The Lambda Calculus, its Syntax and Semantics 40 years later

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Theoretical Foundations of Computer Science

Turing Machines

Alan Turing $\simeq$1936
Manipulating symbols via primitive instructions.

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The Lambda Calculus

Church $\simeq$1932
Based on a primitive notion of function.

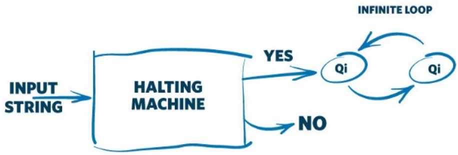
- variable: x
- abstraction: $\lambda x.P$, read $f(x) = P$
- application: PQ, read $P(Q)$

Computation becomes substitution: $(\lambda x.P)Q \to_\beta P[x := Q]$

Self-reference

Russell's paradox. Let $R = \{x \mid x \notin x\}$, then $R \in R \iff R \notin R$.

Turing Machines and the Halting Problem



Key point: Encode programs with natural numbers: $P(n)$ where $n = \#P$

Lambda calculus

Programs can be seen both as functions and as arguments.

$$\Omega = (\lambda x.xx)(\lambda x.xx) \to_\beta \Omega \to_\beta \Omega \to_\beta \cdots$$

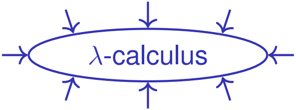
A wealth of techniques are employed

Category Theory

Recursion Theory

Topology

Logic



Universal Algebra

Term Rewriting

Proof Theory

Sharing Graphs

Complexity

Type Theory

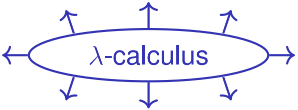
Lambda calculus has many applications

Programming Languages

Set Theory

Proof Assistants

Logic



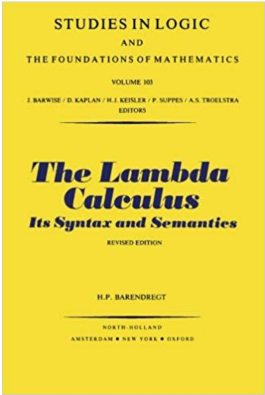
Universal Algebra

Infinitary Rewriting

Cost Models

Type Systems

The “Bible” of Lambda Calculus — 1981/84



The Lambda Calculus
Its Syntax and Semantics

by Henk Barendregt

Best seller, translated in

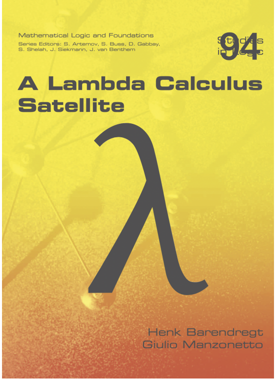
- Russian (MIR),
- Chinese (Nanjing University Press).

$>$ 11 000 copies sold.

Problems & conjectures:

- Complexity of $\boldsymbol{\lambda}\omega$,
- Range property for $\mathcal{H}$,
- Sallé’s conjecture,
- Bijectivity in $\boldsymbol{\lambda}\eta$
- $\cdots$

The “New Testament” of λ-calculus?



A Lambda Calculus Satellite

by Henk Barendregt

& Giulio Manzonetto

Contents in short

Overview of this talk

1. The plane conjecture [Reduction]
2. Bijectivity and invertibility in $\boldsymbol{\lambda}\eta$ [Conversion]
3. Dance of the starlings [Combinatory Logic]

— Reduction —
Leaving a β-reduction plane

A β-reduction plane

Definition

(i) Define an equivalence relation: $M \circlearrowleft N \iff M \twoheadrightarrow_\beta N \twoheadrightarrow_\beta M$.

(ii) Planes = the equivalence classes $[M]_\circlearrowleft$.

(iii) M is an exit (of its plane) if $M \to_\beta N$ and $N \notin [M]_\circlearrowleft$.

Example:

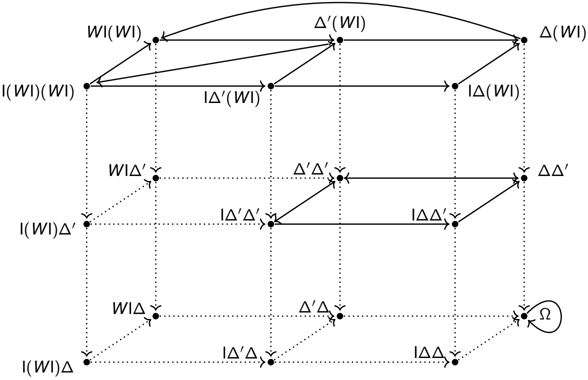
$\mathsf{I} = \lambda x.x$

$\mathsf{W} = \lambda xy.xyy$

$\Delta = \lambda x.xx$

$\Omega = \Delta\Delta$

$\Delta' = \lambda y.\mathsf{I}yy \to_\beta \Delta$



The Plane Conjecture

"If a plane has an exit point, then every point of the plane is an exit."
Klop's conjecture (1980)

Theorem (Mulder 1984 / Sekimoto-Hirokawa 1986)

The plane conjecture is invalid.

PROOF (MULDER). Define

$$\begin{aligned} H &= \lambda fg.ff(\lambda y.g(gy)), \\ P &= (\lambda x.\mathrm{I})z, \text{for a variable } z, \\ P^\eta &= \lambda x.Px. \end{aligned}$$

Then $M = HH(\lambda y.P(Py))$ is not an exit, but reduces to one. so the variable z is erased forever,

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One has $P \to_\beta \mathsf{I}$, so the variable z is erased forever,
$P^\eta = \lambda x.Px \to_\beta \lambda x.\mathsf{I}x \to_\beta \mathsf{I}$
$P^\eta X \to_\beta PX \to_\beta \mathsf{I}X \to_\beta X$

Mulder's Proof (continues. . .)

Consider $M = HH(\lambda y.P(Py))$. The only three 1-step reducts are the following

$$(\lambda g.HH(\lambda y.g(gy))(\lambda y.P(Py)),\ HH(\lambda y.\mathsf{I}(Py)),\ HH(\lambda y.P(\mathsf{I}y))$$

These all three flow back to M and hence are in its plane:

$$\begin{tikzcd}
M = HH(\lambda y.P(Py)) \arrow[r] \arrow[d] \arrow[dr] & HH(\lambda y.\mathsf{I}(Py)) \arrow[r, two heads, "\text{exit}"] \arrow[dd, bend left] & HH\mathsf{I} \\
(\lambda g.HH(\lambda y.g(gy))(\lambda y.P(Py)) \arrow[d, two heads] & HH(\lambda y.P(\mathsf{I}y)) \arrow[d, two heads] & \\
(\lambda g.HH(\lambda y.g(gy))P^{\eta} \arrow[d] & HHP^{\eta} \arrow[dl, two heads] & \\
HH(\lambda y.P^{\eta}(P^{\eta}y)) \arrow[uuu, bend left, two heads] & &
\end{tikzcd}$$

Hence M is not an exit. But $N = HH(\lambda y.\mathsf{I}(Py)) \to HH(\lambda y.\mathsf{I}(\mathsf{I}y)) \twoheadrightarrow HH\mathsf{I}$, and the latter misses the free variable z, so cannot be in the plane of M.
So N is an exit.$\square$

— Conversion —

Does bijectivity correspond to invertibility modulo $\beta\eta$?

Bijectivity vs Invertibility

In Set Theory:

$$f \text{ is bijective} \iff f \text{ is invertible}$$

More precisely:

(i) f is injective $\iff$ f is left-invertible (assuming the excluded middle).

(ii) f is surjective $\iff$ f is right-invertible (assuming AC).

In λ-calculus:

A (closed) λ-term F is bijective if it is

- injective: $\forall X, Y \in \Lambda^o . FX = FY \Rightarrow X = Y$;
- and surjective: $\forall Y \in \Lambda^o, \exists X \in \Lambda^o . FX = Y$.

A (closed) λ-term F is invertible if it is

- left-invertible: $\exists L \in \Lambda^o . L \circ F = \mathrm{I}$;
- and right-invertible: $\exists R \in \Lambda^o . \mathrm{I} = F \circ R$.

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- left-invertible: $\exists L \in \Lambda^o \,.\, L \circ F = \mathsf{I}$;
- and right-invertible: $\exists R \in \Lambda^o \,.\, \mathsf{I} = F \circ R$.

Does bijectivity correspond to invertibility for $=_{\beta\eta}$?

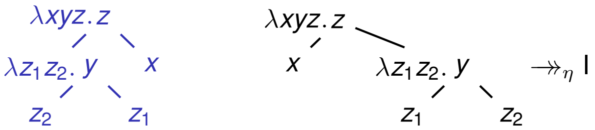
For set-theoretic reasons:

$$F \text{ invertible} \quad \Rightarrow \quad F \text{ bijective}$$

Question: F bijective $\quad \Rightarrow \quad$ F invertible?

Theorem [Dezani 1974 & Bergstra-Klop 1982]

F is invertible $\iff$ F is a hereditary permutation of a finite η-expansion of I.



Proposition [Batenburg-Velmans 1983]

F is injective and right-invertible $\quad \Rightarrow \quad$ F invertible

A positive answer by Folkerts

Theorem [Folkerts 1995]

Modulo $\beta\eta$: F bijective $\iff$ F invertible.

PROOF. The usual (injective $\Rightarrow$ L-invertible, surjective $\Rightarrow$ R-inv) doesn't work.

$$
\begin{tikzcd}
\text{bijective} \arrow[rr] \arrow[d] & & \text{injective} \arrow[rr, "X" description] \arrow[d] \arrow[drr] & & \text{left-invertible} \\
\text{surjective} \arrow[r] \arrow[rrr, bend right, "X" description] & \text{regular} \arrow[r] & \text{faithful} \arrow[r, "(*)"] & \text{right-invertible} \arrow[r] & \text{invertible}
\end{tikzcd}
$$

where

- F is regular if $F =_{\beta\eta} \lambda x\vec{y}.xP_1 \cdots P_k$.
- F is faithful if $F =_{\beta\eta} \lambda x\vec{y}.xP_1 \cdots P_k$, such that the P_i's are unsolvable or have one of the y's as free head-variable.

Interesting analysis of unsolvables…

The most difficult part is to prove

$$\text{surjective } \& \text{ faithful} \quad \Rightarrow \quad \text{right-invertible}$$

Folkerts needs infinitely many unsolvables remaining essentially different.
This happens when they have a different 'unsolvable core':

$$\begin{array}{rcl} \mathsf{X}_n & = & \mathsf{W}_n \mathsf{W}_n^{\sim n+1}, \text{ where} \\ \mathsf{W}_n & = & \lambda y_1 \ldots y_n x.xxy_1 \cdots y_n. \end{array}$$

Their looping reduction graph is:

$$\mathsf{X}_n = \begin{tikzcd} \mathsf{W}_n \mathsf{W}_n^{\sim n+1} \arrow[r, "\beta"'] & (\lambda y_2 \ldots y_n x.xx\mathsf{W}_n y_2 \cdots y_n)\mathsf{W}_n^{\sim n} \arrow[d, twoheadrightarrow, "\beta\ (n-2 \text{ steps})"] \\ (\lambda x.xx\mathsf{W}_n^{\sim n})\mathsf{W}_n \arrow[u, "\beta"] & (\lambda y_n x.xx\mathsf{W}_n^{\sim n-1} y_n)\mathsf{W}_n^{\sim 2} \arrow[l, "\beta"'] \end{tikzcd}$$

Dance of the starlings

Combinatory logic

Curry&Schönfinkel's combinatory logic is based on two 'combinators':

$$\begin{array}{lcl} \mathrm{K}xy & \to_w & x \\ \mathrm{S}xyz & \to_w & xz(yz) \end{array}$$

In Smullyan's beautiful fable about combinators figuring as birds in an enchanted forest, S is the starling.

Theorem

Combinatory logic is (almost) as powerful as λ-calculus.

Idea: Every λ-term is expressible as a combination of K and S.

The S-fragment of CL

What about the fragment $\mathcal{S}$ containing exclusively S?

$$\mathsf{S}xyz \to_w xz(yz)$$

Examples:

$$\mathsf{S}, \mathsf{SSS}(\mathsf{SSS})\mathsf{S}, \mathsf{SS}(\mathsf{SS})(\mathsf{SSS}), \dots$$

Some properties:

- Not as powerful as the λ-calculus (no cancellation).
- Reduction $\to_w$ is confluent.
- The anti-reduction ${}_w\!\leftarrow$ is strongly normalizable.
- S-terms tends to grow in size along reduction.
- $(\mathcal{S}, \to_w)$ is acyclic: $\forall P \in \mathcal{S}, \nexists Q \in \mathcal{S} \,.\, P \to_w Q \twoheadrightarrow_w P$.

Is there a non-terminating S-term?

How to construct a non-terminating term?

Look for a 'spiralling' term P, i.e.: $P \to_w^+ C[P]$, for some context $C[\,]$.

Waldmann 2000

There is no spiralling $P \in \mathcal{S}$.

Property

For every $k \in \mathbb{N}$ there exists a $P_k \in \mathcal{S}$ such that

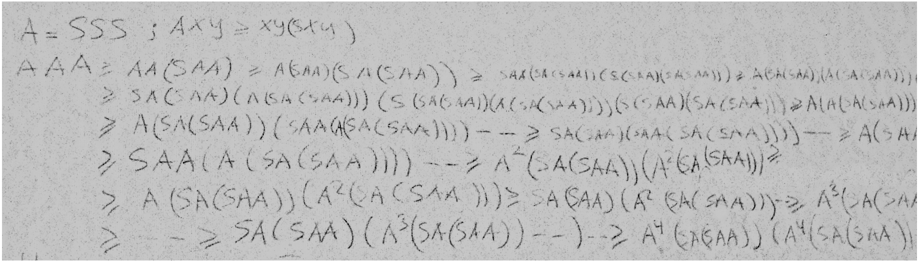
$$k < \mathrm{growth}(P_k) < \infty.$$

where

$$\mathrm{growth}(P) = \begin{cases} \frac{\mathrm{size}(\mathrm{nf}_w(P))}{\mathrm{size}(P)}, & \text{if } P \text{ has a } w\text{-nf;} \\ \infty, & \text{otherwise.} \end{cases}$$

Non-terminating S-terms

P	Year found	By
(SSS)(SSS)(SSS)	1975	Barendregt
S(SS)SSSS	1976	Dubou&Baron
S(SSS)(SSS)(S(SSS)(SSS))	1976	Pettorossi
S(SS)(SS)(S(SS)(SS))	1978	Zachos



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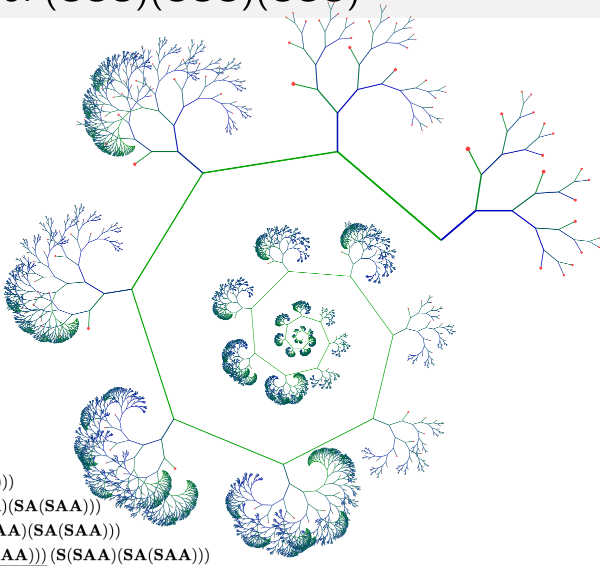
S(SS)(SS)(S(SS)(SS))	1978	Zachos

It makes sense to define the Berarducci tree $\mathrm{BeT}(P)$ of an S-term. Idea:

1. 'push' the reduction into infinity;
2. collect the 'stable pieces' of the term in a tree-like structure.

The Berarducci tree of (SSS)(SSS)(SSS)



$$
\begin{array}{ll}
 & \mathbf{AAA} \\
= & \underline{\mathbf{SSSA}}\ \mathbf{A} \\
\to_h & \boxed{\mathbf{SA(SA)A}} \\
\to_h & \mathbf{AA(SAA)} \\
= & \underline{\mathbf{SSSA}}\ \mathbf{(SAA)} \\
\to_h & \boxed{\mathbf{SA(SA)(SAA)}} \\
\to_h & \mathbf{A(SAA)(SA(SAA))} \\
= & \underline{\mathbf{SSS(SAA)}}\ \mathbf{(SA(SAA))} \\
\to_h & \boxed{\mathbf{S(SAA)(S(SAA))(SA(SAA))}} \\
\to_h & \underline{\mathbf{SAA(SA(SAA))}}\ \mathbf{(S(SAA)(SA(SAA)))} \\
\to_h & \mathbf{A(SA(SAA))(A(SA(SAA)))(S(SAA)(SA(SAA)))} \\
= & \underline{\mathbf{SSS(SA(SAA))}}\ \mathbf{(A(SA(SAA)))(S(SAA)(SA(SAA)))} \\
\to_h & \underline{\mathbf{S(SA(SAA))(S(SA(SAA)))(A(SA(SAA)))}}\ \mathbf{(S(SAA)(SA(SAA)))} \\
\to_h & \underline{\mathbf{SA(SAA)(A(SA(SAA)))}}\ \mathbf{(S(SA(SAA))(A(SA(SAA))))(S(SAA)(SA(SAA)))} \\
\to_h & \cdots
\end{array}
$$

Decidable properties

Theorem (Waldmann 1998)

(Strong) normalization of S*-terms is decidable in linear time.*

Proof: Waldmann constructed a finite state automaton accepting the normalizing S-terms only.

Theorem (Padovani 2020)

Head normalization of S*-terms is decidable.*

Proof: Padovani gave a criterion characterizing head-normalizable terms.

Berarducci trees equality

Define:

$$\begin{array}{rcl} S_1 & = & S, \\ S_{n+1} & = & S(S_n), \quad \text{for } n > 1. \end{array}$$

Theorem (Padovani 2020)

Let

$$\begin{array}{rcl} A & = & S(S_4(S_4S_3))(S(S_2S_3)S_3), \\ B & = & S(S_3S_3)(S_4S_3). \end{array}$$

Then

- $AA \neq_w BB$*, but*
- $\mathrm{BeT}(AA) = \mathrm{BeT}(BB)$.

Open problem

The word problem: Is w-conversion decidable for S-terms?

Padovani's example shows that $=_w$ is not the equality of Berarducci trees.

THE END

(THANKS)