Frobenius structures in ($*-$)autonomous categories

Lambda-Comb meeting 2023

Luigi Santocanale[1] Cédric de Lacroix[2]★

[1]Laboratoire d'Informatique et Système (LIS)
Aix-Marseille Université (AMU)

[2]Laboratoire d'Informatique et Système (LIS)
Aix-Marseille Université (AMU)

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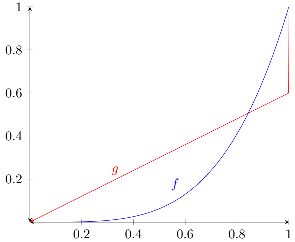
Motivations

The quantale of endomaps of $[0,1]$

We interpret formulas of Linear logic as join-preserving maps $[0,1] \to [0,1]$.
For instance, A is interpreted by f and B by g.



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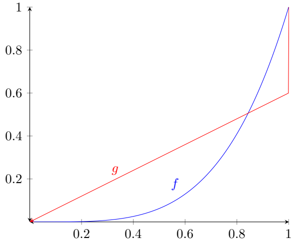
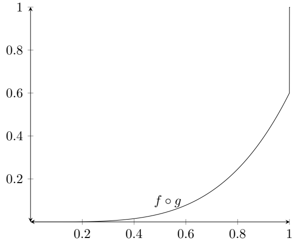
The tensor is interpreted by the composition of maps: $A \otimes B \longrightarrow f \circ g$.





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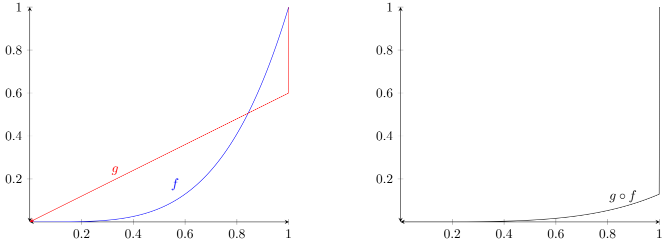
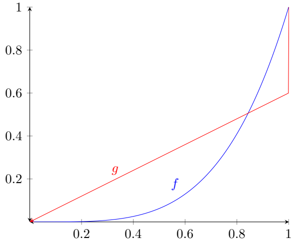
We interpret formulas of Linear logic as join-preserving maps $[0,1] \to [0,1]$.
It is not commutative: $B \otimes A \longrightarrow g \circ f$.

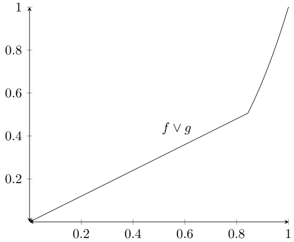


Motivations

The quantale of endomaps of $[0, 1]$

We interpret formulas of Linear logic as join-preserving maps $[0, 1] \to [0, 1]$.
The additive are interpreted with joins and meet: $A \oplus B \longrightarrow f \vee g$.





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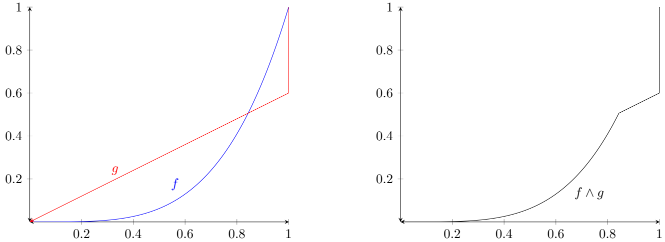

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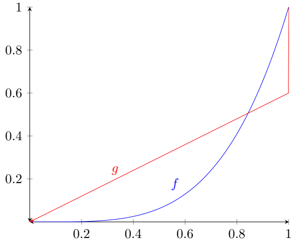
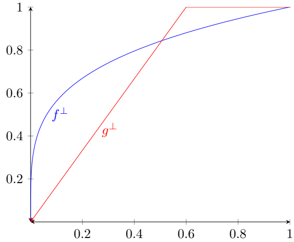
We interpret formulas of Linear logic as join-preserving maps $[0,1] \to [0,1]$.
Here we have one negation which is the symmetric with respect to the diagonal.

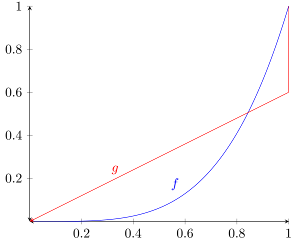


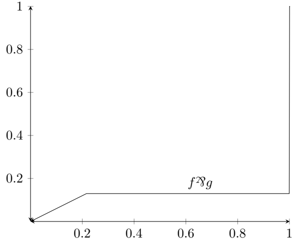


Motivations

The quantale of endomaps of [0, 1]

We interpret formulas of Linear logic as join-preserving maps $[0,1] \to [0,1]$.
We also interpret the par by $A ⅋ B = (B^\perp \otimes A^\perp)^\perp \longrightarrow (g^\perp \circ f^\perp)^\perp$.





Motivations

Let L be a complete lattice.
Under which condition the quantale $([L, L], \circ)$ is a Frobenius quantale?

Motivations

Theorem (Kruml and Paseka 2008, Santocanale 2020)

Let L be a complete lattice. The following are equivalent:

- L is a completely distributive lattice.
- The quantale $[L, L]_{\vee}$ of join-preserving endomaps of L is a Frobenius quantale.

Theorem (Raney 1960, Higgs and Rowe 1989)

The nuclear objects of the category of complete sup-lattices are exactly the completely distributive lattice.

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Motivations

Conjecture

Let A be an object of an autonomous category (symmetric monoidal closed). The following are equivalent:

- A is nuclear.
- The object $[A, A]$ of endomorphisms of A is a Frobenius structure.

Theorem (Raney 1960, Higgs and Rowe 1989)

The nuclear objects of the category of complete sup-lattices are exactly the completely distributive lattices.

Papers available:

For details and many more beautiful properties

- About unitless Frobenius quantales:
  `https://hal-amu.archives-ouvertes.fr/LIS-LAB/hal-03661651v1`
  (Accepted by Applied Categorical Structures (ACS 31.))

- About Frobenius structures :
  `https://hal.archives-ouvertes.fr/hal-03739197/`
  (Accepted by Computer Science Logic 2023(CSL 2023))

Next

Quantales

Definition

A *quantale* $(Q, \star)$ is a complete lattice Q with an associative law

$$\star : Q \times Q \to Q$$

which distributes over the sup on both variables:

$$(\bigvee_{i \in I} x_i) \star y = \bigvee_{i \in I} (x_i \star y) \quad \text{and} \quad x \star (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \star y_i).$$

Remark

- A quantale is a semigroup in the category **SLatt**.
- A quantale is a posetal monoidal bi-closed and complete category (without unit). We have:

$$x \star y \leq z \qquad \text{iff} \qquad y \leq x \multimap z \qquad \text{iff} \qquad x \leq z \multimapinv y\,.$$

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Unitless Frobenius quantales

Definition

A *unitless Frobenius quantale* is a tuple $(Q, \star, {}^{\perp}(-), (-)^{\perp})$ with ${}^{\perp}(-), (-)^{\perp} : Q \to Q^{\mathrm{op}}$ inverse maps such that for all $x, y \in Q$, we have

$$x \multimap {}^{\perp}y = x^{\perp} \multimapinv y \qquad \text{(contraposition law)},$$

$$\text{or equivalently: } \forall x, y, z, \quad x \star z \leq {}^{\perp}y \quad \text{iff} \quad z \star y \leq x^{\perp} \qquad \text{(shift relation)}.$$

Remark

- In a quantale $(Q, \star)$, if 0 is dualizing (*.ie.* $(0 \multimapinv x) \multimap 0 = x = 0 \multimapinv (x \multimap 0)$) then $0 \multimapinv 0 = 0 \multimap 0$ is the unit of $(Q, \star)$.
- If $(Q, \star, {}^{\perp}(-), (-)^{\perp})$ is a Frobenius quantale with a unit 1 then ${}^{\perp}1 = 1^{\perp}$ is a dualizing element.

Proposition

1. There exist non unital Frobenius quantales.
2. There is no extension which preserves the two negations from a unitless quantale to a unital one.

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Examples of unitless Frobenius quantales

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- The Chu construction of a quantale.
- If L is a completely distributive lattice then $[L, L]$ is a Frobenius quantale.
- Given a certain kind of relation on a semigroup S, we can construct a Frobenius quantale on a subquantale of $P(S)$ (Every unitless Frobenius quantale arise this way).

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Now we are going to abstract this structure in a categorical setting.

Next

Dual pair

For an object A of a $*$-autonomous category, we have the two equivalences:

$$\frac{A \otimes X \longrightarrow 0}{X \longrightarrow A^*} \qquad \frac{X \otimes A^* \longrightarrow 0}{X \longrightarrow A^{**} \cong A\,.}$$

Definition

A map $\epsilon : A \otimes B \longrightarrow 0$ in $\mathcal{V}$ is said to be a *dual pairing* (w.r.t. the object 0) if the two induced natural transformations are isomorphims.

$$\hom(X, B) \longrightarrow \hom(A \otimes X, 0)\,, \qquad \hom(X, A) \longrightarrow \hom(X \otimes B, 0)\,.$$

Example

- In a $*$-autonomous category, $(A, A^*, ev_{A,0})$ is a dual pair.
- In **SLatt**, $(L, L^{\mathrm{op}}, \epsilon)$, $\epsilon(x, y) = \bot$ if $x \leq y$, and $\epsilon(x, y) = \top$ otherwise.

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Other examples of dual pairs

Examples

- In **Coh**, $X^{\mathrm{op}} \cong X^*$ so (X, X^{op}) is also a dual pair.
- In a $*$-autonomous category, $A^* \otimes A \cong [A, A]^*$ so $(A^* \otimes A, [A, A], \epsilon)$ is a dual pair with $\epsilon := ev \circ \sigma \circ ev$.

$$A^* \otimes A \otimes [A, A] \xrightarrow{A^* \otimes ev_{A,A}} A^* \otimes A \xrightarrow{\sigma_{A^*,A}} A \otimes A^* \xrightarrow{ev_{A,0}} 0$$

- In **Hilb**, H and $\overline{H}$ is a dual pair with pairing $\langle -, - \rangle : H \otimes \overline{H} \to \mathbb{C}$ the linear extension of the inner product of H.

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Usual adjunction between lattices

For a join preserving map $f : L \to M$, the right adjoint to it $\widetilde{f} : M^{\mathrm{op}} \to L^{\mathrm{op}}$ is the only map s.t:

$$f(x) \leq y \qquad \text{iff} \qquad x \leq \widetilde{f}(y)$$

$$\begin{tikzcd}
L \otimes M^{\mathrm{op}} \arrow[r, "f \otimes M^{\mathrm{op}}"] \arrow[d, "L \otimes \widetilde{f}"'] & M \otimes M^{\mathrm{op}} \arrow[d, "\epsilon_M"] \\
L \otimes L^{\mathrm{op}} \arrow[r, "\epsilon_L"] & 0\,.
\end{tikzcd}$$

Adjoints in dual pair

Let (A_0, B_0), (A_1, B_1) be two dual pairs. For every morphism $f : A_0 \longrightarrow A_1$ we define $\widetilde{f} : B_1 \longrightarrow B_0$ by transposing:

$$\frac{\dfrac{A_0 \longrightarrow A_1}{A_0 \otimes B_1 \longrightarrow 0}}{B_1 \longrightarrow B_0}$$

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A_0 \otimes B_1 \arrow[r, "f \otimes B_1"] \arrow[d, "A_0 \otimes \widetilde{f}"'] & A_1 \otimes B_1 \arrow[d, "\epsilon_1"] \\
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Definition

We say that (f, g) is an adjoint pair if $g = \widetilde{f}$.

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Next

Semigroups over a monoidal category

Definition

A *semigroup* is a pair (A, μ_A) such that

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "A \otimes \mu_A"] \arrow[d, "\mu_A \otimes A"'] & A \otimes A \arrow[d, "\mu_A"] \\
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Quantales

Definition

A *quantale* $(Q, \star)$ is a semigroup in the category **SLatt**.

Remark

In a quantale, $(x \star -) : Q \to Q$ and $(- \star y) : Q \to Q$ both have a right adjoint, the left and right implications:

$$x \star y \leq z \quad \text{iff} \quad y \leq x \multimap z \quad \text{iff} \quad x \leq z \multimapinv y$$

We have

$$- \multimapinv - : Q \otimes Q^{\mathrm{op}} \longrightarrow Q^{\mathrm{op}} \qquad \text{and} \qquad - \multimap - : Q^{\mathrm{op}} \otimes Q \longrightarrow Q^{\mathrm{op}}$$

$$z \multimapinv (y \star x) = (z \multimapinv y) \multimapinv x \qquad \text{and} \qquad (x \star y) \multimap z = x \multimap (y \multimap z)$$

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Implications as actions

Let (A, B) be a dual pair such that (A, μ_A) is a semigroup.
We define $\alpha_A^\ell : A \otimes B \to B$ and $\alpha_A^\rho : B \otimes A \to B$ as the only morphisms such that

$$
\begin{tikzcd}
A \otimes A \otimes B \arrow[r, "A \otimes \alpha_A^\ell"] \arrow[d, "\mu_A \otimes B"] & A \otimes B \arrow[d, "\epsilon"] \\
A \otimes B \arrow[r, "\epsilon"] & 0
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B \otimes A \otimes A \arrow[r, "B \otimes \mu_A"] \arrow[d, "\alpha_A^\rho \otimes A"] & B \otimes A \arrow[r, "\sigma"] & A \otimes B \arrow[d, "\epsilon"] \\
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$$\begin{tikzcd}
A \otimes A \otimes X \arrow[r, shift left, "\mu_A\otimes X"] \arrow[r, shift right, "A\otimes\alpha^\ell"'] & A \otimes X \arrow[r, "\alpha^\ell"] & X.
\end{tikzcd}$$

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The case of Frobenius quantales

In a Frobenius quantale $(Q, \star, {}^{\perp}(-), (-)^{\perp})$, we have

- $(Q, Q^{\mathrm{op}}, \epsilon)$ is a dual pair;
- $(Q, \star)$ is a semigroup;
- ${}^{\perp}(-), (-)^{\perp} : Q \to Q^{\mathrm{op}}$ and $x \leq {}^{\perp}y$ iff $y \leq x^{\perp}$;

$$x \multimap {}^{\perp}y = x^{\perp} \multimapinv y$$

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Frobenius structures

Definition

A *Frobenius structure* is a tuple $(A, B, \epsilon, \mu_A, l, r)$ where

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We use the usual definition of the $\parr$:

$$A \parr B := A^{\perp} \multimap B = A \multimapinv {}^{\perp}B$$

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The diagram on the left commutes iff the diagram on the right does,

defining a multiplication on B.

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1. (B, μ_B) is a semigroup ;
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Nuclearity

From here, C is symmetric monoidal closed and $0 = I$.

Definition

For every object A of C, there exists a canonical arrow

$$\mathtt{mix}_A : A^* \otimes A \longrightarrow [A, A].$$

An object A is *nuclear* if $\mathtt{mix}_A$ is an isomorphism.

Example

- In k-**Vect** they are the vector spaces of finite dimension.
- In a commutative unital quantale $(Q, \star, 1)$, they are the invertible elements.
- In **Coh** they are necessarily the trivial coherent space.
- In **HypCoh** there is no nuclear object.

Theorem (Raney 1960, Higgs and Rowe 1989)

The nuclear objects of **SLatt** are exactly the completely distributive lattices.

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Adjunction and Nuclearity

Definition

For $\eta : I \to B \otimes A$, and $\epsilon : A \otimes B \to I$, (A, B, ϵ, η) is an *adjunction* if

$$
\begin{tikzcd}
A \otimes B \otimes A \arrow[d, "\epsilon \otimes A"'] & A \otimes I \arrow[l, "A \otimes \eta"'] \arrow[d, "\rho_A"] \\
I \otimes A \arrow[r, "\ell_A"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes B \arrow[r, "\eta \otimes B"] \arrow[d, "\ell_B"'] & B \otimes A \otimes B \arrow[d, "B \otimes \epsilon"] \\
B & B \otimes I \arrow[l, "\rho_B"'] .
\end{tikzcd}
$$

Proposition

An object is nuclear iff there exist a (right or left) adjoint to it.

Next

From Nuclearity to Frobenius structure

Theorem (LS and CL)

In a symmetric monoidal closed category, if A is nuclear then $[A, A]$ can be endowed with a Frobenius structure.

Sketch of the proof

- If $\mathtt{mix}$ is invertible, then $(A^* \otimes A, [A, A], \epsilon, \mu_{A^* \otimes A}, \mathtt{mix}, \mathtt{mix})$ is a Frobenius structure.
- As $A^* \otimes A$ is isomorphic to $[A, A]^*$ and Frobenius structures are closed under iso, we obtain the desired theorem.

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Conjecture

Let $([A,A],[A,A]^*,\mu,r,I)$ be a Frobenius structure in an autonomous category. Then A is a nuclear object.

We actually need to add a technical hypothesis.

Sketch of a proof

We use the caracterisation of nuclearity with adjoints. So we want:

$$\eta : I \longrightarrow A^* \otimes A \text{ (unit)} \qquad \epsilon : A \otimes A^* \longrightarrow I \text{ (co-unit)}$$

such that (A, A^*, ϵ, η) is an adjunction.

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Continuing the proof

- We identify $[A, A]^*$ with $A^* \otimes A$. Suppose $([A, A], A^* \otimes A, ev, \mu, r, l)$ is a Frobenius structure.

- We can easily find a candidate for the unit of the adjunction. Indeed, $[A, A]$ is a monoid,and as $r : [A, A] \to A^* \otimes A$ is an iso, $A^* \otimes A$ is also a monoid with unit $\eta : I \to A^* \otimes A$.

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The multiplication on $A^* \otimes A$ is given by

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From Frobenius structure to nuclearity

Continuing the proof

That is, we have a diagram of the shape

$$
\begin{tikzcd}
A^* \otimes A \otimes A^* \otimes A \arrow[d, bend right=60, "A^* \otimes g"'] \arrow[d, bend left=60, "h \otimes A"] \\
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This map actually exits if A is a pseudoaffine object.

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- **SLatt**
- k-**Vect**
- **Coh**
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Theorem (LS and CL)

If $\mathcal{C}$ is pseudoaffine and $([A, A], [A, A]^*, ev, \mu, r, I)$ is a Frobenius structure then A is a nuclear object.

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Discussing the pseudoaffine condition

To understand the role of the pseudoaffine condition we build categories which are not pseudoaffine. This construction is actually a case of the one described in (Schalk and de Paiva 2004).

Let $(P, \leq)$ be a poset (the base category). We define the category P-**Set**:

- Objects: pairs (X, A) with X a set and $A : X \to P$ a map;
- Arrows $A \to B$: relations $R \in P(X \times Y)$ such that xRy implies $A(x) \leq B(y)$.

Theorem (Schalk and de Paiva 2004)

If $(Q, \star, 1)$ is a unital commutative Frobenius quantale, the category Q-**Set** is a $*$-autonomous category.

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If $(Q, \star, 1)$ is a unital commutative Frobenius quantale, the category Q-**Set** is a $*$-autonomous category.

Examples

The categories **Coh** and **HypCoh** are subcategories of similar categories (In those case, we also need an endofunctor of **Rel**).

Discussing the pseudoaffine condition

To understand the role of the pseudoaffine condition we build categories which are not pseudoaffine. This construction is actually a case of the one described in (Schalk and de Paiva 2004).

Let $(P, \leq)$ be a poset (the base category). We define the category P-**Set**:

- Objects: pairs (X, A) with X a set and $A : X \to P$ a map;
- Arrows $A \to B$: relations $R \in P(X \times Y)$ such that xRy implies $A(x) \leq B(y)$.

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Discussing the pseudoaffine condition

To study nuclearity, we ask that $1 = 0$ in Q which implies that $I = 0$ in Q-**Set**.

Lemmas

- A Q-**Set** A is nuclear if the image of A is included in the invertible element of Q, ie if for all $x, y \in X$,

$$A(x) \multimap A(y) = A(x)^{\perp} \star A(y)\,.$$

- A Frobenius structure on $[A, A]$ in Q-**Set** is given by a pair of inverse map (f, g) over the underlyng set X such that for all $x, y \in X$:

$$A(x) \multimap A(y) = A(fx)^{\perp} \star A(y) = A(x)^{\perp} \star A(gy)\,.$$

Theorem(LS and CL)

In Q-**Set**, the statement that $[A, A]$ endows a Frobenius structure is equivalent to A being nuclear if one of the following conditions is true:

- The Frobenius quantale Q has no infinite chain;
- The underlyng set X is finite;
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And in general?

We found no reason why it should be true in general.
After some time we were able to construct an infinite quantale Q and a Frobenius structure on $[A, A]$ in Q-**Set** with A not being nuclear !

Counterexample

It is just the infinite chain $\mathbb{Z}$ with ∞ and $-\infty$ and another unit between -1 and 0. Then X could be $\mathbb{Z}$, A the inclusion and f and g the antecedent and successor (cf drawing).

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Conclusion

Results

- A new definition of Frobenius quantale which does not involve unit and its study;
- A definition of Frobenius structures in autonomous categories;
- Generalisation of the double negation construction;
- Proof of our conjecture up to a technical (but quite natural) hypothesis.

What we will do next

- Connect with linear logic semantic;
- Study the logic of pseudoaffine category;
- Understand "how much" we need $*$-autonomous categories;
- Use our results on differents categories such as Banach spaces.

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Thank you!

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Next

Results on tight maps

Proposition (LS and CL)

For every complete lattice L, $([L,L]^t_\vee, \circ, (-)^\perp, (-)^\perp)$ is a Frobenius quantale with $f^\perp = l(f^\wedge)$.

Theorem

Let L be a complete lattice. The following are equivalent:

1. The lattice L is completely distributive;
2. $[L,L]^t_\vee = [L,L]$ (Raney, 1960);
3. L is a nuclear object of **SLatt** (Higgs Rowe 1989);
4. There is a unique sup-preserving map $0 : L \to L$ such that $([L,L], \circ, 0)$ is a Frobenius quantale. (Kruml Paseka 2008, Santocanale 2020);
5. *The Frobenius quantale* $([L,L]^t_\vee, \circ, (-)^\perp, (-)^\perp)$ *has a unit.* (LS CL).

Frobenius structure and associative bracketed semigroups

Proposition

For a Frobenius structure $(A, B, \epsilon, \mu_A, l, r)$, we can define

$$\pi^l_A := \epsilon \circ (A \otimes l) : A \otimes A \to 0\,,$$

We have :

- (A, μ_A, π^l_A) is an associative bracketed semigroup;
- π^l_A is a dual pairing.

Conversely, from an associative bracketed semigroup (A, μ_A, π_A) for which π_A is a dual pairing, one obtains a Frobenius structure.

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Result

Theorem (LS and CL)

Let C be a $*$-autonomous category such that $\mathbf{Sem}_C$ has an epi-mono factorization system and A an object of C.
The image of $\mathtt{mix}_A$ can always be endowed with a Frobenius structure.

$$\begin{tikzcd}
A^* \otimes A \arrow[rr, "\mathtt{mix}_A"] \arrow[dr, two heads] & & {[A, A]} \\
& \Im\mathtt{mix}_A \arrow[ur, tail] &
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Corollary

If A is nuclear then $[A, A]$ can always be endowed with a Frobenius structure.

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