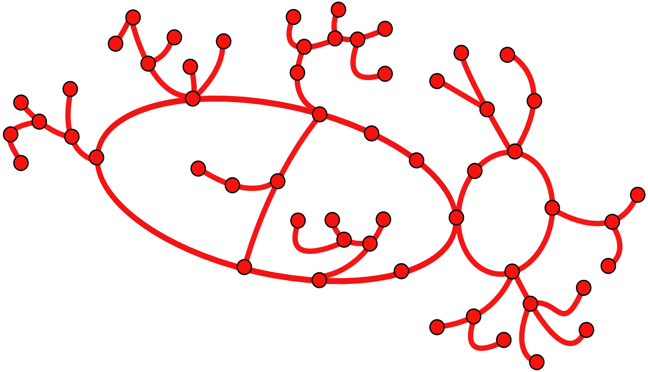

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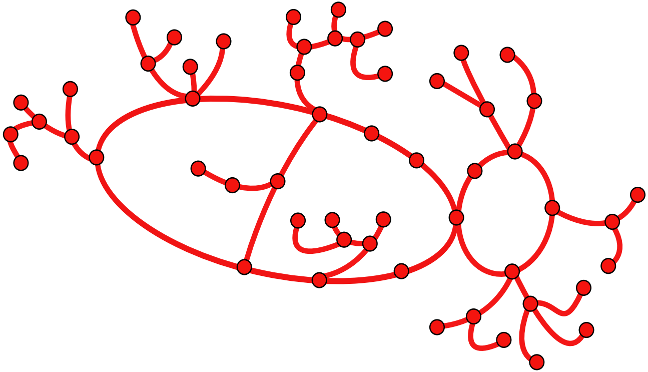

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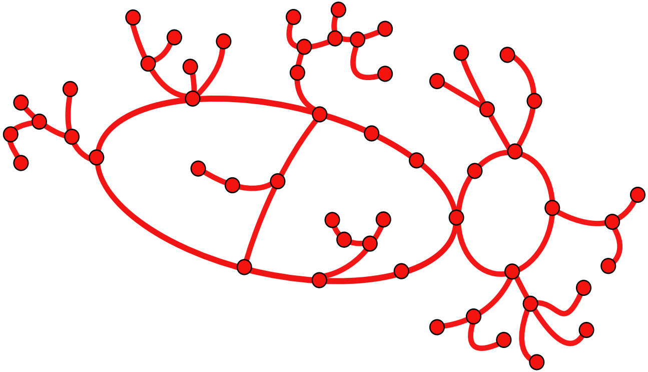

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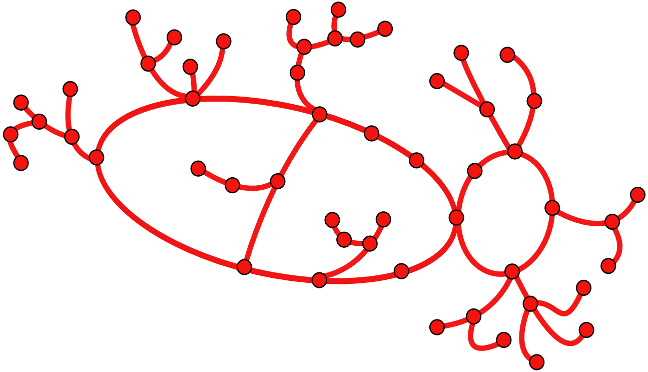

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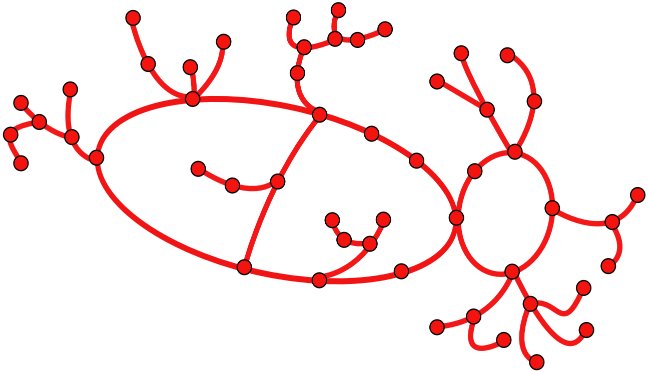

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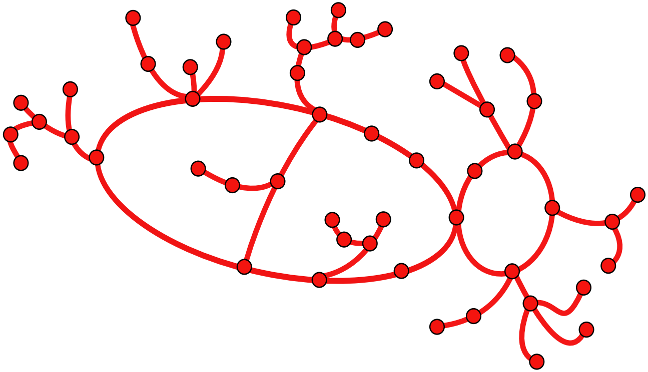

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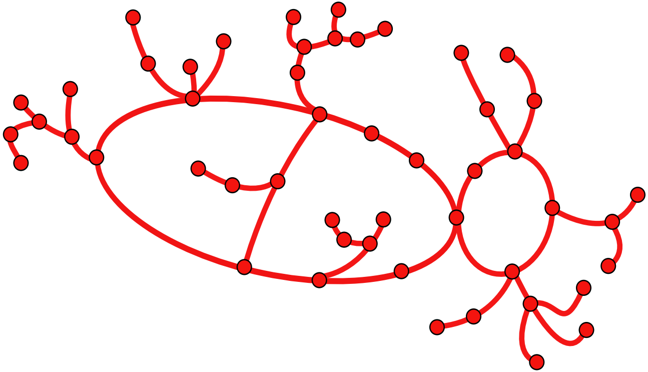

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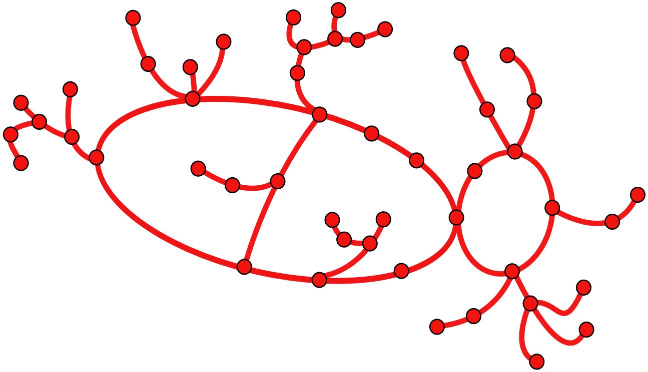

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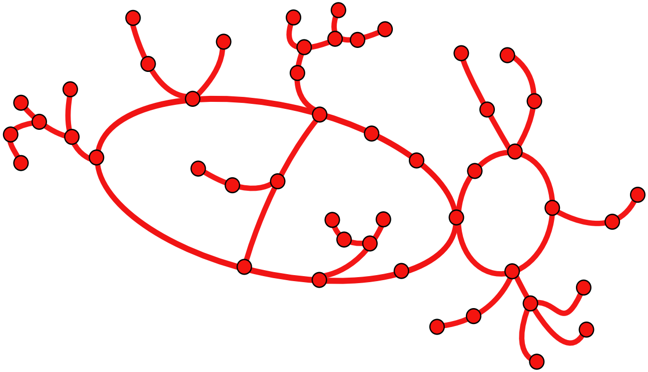

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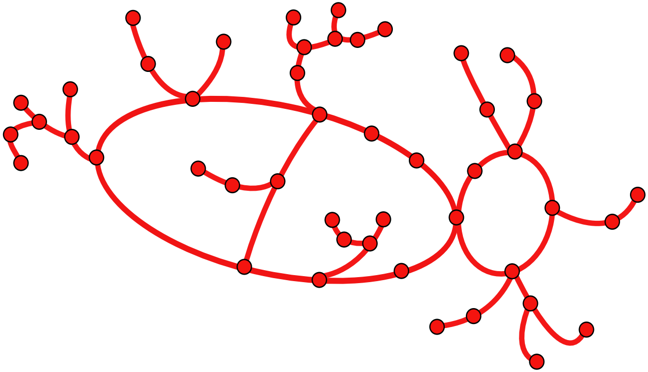

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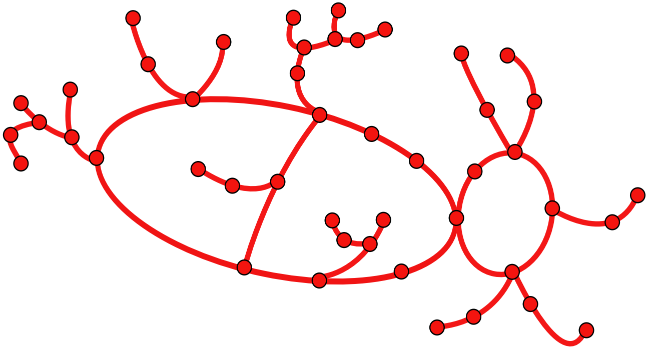

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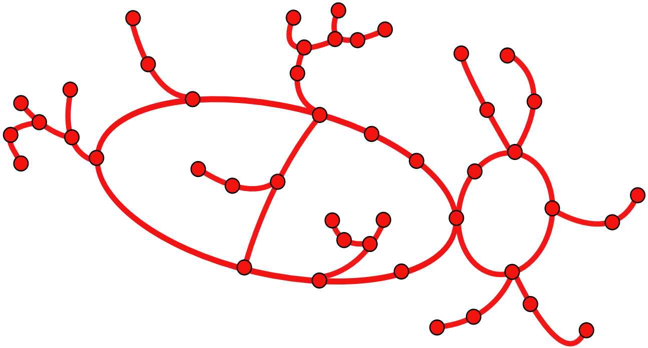

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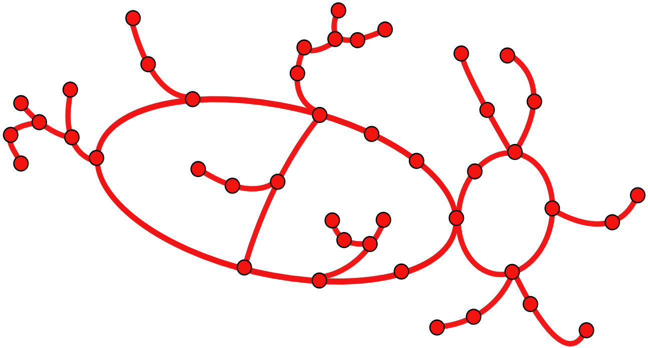

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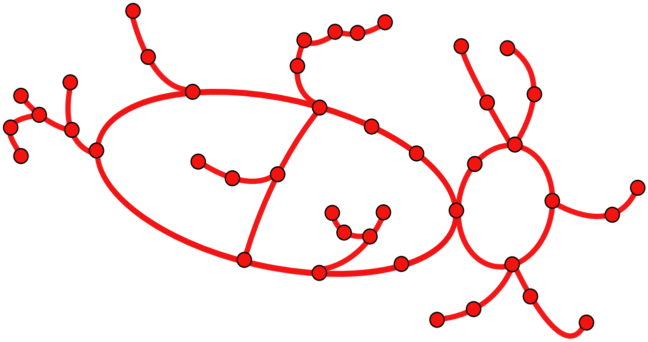

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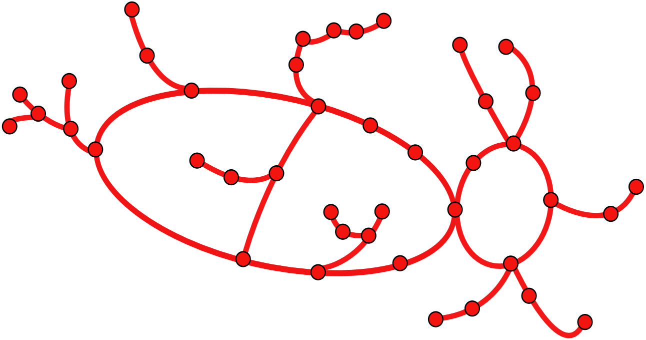

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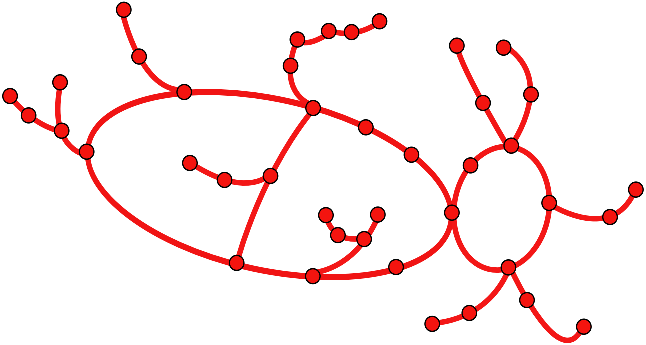

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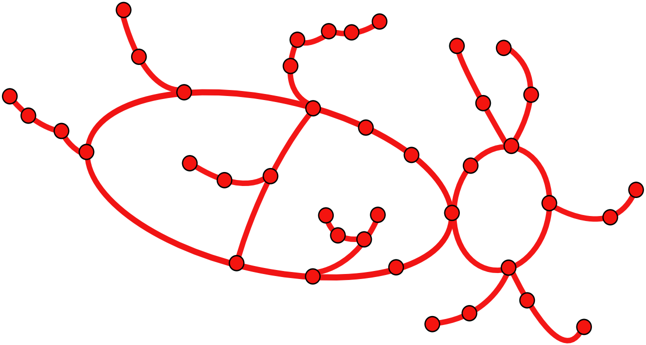

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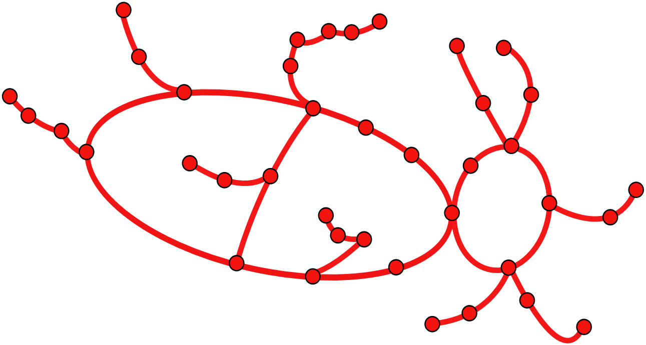

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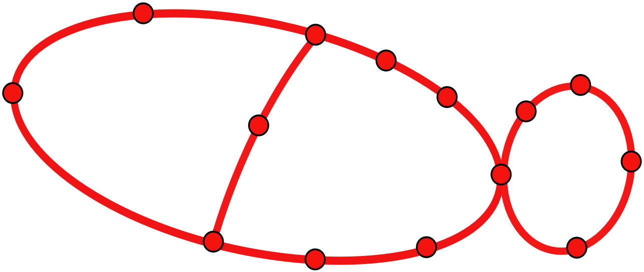

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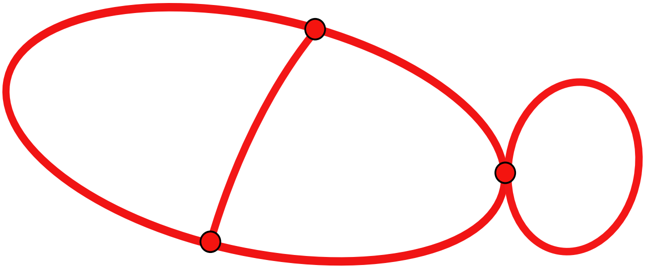
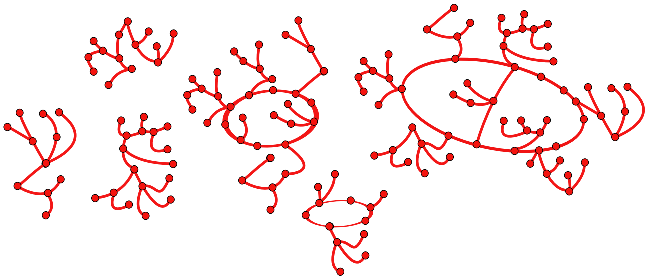


The resulting *multigraph* is the **core** of the initial graph

The strategy (and II): appearance in the critical window

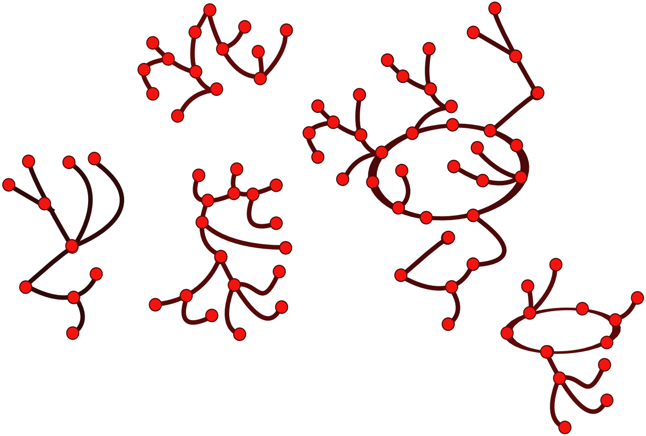
Łuczak, Pittel, Wierman (1994):
the structure of a random graph in the critical window



$$p(\lambda) = \frac{\text{number of planar graphs with } \frac{n}{2}(1+\lambda n^{-1/3}) \text{ edges}}{\binom{\binom{n}{2}}{\frac{n}{2}(1+\lambda n^{-1/3})}}$$

Hence...**We need to count!**

Generating Functions: algebraic methods



The symbolic method à la Flajolet

COMBINATORIAL RELATIONS between CLASSES

$$\Updownarrow$$

EQUATIONS between GENERATING FUNCTIONS

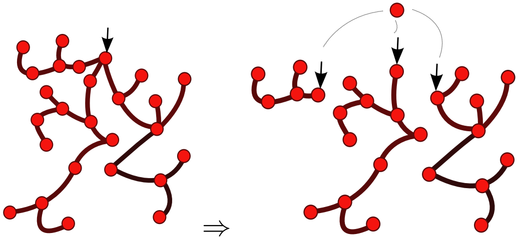
Class	Relations
$\mathcal{C} = \mathcal{A} \cup \mathcal{B}$	$C(x) = A(x) + B(x)$
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$C(x) = A(x) \cdot B(x)$
$\mathcal{C} = \mathrm{Seq}(\mathcal{B})$	$C(x) = (1 - B(x))^{-1}$
$\mathcal{C} = \mathrm{Set}(\mathcal{B})$	$C(x) = \exp(B(x))$
$\mathcal{C} = \mathcal{A} \circ \mathcal{B}$	$C(x) = A(B(x))$

All GF are *exponential* $\equiv$ *labelled* objects

$$A(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n.$$

First application: Trees

We apply the previous grammar to count *rooted* trees



$$\mathcal{T} = \bullet \times \mathrm{Set}(\mathcal{T}) \to T(x) = xe^{T(x)}$$

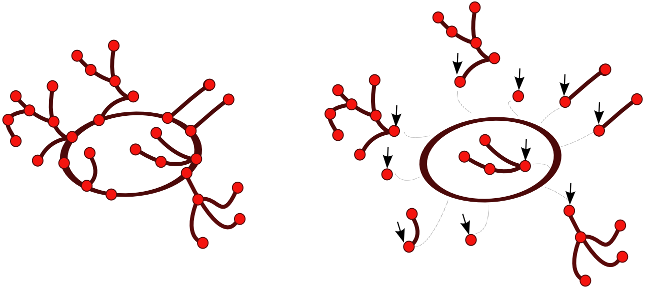
To forget the root, we just integrate: $(xU'(x) = T(x))$

$$\int_0^x \frac{T(s)}{s} ds = \left\{ \begin{array}{c} T(s) = u \\ T'(s)\, ds = du \end{array} \right\} = \int_{T(0)}^{T(x)} 1 - u\, du = T(x) - \frac{1}{2} T(x)^2$$

and the general version

$$e^{U(x)} = e^{T(x)} e^{-\frac{1}{2}T(x)^2}$$

Second application: Unicyclic graphs

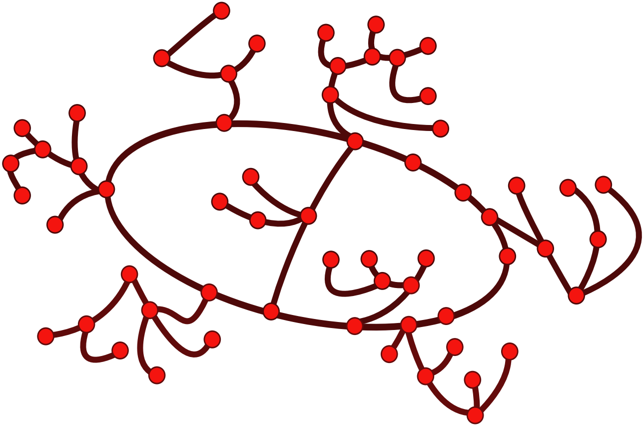


$$\mathcal{V} = \bigcirc_{\geq 3}(\mathcal{T}) \to V(x) = \sum_{n=3}^{\infty} \frac{1}{2} \frac{(n-1)!}{n!} (T(x))^n$$

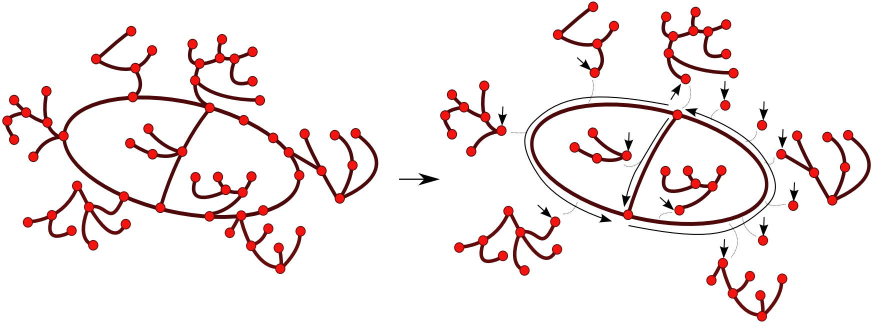
We can write $V(x)$ in a compact way:

$$\frac{1}{2}\left(-\log\left(1-T(x)\right) - T(x) - \frac{T(x)^2}{2}\right) \to e^{V(x)} = \frac{e^{-T(x)/2-T(x)^2/4}}{\sqrt{1-T(x)}}.$$

Cubic planar multigraphs



Planar graphs arising from cubic multigraphs



In an informal way:

$$\mathcal{G}(\bullet \leftarrow \mathcal{T}, \bullet - \bullet \leftarrow \mathrm{Seq}(\mathcal{T}))$$

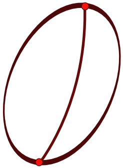
Weighted planar cubic multigraphs

Cubic multigraphs have $2r$ vertices and $3r$ edges (Euler relation)

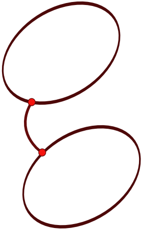
$$G(x,y) = \sum_{r\geq 1} \frac{g_r(2r)!}{(2r)!} x^{2r} y^{3r} = G(x^2y^3)$$

We need to remember the number of loops and the number of multiple edges to avoid symmetries:

weights $2^{-f_1-f_2}(3!)^{-f_3}$



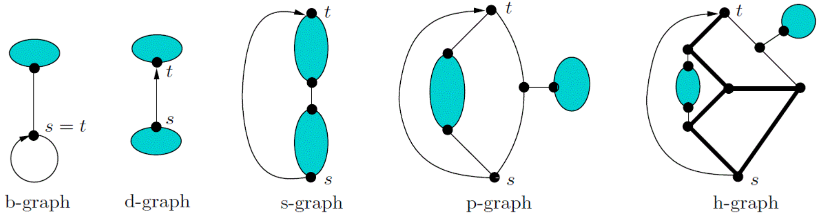
$\frac{1}{2!}\frac{1}{6}x^2y^3$



$\frac{1}{2!}\frac{1}{2^2}x^2y^3$

The decomposition

- We consider *rooted* multigraphs (namely, an edge is oriented).
- *Rooted* cubic planar multigraphs have the following form:



(From Bodirsky, Kang, Löffler, McDiarmid *Random Cubic Planar Graphs*)

The equations

We can relate different families of rooted cubic planar graphs between them:

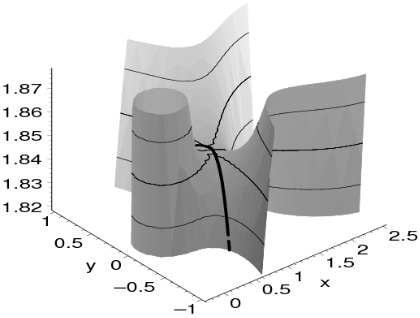
$$
\begin{aligned}
G(z) &= \exp G_1(z) \\
3z\,\frac{dG_1(z)}{dz} &= D(z) + C(z) \\
B(z) &= \frac{z^2}{2}(D(z) + C(z)) + \frac{z^2}{2} \\
C(z) &= S(z) + P(z) + H(z) + B(z) \\
D(z) &= \frac{B(z)^2}{z^2} \\
S(z) &= C(z)^2 - C(z)S(z) \\
P(z) &= z^2C(z) + \frac{1}{2}z^2C(z)^2 + \frac{z^2}{2} \\
2(1 + C(z))H(z) &= u(z)(1 - 2u(z)) - u(z)(1 - u(z))^3 \\
z^2(C(z) + 1)^3 &= u(z)(1 - u(z))^3.
\end{aligned}
$$

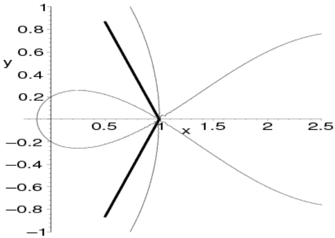
The equations: an appetizer

All GF obtained (except $G(z)$) are *algebraic* GF; for instance:

$$
\begin{aligned}
&1048576\, z^6 + 1034496\, z^4 - 55296\, z^2 + \\
&\left(9437184\, z^6 + 6731264\, z^4 - 1677312\, z^2 + 55296\right) C + \\
&\left(37748736\, z^6 + 18925312\, z^4 - 7913472\, z^2 + 470016\right) C^2 + \\
&\left(88080384\, z^6 + 30127104\, z^4 - 16687104\, z^2 + 1622016\right) C^3 + \\
&\left(132120576\, z^6 + 29935360\, z^4 - 19138560\, z^2 + 2928640\right) C^4 + \\
&\left(132120576\, z^6 + 19314176\, z^4 - 12429312\, z^2 + 2981888\right) C^5 + \\
&\left(88080384\, z^6 + 8112384\, z^4 - 4300800\, z^2 + 1720320\right) C^6 + \\
&\left(37748736\, z^6 + 2097152\, z^4 - 614400\, z^2 + 524288\right) C^7 + \\
&\left(9437184\, z^6 + 262144\, z^4 + 65536\right) C^8 + 1048576\, C^9 z^6 = 0.
\end{aligned}
$$

Computing large powers: analytic methods





Singularity analysis on generating functions

GFs: analytic functions in a neighbourhood of the origin.

The smallest singularity of $A(z)$ determines the asymptotics of the coefficients of $A(z)$.

- POSITION: exponential growth ρ.
- NATURE: subexponential growth
- Transfer Theorems: Let $\alpha \notin \{0, -1, -2, \dots\}$. If

$$A(z) = a \cdot (1 - z/\rho)^{-\alpha} + o((1 - z/\rho)^{-\alpha})$$

then

$$a_n = [z^n]A(z) \sim \frac{a}{\Gamma(\alpha)} \cdot n^{\alpha-1} \cdot \rho^{-n}(1 + o(1))$$

Our estimates

- The **excess** of a graph ($ex(G)$) is the number of edges minus the number of vertices

$$n![z^n]\ \overbrace{\frac{U(z)^{n-M+r}}{(n-M+r)!}}^{\text{Trees, } ex=-1}\ \overbrace{\frac{e^{-T(z)/2-T(z)^2/4}}{\sqrt{1-T(z)}}}^{\text{Unicyclic, } ex=0}\ \overbrace{\frac{P_{5r}(T(z))}{(1-T(z))^{3r}}}^{\text{Cubic, } ex=3r-2r=r}$$

Where $P_{5r}(x)$ is a polynomial of degree $\leq 5r$.

- We then apply a *sandwich* argument to get the estimates
- We use saddle point estimates (*a la Van der Corput*).

Without many details...

We estimate the constant using Stirling:

$$\frac{n!}{\binom{\binom{n}{2}}{M}} \frac{1}{(n-M+r)!} = \sqrt{2\pi n}\, \frac{2^{n-M+r}}{n^r} e^{-\lambda^3/6+3/4-n} \left(1 + O\left(\frac{\lambda^4}{n^{1/3}}\right)\right).$$

For every a, we study the asymptotic behavior of

$$[z^n] U(z)^{n-M+r} \frac{T(z)^{a} e^{V(z)}}{(1-T(z))^{3r}} = \frac{1}{2\pi i} \oint U(z)^{n-M+r} \frac{T(z)^{a}\, e^{V(z)}}{(1-T(z))^{3r}} \frac{dz}{z^{n+1}}$$

We write the integrand as $g(u)\, e^{nh(u)}$ $(u = T(z))$; relate with:

$$A(y,\lambda) = \frac{1}{2\pi i} \int_{\Pi} s^{1-y} e^{K(\lambda,s)} ds,\ K(\lambda,s) = \frac{s^3}{3} + \frac{\lambda s^2}{2} - \frac{\lambda^3}{6}$$

and Π is the following path in the complex plane:

$$s(t) = \begin{cases} -e^{-\pi i/3}\, t, & \text{for } -\infty < t \le -2, \\ 1 + it \sin \pi/3, & \text{for } -2 \le t \le +2, \\ e^{+\pi i/3}\, t, & \text{for } +2 \le t < +\infty. \end{cases}$$

Nice cancelations of $n \ldots$

Other applications

General families of graphs

Many families of graphs admit an straightforward analysis:

(Noy, Ravelomanana, R.)
Let $\mathcal{G} = \mathrm{Ex}(H_1, \ldots, H_k)$ and assume all the H_i are 3-connected. Let $h_r(2r)!$ be the number of cubic multigraphs in $\mathcal{G}$ with $2r$ vertices. Then the limiting probability that the random graph $G(n, \frac{n}{2}(1 + \lambda n^{-1/3}))$ is in $\mathcal{G}$ is

$$p_{\mathcal{G}}(\lambda) = \sum_{r \geq 0} \sqrt{2\pi}\, h_r A(3r + \frac{1}{2}, \lambda).$$

In particular, the limiting probability that $G(n, \frac{n}{2})$ is in $\mathcal{G}$ is

$$p_{\mathcal{G}}(0) = \sum_{r \geq 0} \sqrt{\frac{2}{3}} \left(\frac{4}{3}\right)^r h_r \frac{r!}{(2r)!}.$$

Moreover, for each λ we have

$$0 < p_{\mathcal{G}}(\lambda) < 1.$$

Examples...please

Some interesting families fit in the previous scheme:

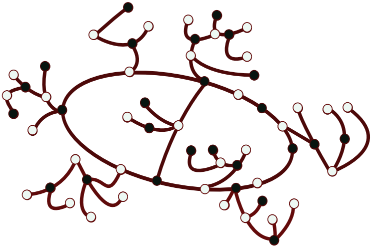
- $\mathsf{Ex}(K_4)$:series-parallel graphs: there are not 3-connected elements in the family!
- $\mathsf{Ex}(K_{2,3}, K_4)$: outerplanar graphs: need to adapt the equations for cubics.
- $\mathsf{Ex}(K_{3,3})$: The same limiting probability as planar...K_5 does not appear as a core!
- Many others: $\mathsf{Ex}(K_{3,3}^+)$, $\mathsf{Ex}(K_5^-)$, $\mathsf{Ex}(K_2 \times K_3)\ldots$

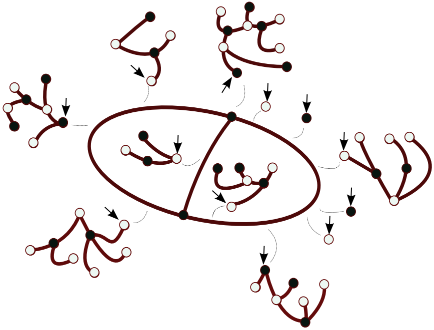
Further research

Bipartite planar graphs and the Ising model

What about ***bipartite*** planar graphs in the critical window?

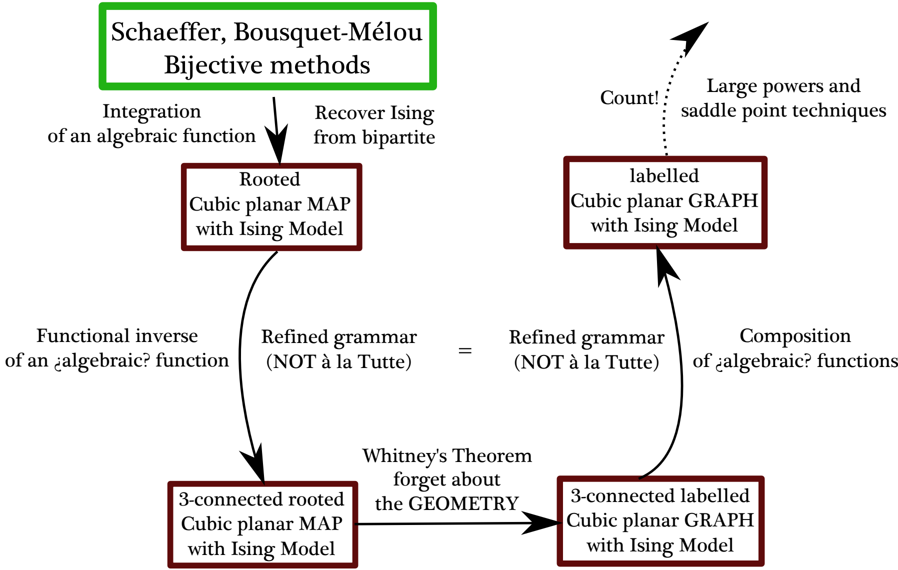
- Trees are always bipartite!
- Unicyclic bipartite graphs are characterized by a cycle of even lenght
- But...What about cubic multigraphs?





We need something more complicated: **ISING MODEL**

A program


Schaeffer, Bousquet-Mélou
Bijective methods
Integration
of an algebraic function
Recover Ising
from bipartite
Rooted
Cubic planar MAP
with Ising Model
Count!
Large powers and
saddle point techniques
labelled
Cubic planar GRAPH
with Ising Model
Functional inverse
of an ¿algebraic? function
Refined grammar
(NOT à la Tutte)
=
Refined grammar
(NOT à la Tutte)
Composition
of ¿algebraic? functions
Whitney's Theorem
forget about
the GEOMETRY
3-connected rooted
Cubic planar MAP
with Ising Model
3-connected labelled
Cubic planar GRAPH
with Ising Model

More problems (I)

Main result: structural behavior in the critical window

$$\Downarrow\downarrow\Downarrow$$

Can we say similar things for *planar* graphs with bounded vertex degree?

- Enumeration of 4-regular and $\{3,4\}-$regular planar graphs (**To be done**).
- Study of parameters: Airy distributions (**To be done**).
- Extend to the bipartite setting (**To be done**).

More problems (and II)

The asymptotic enumeration of *bipartite* planar graphs seems technically complicated (Bousquet-Mélou, Bernardi, 2009)

- Refine the grammar introduced by Chapuy, Fusy, Kang, Shoilekova, and study SP-graphs (**Work in progress**).
- Extend the formulas by Bousquet-Mélou, Bernardi to get the 3-connected planar components (Computationally involved!) (**??**)
- Study the full planar case ...

Gràcies!

The probability of planarity of a random graph near the critical point

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