The combinatorics of free bifibrations

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What is a bifibration?

One category living over another category, such that *objects* of the category above may be pushed and pulled along *arrows* of the category below.

Formally:

$$\begin{tikzcd} \mathcal{D} \arrow[d, "p"'] \\ \mathcal{C} \end{tikzcd}$$

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\begin{tikzcd}
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...and these liftings should be "universal" in an appropriate sense.

What is a bifibration? (cont.)

Pushing and pulling along an arrow $f : A \to B$ of $\mathcal{C}$ induces an *adjunction*

$$\begin{tikzcd} \mathcal{D}_A \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & \mathcal{D}_B \arrow[l, bend left, "f^*"] \end{tikzcd}$$

between the *fibers* of A and B.

This leads to an equivalent way of seeing bifibrations $\mathcal{D} \to \mathcal{C}$, as pseudofunctors $\mathcal{C} \to \mathcal{A}\mathrm{dj}$ into the category of small categories and adjunctions.

A simple example

Let $\mathcal{S}\mathrm{et}$ be the category of sets and functions.

Let $\mathcal{S}\mathrm{ubset}$ be the category whose objects are subsets, and whose arrows $(S \subseteq A) \longrightarrow (T \subseteq B)$ are functions $f : A \to B$ such that $a \in S$ implies $f(a) \in T$.

The evident forgetful functor $\mathcal{S}\mathrm{ubset} \to \mathcal{S}\mathrm{et}$ is a bifibration:

$$
\begin{array}{ccccccc}
\mathcal{S}\mathrm{ubset} & \quad & S \xrightarrow{f_S} f(S) & \quad & g^{-1}(T) \xrightarrow{\bar{g}_T} T \\
\big\downarrow & & \rotatebox{90}{$\subseteq$} \qquad\quad \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \qquad\qquad \rotatebox{90}{$\subseteq$} \\
\mathcal{S}\mathrm{et} & & A \xrightarrow{\quad f \quad} B & & B \xrightarrow{\qquad g \qquad} C
\end{array}
$$

(Adjunction property: $f(S) \subseteq R \iff S \subseteq f^{-1}(R)$.)

Other motivating examples, from logic

Pushforward and pullback may be used to express:

- strongest postconditions and weakest preconditions in program logic
- existential and universal quantification in predicate logic
- diamond and box in modal logic
- $\otimes$ and $\parr$ in linear logic

Our problem

Most functors are not bifibrations.
Given a functor $p : \mathcal{D} \to \mathcal{C}$, how do we construct the **free bifibration** over p?

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "\eta_p"] \arrow[dr, "p"'] & & \mathcal{B}\mathrm{Fib}(p) \arrow[dl, "\tilde{p}\ \text{bifibration}"] \\
& \mathcal{C} &
\end{tikzcd}$$

A relatively little-studied problem:

- Robert Dawson, Robert Paré, and Dorette Pronk. Adjoining adjoints. *Advances in Mathematics*, 178(1):99–140, 2003.
- François Lamarche. Path functors in Cat. Unpublished, 2010. `https://hal.inria.fr/hal-00831430`.

Overview of our work

Developed alternative constructions of the free bifibration over a functor $p : \mathcal{D} \to \mathcal{C}$

- a *proof-theoretic* construction, using sequent calculus
- an *algebraic* construction, using double categories
- a *topological* construction, using string diagrams

(These provide three different perspectives, but all closely related.)

We also discovered examples of specific functors $p : \mathcal{D} \to \mathcal{C}$, such that the free bifibration over p has some surprisingly nice combinatorics.

A sequent calculus for the free bifibration over $p : \mathcal{D} \to \mathcal{C}$

Formulas ($S \sqsubset A$):

$$\frac{X \in \mathcal{D} \quad p(X) = A}{X \sqsubset A} \qquad \frac{S \sqsubset A \quad f : A \to B}{f_* S \sqsubset B} \qquad \frac{f : A \to B \quad T \sqsubset B}{f^* T \sqsubset A}$$

Proofs ($S \underset{h}{\Longrightarrow} T$):

$$\frac{S \underset{fg}{\Longrightarrow} T}{f_* S \underset{g}{\Longrightarrow} T}\ L_f \qquad \frac{S \underset{g}{\Longrightarrow} T}{S \underset{gf}{\Longrightarrow} f_* T}\ R_f \qquad \frac{S \underset{g}{\Longrightarrow} T}{f^* S \underset{fg}{\Longrightarrow} T}\ L_{\bar{f}} \qquad \frac{S \underset{gf}{\Longrightarrow} T}{S \underset{g}{\Longrightarrow} f^* T}\ R_{\bar{f}}$$

$$\frac{\alpha : X \longrightarrow Y \in \mathcal{D} \quad p(\alpha) = g}{X \underset{g}{\Longrightarrow} Y}\ \alpha$$

Equational theory on derivations

Need to impose four **permutation equivalences** on derivations, including

$$\dfrac{\dfrac{S \underset{fg}{\Longrightarrow} T}{S \underset{fgh}{\Longrightarrow} h_* T}\; R_h}{f_* S \underset{gh}{\Longrightarrow} h_* T}\; L_f \sim \dfrac{\dfrac{S \underset{fg}{\Longrightarrow} T}{f_* S \underset{g}{\Longrightarrow} T}\; L_f}{f_* S \underset{gh}{\Longrightarrow} h_* T}\; R_h \qquad \dfrac{\dfrac{S \underset{g}{\Longrightarrow} T}{S \underset{gh}{\Longrightarrow} h_* T}\; R_h}{f^* S \underset{fgh}{\Longrightarrow} h_* T}\; L_{\bar{f}} \sim \dfrac{\dfrac{S \underset{g}{\Longrightarrow} T}{S^* \underset{fg}{\Longrightarrow} T}\; L_{\bar{f}}}{f^* S \underset{fgh}{\Longrightarrow} h_* T}\; R_h$$

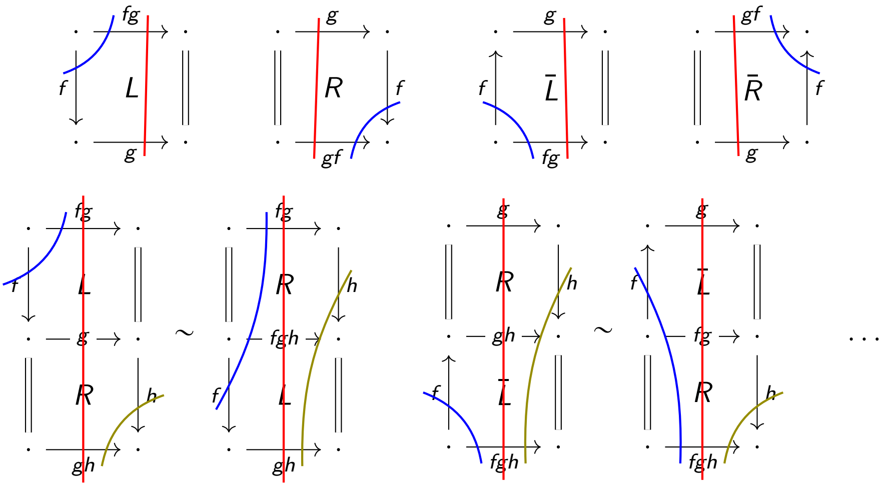
plus their symmetric versions with pushforward and pullback swapped.

Arrows of $\mathcal{B}\mathrm{Fib}(p)$ are equivalence classes of proofs. Composition is by cut-elimination.

Example derivations

$$\cfrac{\cfrac{\cfrac{}{S \underset{A}{\Longrightarrow} S}\ \mathrm{id}_S}{S \underset{f}{\Longrightarrow} f_*\, S}\ R_f}{S \underset{A}{\Longrightarrow} f^*\, f_*\, S}\ R_{\bar{f}} \qquad \cfrac{\cfrac{S \underset{A}{\Longrightarrow} S'}{S \underset{f}{\Longrightarrow} f_*\, S'}\ R_f}{f_*\, S \underset{B}{\Longrightarrow} f_*\, S'}\ L_f \qquad \cfrac{\cfrac{T \underset{B}{\Longrightarrow} T'}{f^*\, T \underset{f}{\Longrightarrow} T'}\ L_{\bar{f}}}{f^*\, T \underset{A}{\Longrightarrow} f^*\, T'}\ R_{\bar{f}} \qquad \cfrac{\cfrac{\cfrac{}{T \underset{B}{\Longrightarrow} T}\ \mathrm{id}_T}{f^*\, T \underset{f}{\Longrightarrow} T}\ L_{\bar{f}}}{f_*\, f^*\, T \underset{B}{\Longrightarrow} T}\ L_f$$

Construction via the double category of zigzags, and via string diagrams



Canonical forms in general

A challenge in understanding free bifibrations is getting a handle on the equivalence classes (of proofs/double cells/string diagrams) induced by the permutation relations.

Note that equivalence is in general undecidable![1]

Nevertheless, we (believe we) have a normal form based on *maximal multifocusing*...

$$\frac{N \underset{\pi f}{\Longrightarrow} P}{\pi_* N \underset{f}{\Longrightarrow} P}\ L_\pi \qquad \frac{N \underset{\pi f \rho}{\Longrightarrow} P}{\pi_* N \underset{f}{\Longrightarrow} \rho^* P}\ L_\pi R_{\bar\rho} \qquad \frac{N \underset{f \rho}{\Longrightarrow} P}{N \underset{f}{\Longrightarrow} \rho^* P}\ R_{\bar\rho}$$

$$\frac{P \underset{f}{\Longrightarrow} Q}{\pi^* P \underset{\pi f}{\Longrightarrow} Q}\ L_{\bar\pi} \qquad \frac{P \underset{f}{\Longrightarrow} N}{\pi^* P \underset{\pi f \rho}{\Longrightarrow} \rho_* N}\ L_{\bar\pi} R_\rho \qquad \frac{M \underset{f}{\Longrightarrow} N}{M \underset{f \rho}{\Longrightarrow} \rho_* N}\ R_\rho$$

[1] By adapting a construction in: Robert Dawson, Robert Paré, and Dorette Pronk. Undecidability of the free adjoint construction. *Applied Categorical Structures*, 11:403–419, 2003.

Now for some examples!

Example #1

Consider the following functor:

$$\begin{tikzcd} 1 \arrow[d, "p_0"'] & 0 \arrow[d, dotted, no head] & \\ 2 & 0 \arrow[r, "f"] & 1 \end{tikzcd}$$

Build the free bifibration $\mathcal{B}\mathrm{Fib}(p_0) \to 2$, and look at the fiber of 0.

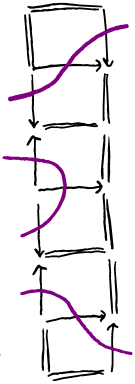
Objects are isomorphic to even-length alternating push/pull sequences $f^* f_* \cdots f^* f_* 0$

Let $d_{m,n}$ be the number of arrows $(f^* f_*)^m 0 \longrightarrow (f^* f_*)^n 0$?

Puzzle: what is $d_{m,n}$?

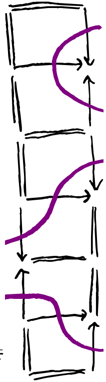
$d_{2,1} = 1$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0 \xRightarrow[0]{} 0}\ id}{0 \xRightarrow[f]{} f_*\, 0}\ R_f}{f_*\, 0 \xRightarrow[1]{} f_*\, 0}\ L_f}{f^*\, f_*\, 0 \xRightarrow[f]{} f_*\, 0}\ L_{\bar{f}}}{f_*\, f^*\, f_*\, 0 \xRightarrow[1]{} f_*\, 0}\ L_f}{f^*\, f_*\, f^*\, f_*\, 0 \xRightarrow[f]{} f_*\, 0}\ L_{\bar{f}}}{f^*\, f_*\, f^*\, f_*\, 0 \xRightarrow[0]{} f^*\, f_*\, 0}\ R_{\bar{f}}
$$



$d_{1,2} = 2$

$$\begin{array}{cl}
\hline 0 \xRightarrow[0]{} 0 & \mathrm{id}_0 \\
\hline 0 \xRightarrow[f]{} f_* \, 0 & R_f \\
\hline 0 \xRightarrow[0]{} f^* \, f_* \, 0 & R_{\bar{f}} \\
\hline 0 \xRightarrow[f]{} f_* \, f^* \, f_* \, 0 & R_f \\
\hline f_* \, 0 \xRightarrow[1]{} f_* \, f^* \, f_* \, 0 & L_f \\
\hline f^* \, f_* \, 0 \xRightarrow[f]{} f_* \, f^* \, f_* \, 0 & L_{\bar{f}} \\
\hline f^* \, f_* \, 0 \xRightarrow[0]{} f^* \, f_* \, f^* \, f_* \, 0 & R_{\bar{f}}
\end{array}$$



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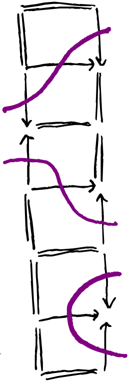
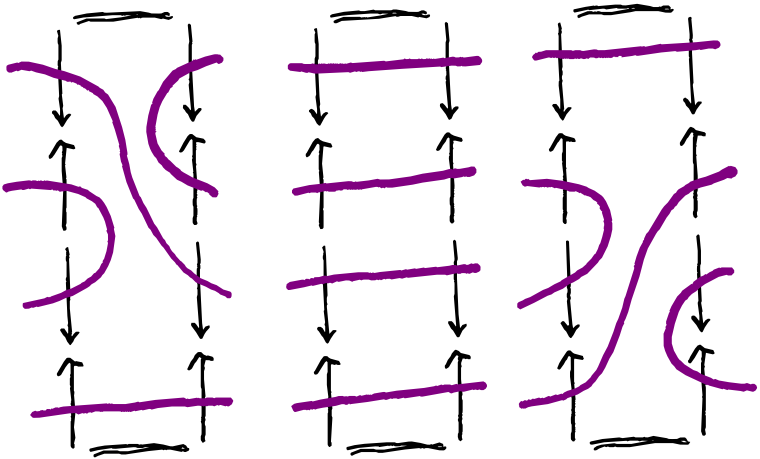
\hline 0 \xRightarrow[f]{} f_* \, 0 & R_f \\
\hline f_* \, 0 \xRightarrow[1]{} f_* \, 0 & L_f \\
\hline f^* \, f_* \, 0 \xRightarrow[f]{} f_* \, 0 & L_{\bar{f}} \\
\hline f^* \, f_* \, 0 \xRightarrow[0]{} f^* \, f_* \, 0 & R_{\bar{f}} \\
\hline f^* \, f_* \, 0 \xRightarrow[f]{} f_* \, f^* \, f_* \, 0 & R_f \\
\hline f^* \, f_* \, 0 \xRightarrow[0]{} f^* \, f_* \, f^* \, f_* \, 0 & R_{\bar{f}}
\end{array}$$



$d_{2,2} = 3$



$$f^* \, f_* \, f^* \, f_* \, 0 \underset{0}{\Longrightarrow} f^* \, f_* \, f^* \, f_* \, 0$$

Example #1 continued

Arrows $(f^* f_*)^m 0 \longrightarrow (f^* f_*)^n 0$ correspond to monotone maps $\mathsf{m} \to \mathsf{n}$!

Indeed, the free bifibration over $p_0 : 1 \to 2$ captures the adjunction

$$\begin{tikzcd} \Delta \arrow[r, bend left, "f_*"] \arrow[r, phantom, "\perp"] & \Delta_\perp \arrow[l, bend left, "f^*"] \end{tikzcd}$$

between the category Δ of finite ordinals and order-preserving maps, and the category $\Delta_\perp$ of non-empty finite ordinals and order-and-least-element-preserving maps.

. . . So what's the answer to the puzzle?

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...So what's the answer to the puzzle? $d_{m,n} = \binom{n+m-1}{m}$

Example #2

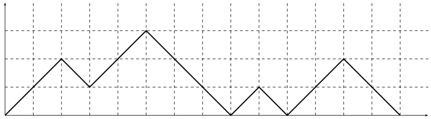
Now consider the following functor:

$$
\begin{tikzcd}
1 \arrow[d, "p_0"'] & 0 \arrow[d, dotted, no head] & & & \\
\mathbb{N} & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \dots
\end{tikzcd}
$$

Build the free bifibration $\mathcal{B}\mathrm{Fib}(p_0) \to \mathbb{N}$, and look at the fiber of 0.

Puzzle: what are its objects?

A category with Dyck walks as objects!

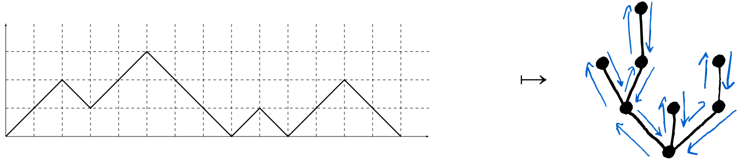
$f^* f^* f_* f_* f^* f_* f^* f^* f^* f_* f_* f^* f_* f_* 0 \quad =$ 

But what is a *morphism* of Dyck walks??

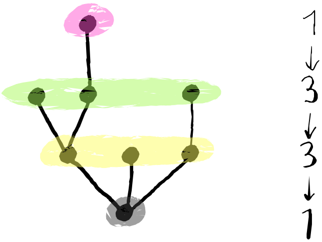
The $\mathcal{B}\mathrm{Fib}(-)$ construction gives an answer. Is it something natural/known?

Reconstructing the Batanin-Joyal category of trees

Dyck paths have a well-known, canonical bijection with (finite rooted plane) trees.



Trees may also be encoded as *functors* $T : \mathbb{N}^{\mathrm{op}} \to \Delta$.



Reconstructing the Batanin-Joyal category of trees

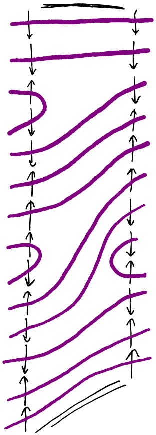
Consider *natural transformations* $\theta : S \Rightarrow T$.

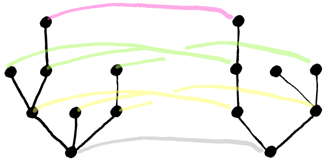
$$
\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] \\
S(2) \arrow[rr, "\theta_2"] \arrow[d] & & T(2) \arrow[d] \\
S(1) \arrow[rr, "\theta_1"] \arrow[d] & & T(1) \arrow[d] \\
S(0) = 1 \arrow[rr, equal] & & 1 = T(0)
\end{tikzcd}
$$

In other words, map nodes to nodes of the same height, respecting parents.

Reconstructing the Batanin-Joyal category of trees

Theorem: $\mathcal{B}\mathrm{Fib}(p_0 : 1 \to \mathbb{N})_0 \cong \mathsf{PTree}$.





(More generally, $\mathcal{B}\mathrm{Fib}(p_0)_k \cong \mathsf{PTree}_k =$ category of finite rooted plane trees whose rightmost branch is pointed by a node of height k.)

Example #2 continued

Fix a walk W, and consider the following pair of sequences:

$$in[W]_n = \#\{\, \theta : S \Rightarrow W \mid |S| = n \,\} \qquad out[W]_n = \#\{\, \theta : W \Rightarrow T \mid |T| = n \,\}$$

These seem to be always nice!

W	$out[W]$	$in[W]$
ϵ	A000108	A000007
UD	A000245	A000012
$UUDD$	A000344	A011782
$UDUD$	A099376	A000027

W	$out[W]$	$in[W]$
$UUUDDD$	A000588	A001519
$UUDUDD$	A003517	A001792
$UUDDUD$	A003517	A000079
$UDUUDD$	A003517	A000079
$UDUDUD$	A000344	A000217
$UUDUUDDD$	A003518	A061667

Conclusion

We have a clean and simple construction of the free bifibration over a functor.

An application of proof theory, w/complementary algebraic & topological perspectives.

Some surprisingly rich combinatorics emerges as if out of thin air.

Thanks!