Exam of the course 2-38-2 of MPRI 2022
Algorithms and combinatorics of geometric graphs

Handouts and handwritten course notes allowed. Electronic devices prohibited.

Prepare three separate copies for the three parts of the exam.

You can skip the questions that block you. However, it is recommended that you try to deal with a coherent part of the topic, even incomplete, rather than dealing sporadically with questions that seem easier.

The writting and presentation of your solutions will be an important criterion of the evaluation.

1 Minimal and maximal cuts in polytopes (VP)

A *cut* in a graph $G = (V, E)$ is a partition $V = V_\circ \sqcup V_\bullet$ of the vertices of G with $V_\circ \neq \varnothing \neq V_\bullet$. We then denote by $E_{\circ\circ}$ (resp. $E_{\bullet\bullet}$) the set of edges of E whose two vertices are in $V_\circ$ (resp. in $V_\bullet$), and by $E_{\circ\bullet}$ the set of edges of E connecting a vertex of $V_\circ$ to a vertex of $V_\bullet$. We define $v = |V|$, $v_\circ = |V_\circ|$, $v_\bullet = |V_\bullet|$, $e = |E|$, $e_{\circ\circ} = |E_{\circ\circ}|$, $e_{\bullet\bullet} = |E_{\bullet\bullet}|$, and $e_{\circ\bullet} = |E_{\circ\bullet}|$. The cut is *trivial* if $v_\circ = 1$ or $v_\bullet = 1$. The *size* of the cut is $e_{\circ\bullet}$, and a cut is *minimum* (resp. *maximum*) if its size is minimum (resp. maximum) among all cuts of G. The objective of this exercise is to study the cuts in the graph $G = (V, E)$ of a polytope of dimension d.

If P and Q are two polytopes of dimension d and F and G are facets of P and Q which are simplices, we denote by $P\,{}_F\#_G\,Q$ the polytope obtained by gluing the polytopes P and Q along their facets F and G. Since F and G are simplices, this glueing is always possible up to an affine transformation of P and Q. The result can moreover be assumed to be a convex polytope up to a projective transformation of P and Q. We thus obtain a polytope $P\,{}_F\#_G\,Q$ whose graph is obtained by gluing the graphs of P and Q along the graphs of F and G. See Figure 1. You can use this operation in this problem.

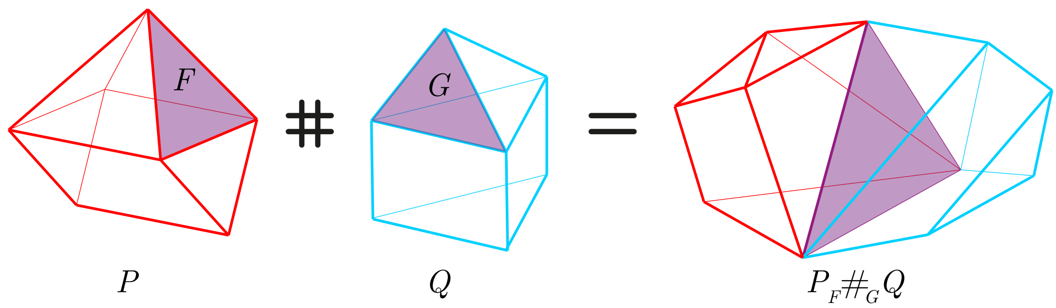

Figure 1: The operation $P_F\#_G Q$.

Dimension 2. We first suppose that $d = 2$.

Q1. What sizes can cuts in dimension $d = 2$ have? What is the minimum and maximum size?

Dimension 3. We now assume that $d = 3$.

Q2. Show that the minimum cuts must have size 3, 4 or 5. Give a family of examples for each of these sizes.

Q3. Show that if P is simplicial (*i.e.* all facets of P are triangles), then any minimal cut is trivial. [*Hint: Treat separately the case where* $v_\circ = 2$ *or* $v_\bullet = 2$. *When* $v_\circ \geq 3$ *and* $v_\bullet \geq 3$, *compute* e *and bound* $e_{\circ\circ}$ *and* $e_{\bullet\bullet}$. *You can consider the graph* $G_\circ = (V_\circ, E_{\circ\circ})$.]

Q4. Give an example of a polytope where minimal cuts are not trivial.

Q5. Show that if P is cubic (*i.e.* all facets of P are squares), then the maximal cuts have size $2v-4$. [*Hint: You can show that any cycle of* G *is of even length, by considering the planar graph formed by the faces inside the cycle.*]

Q6. Show that if P is simplicial (*i.e.* all facets of P are triangles), then the maximal cuts have size $2v - 4$.
[*Hint: For the existence of a cut of size* $2v - 4$, *you can use the* 4*-color theorem. From a* 4*-coloring* $V = V_\spadesuit \sqcup V_\heartsuit \sqcup V_\diamondsuit \sqcup V_\clubsuit$ *such that no edge of* G *is monochromatic, you can consider the partition* $V = V_\circ \sqcup V_\bullet$ *where* $V_\circ = V_\heartsuit \sqcup V_\diamondsuit$ *and* $V_\bullet = V_\spadesuit \sqcup V_\clubsuit$ *and show that no triangle of* G *is monochromatic.*]

Q7. Show that there is always a cut of size at least $2e/3$.
[*Hint: From a* 4*-coloring* $V = V_\spadesuit \sqcup V_\heartsuit \sqcup V_\diamondsuit \sqcup V_\clubsuit$ *such that no edge of* G *is monochromatic, you can consider all the partitions* $V = V_\circ \sqcup V_\bullet$ *where* $V_\circ$ *and* $V_\bullet$ *are obtained as unions of exactly two parts among* $\{V_\spadesuit, V_\heartsuit, V_\diamondsuit, V_\clubsuit\}$.]

Q8. Show that the size of the maximum cuts is always between v and $2v - 4$.
[*Hint: For the lower bound, you can either use the Question 7, or color the vertices one by one in an order given by a linear direction. For the upper bound, you can either use Question 6, or show that the number of edges is at most* $2v - 4 + t/2$, *where* t *denotes the number of triangular faces. Bonus points will be awarded if all of these approaches are presented.*]

Dimension d. We finally assume that $d \geq 4$.

Q9. Give bounds on the size of cuts of G, and show that these bounds are reached.

Q10. For each dimension d, give an example of a polytope where the minimal cuts are not trivial.

Q11. For any dimension d, give an example of a simplicial polytope where the minimal cuts are not trivial.

2 Schnyder woods of 4-connected planar triangulations (LCA)

Let us remind that any simple[1] plane triangulation $\mathcal{T}$ with root face[2] (V_0, V_1, V_2) can be endowed with a Schnyder wood which defines an orientation and coloration of the inner edges of $\mathcal{T}$ such that every inner vertex has exactly three outgoing edges (and each inner edge has a color in $\{0, 1, 2\}$). Please refer to lecture 6 for more details.
Remark: from now on we will assume that the input plane triangulation $\mathcal{T}$ is endowed with a Schnyder wood (Fig. 2(left) shows two Schnyder woods of the same triangulation).

A *cycle* is defined by an ordered list of edges $\{(u_0, u_1), (u_1, u_2), \ldots (u_{k-1}, u_0)\}$ such that each vertex is shared by exactly two consecutive edges (no self-intersections). A cycle partitions the set of faces of $\mathcal{T}$ into two regions: the *outer region* (containing the root outer face) and the *inner region*

[1] Loops and multiples edges are not allowed.
[2] In our drawings the root face does coincide with the (infinite) outer face.

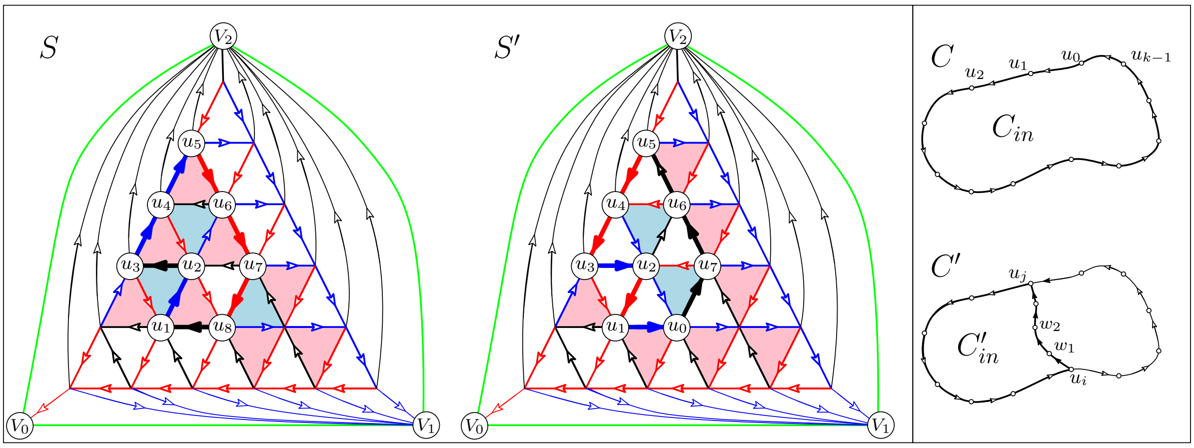

Figure 2: (Left) A (4-connected) planar triangulation $\mathcal{T}$ with root face (V_0, V_1, V_2), endowed with two distinct Schnyder woods S and S'. The 3-colored faces are shaded: *cw*-oriented triangles are red and *ccw*-oriented triangles are blue. Bold lines represent the edges of a cycle of $C = \{(u_0, u_1), \ldots, (u_6, u_7), (u_7, u_0)\}$ of length 8, whose inner region C_{in} contains 6 triangles. The cycle C is *cw*-oriented in S and *ccw*-oriented in S'. (Right) Illustration of Question 4.

C_{in} containing all remaining faces (inside C). The cycles of length 3 are called *triangles*. A triangle does not necessarily defines a face in the embedding of $\mathcal{T}$ (such a triangle is called *separating*). Consider a cycle $C = \{(u_0, u_1), \ldots (u_{k-1}, u_0)\}$ and for an edge $e = (u, v) \in C$, oriented from u to v, and denote by $t_{in}(e)$ the face containing e that is lying in the inner region C_{in}. We say that C is *cw*-oriented (i.e. oriented in clockwise direction) if for each edge $e \in C$ the corresponding face $t_{in}(e)$ is on the right of e. Similarly, the cycle C is *ccw*-oriented (i.e. oriented in counter-clockwise direction) if for each edge $e \in C$ the face $t_{in}(e)$ is on the left of e). A triangle is 3-*colored* if the three edges $\{(u_0, u_1), (u_1, u_2), (u_2, u_0)\}$ have different colors (see Fig. 2(left) for an illustration).

1. Let $C = \{(u_0, u_1), \ldots (u_{k-1}, u_0)\}$ be a cycle in $\mathcal{T}$ and assume that no edge in the inner region of C is oriented outgoing from a vertex u_i of C. Montrer that C must have length 3.

[*Hint: use a double counting of the edges in the inner region of* C]

2. Montrer that the boundary of a 3-colored triangle must define an oriented cycle (either *cw*-oriented or *ccw*-oriented).

3. Montrer that if $\mathcal{T}$ contains an oriented cycle C then the inner region of C also contains a 3-colored triangle.

Flips of faces in 4-connected planar triangulations

From now on we will assume that the input plane triangulation $\mathcal{T}$ is 4-*connected* (and endowed with a Schnyder wood), which means that $\mathcal{T}$ **does not contain separating triangles**.

4. Let C be an oriented cycle of $\mathcal{T}$. Montrer that there exists a path of $p \geq 1$ directed edges $P = \{(u_i, w_1), (w_1, w_2), \ldots (w_{p-1}, u_j)\}$ (where $u_i, u_j \in C$ and $i \neq j$) whose edges are all lying in the inner region of C and define a new oriented cycle C' such its inner region C'_{in} is strictly contained in the inner region of C (see Fig. 2(right) for an illustration).

Given a Schnyder wood S of $\mathcal{T}$ and a 3-colored face $t = \{(u_0, u_1), (u_1, u_2), (u_2, u_0)\}$ the operation of *flip* of t consists in changing the orientation of its boundary edges. If the boundary of t is *cw*-oriented then we have a so-called *cw-flip*, and the edge of color i gets color $i+2$ (indices are modulo 3). Otherwise, if t is *cww*-oriented, we have a *ccw-flip* (the edge of color i gets color $i+1$). Observe that after flipping a 3-colored face in S we obtain another valid Schnyder wood S' (distinct from

S). So, starting from an initial Schnyder wood S_0 we can construct a sequence Schnyder woods $\mathcal{S} = \{S_0, S_1, S_2, \ldots\}$ by flipping 3-colored faces (where S_{i+1} is obtained from S_i by flipping exactly one face). Observe that, in principle, the sequence $\mathcal{S}$ may contain the same Schnyder wood more than once.

5. Consider the two Schnyder woods S and S' of Fig. 2. Give a list of face cw-flips that allows to obtain the Schnyder wood S' from S (the expected result is an ordered list of triangles, whose vertices are labeled as in Fig. 2).

[*Remark: observe that the edge orientations of* S *and* S' *only differ on the oriented cycle* C]

Let us denote by $|C_{in}|$ the number of faces in the inner region bounded by C.

6. Let us consider two Schnyder woods S and S' of $\mathcal{T}$ and a cw-oriented cycle C in S. Assume that C is ccw-oriented in S' and that all remaining edges in $\mathcal{T} \setminus C$ have the same orientation in S and S'. Montrer that there is a sequence of face cw-flips [3] of length $|C_{in}|$ in which every inner face of C appears exactly once, that allows to reverse the orientation of C and to obtain S' starting from S.

The graph of Schnyder woods (of a 4-connected plane triangulation)

Our goal is to show that flipping 3-colored faces we can obtain all distinct Schnyder woods of $\mathcal{T}$ and, moreover, we want to get a bound on the length of flipping sequences. Observe that performing both cw-flips and ccw-flips of 3-colored faces one can get flip sequences of infinite length: this does not occur if we restrict ourselves to only cw-flips (or to only ccw-flips).

7. Let t be a face of $\mathcal{T}$ and consider a sequence of Schnyder woods $\{S_0, S_1, S_2, \ldots\}$ obtained performing cw-flips of 3-colored faces. Assume that the face t has been flipped more than once, which means that S_{i+1} is obtained from S_i flipping t and also S_{j+1} is obtained from S_j after flipping t (for some $j > i \geq 0$). Montrer that after cw-flipping t the first time and before cw-flipping t the second time there must occur a cw-flip of each neighboring face of t.

8. Montrer that the number of times a given face of $\mathcal{T}$ can be cw-flipped is bounded.

[*Hint: use previous question and observe that the faces incident to the outer vertices* V_0, V_1 *and* V_2 *cannot be flipped.*]

Given a fixed 4-connected planar triangulation $\mathcal{T}$, let $G_{\mathcal{T}}$ be the graph whose nodes are all Schnyder woods of $\mathcal{T}$ and such that two Schnyder woods S_1 and S_2 are neighbors if and only if one can be obtained from another by a single face flip (either a cw-flip or a ccw-flip).

9. Montrer that $\mathcal{T}$ admits a Schnyder wood without cw-oriented cycles (and, similarly, a Schnyder wood with no ccw-oriented cycles).

10. Montrer that the graph $G_{\mathcal{T}}$ of the Schnyder woods is connected (given two Schnyder woods S and S', one can go from S to S' with a sequence of face flips).

[*Hint: you may use the fact (**not** to be proved) that the Schnyder wood without cw-oriented (resp. ccw-oriented) triangles is unique.*]

[3] A sequence of cw-flips is just a defined by a sequence of 3-colored and cw-oriented faces.

3 Forcing uniqueness of geodesics on surfaces (ADM)

Let G be a loopless graph cellularly embedded on a surface S, with a function $\ell : E \to \mathbb{R}_+$ indicating a positive length for each edge. A shortest path between two vertices is a path of minimal length (i.e., sum of lengths) between those two vertices. In general, shortest paths are not unique, but in many applications, it is convenient to be able to to assume that any pair of vertices are connected by a unique shortest path. In other words, for each pair of vertices, one would like to be able to choose one favorite shortest path among the possible choices. This could be done in quadratic time by making an arbitrary choice for each pair of vertices, but we want something faster. One classical way to ensure that is to perturb randomly the length of each edge by a very small quantity. This exercise explores an alternative way of doing so in a deterministic fashion.

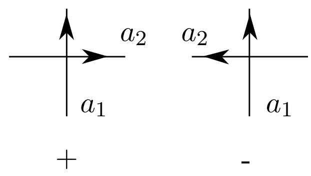

Figure 3: Left: a positive crossing. Right: a negative crossing.

Throughout the exercise, we use the word **cycle** in the graph-theoretical sense: a walk in the graph starting and ending at the same vertex with, apart from that vertex, no repeated vertices nor edges. An **arc** of G is an edge with a choice of orientation[4], for example each edge (u, v) corresponds to two arcs: the arc $u \to v$ and the arc $v \to u$. For each crossing between two arcs a_1 and a_2, we say that the crossing is positive if a_2 crosses a_1 from left to right (Figure 3, left), and negative in the other case (Figure 3, right). The **oriented crossing number** of two arcs, or more generally of two paths, is the number of positive crossings minus the number of negative crossings.

We first consider the case where G is a loopless connected plane graph, i.e., a loopless connected planar graph with a fixed embedding in $\mathbb{R}^2$. Let T^* denote a spanning tree of G^*, the graph dual to G. We define a weight function w on the arcs of G as follows:

- If a does not cross T^*, $w(a) = 0$.
- If a crosses T^*, it separates T^* into two subtrees: we denote by T_1^* the one that does **not** contain the outer face. We set $w(a)$ to be the number of vertices in T_1^* if that tree lies to the right of a, or minus that number if T_1^* lies to the left of a.

For a path or a cycle P in the graph G, we denote by $w(P)$ the sum of the weights of the arcs of P.

Q1. Let C be a cycle of G that rotates clockwise, and let F be a face enclosed by the cycle C. Let P^* be the path from F to the outer face in T^*, show that the oriented crossing number of P^* and C is exactly one. In contrast, show that for a face F not enclosed by the cycle C, that number is zero. Deduce that $w(C)$ is equal to the number of faces enclosed by C. By symmetry, this implies that if C rotates counter-clockwise, $w(C)$ is equal to minus that number.

For any path P of G, we define its **modified length** to be the pair $(\ell(P), w(P))$. We compare modified lengths lexicographically, that is $(\ell(P_1), w(P_1)) \leq (\ell(P_2), w(P_2))$ if and only if $\ell(P_1) < \ell(P_2)$ or $\ell(P_1) = \ell(P_2)$ and $w(P_1) \leq w(P_2)$.

Q2. Show that in any plane graph G, shortest paths for the modified length are unique.

Q3. Show that one can compute the weight function w in linear time.

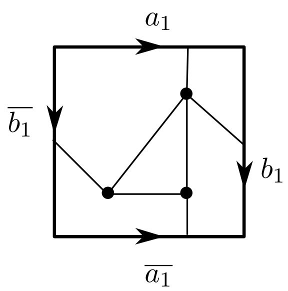

Figure 4: An example of a cellular embedding on top of a canonical polygonal scheme of the torus.

Now, let G be a loopless graph cellularly embedded on an orientable surface S of genus g. We assume that the embedding is described by providing an embedding of G on top[5] of the canonical polygonal scheme $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$ for the surface S, see Figure 4. We define the **signature** $s(a)$ of an arc a to be the $2g$-dimensional vector $(s_1, \dots, s_{2g})$, where, if i is even, s_i denotes the oriented crossing number of a and $a_{i/2}$, and if i is odd, s_i denotes the oriented crossing number of a and $b_{(i+1)/2}$.

Let T^* denote a spanning tree of G^*, the dual of G, and let r be a fixed face of G, or equivalently a fixed vertex of G^*. We define a weight function on the arcs of G as follows:

- If a does not cross T^*, $w(a)$ is the $2g+1$-dimensional vector $(0, s(a))$.
- If a crosses T^*, it separates T^* into two subtrees: we denote by T_1^* the one that does not contain the vertex r. We set the first coordinate of $w(a)$ to be the number of vertices in T_1^* if T_1^* lies to the right of a, or minus that number otherwise. The remaining coordinates are given by $s(a)$.

As before, for a path or a cycle P in the graph G, denote by $w(P)$ the sum of the weights of the arcs of P.

Q4. Choose a face r and a spanning tree T^* and compute the weight function for all the arcs in the example on Figure 4.

Q5. Let C be a separating cycle of G. Show that the first coordinate of $w(C)$ is equal to the number of faces to the right of C if r is on the left side of C, or minus the number of faces to the left of C if r is on the right side of C.

Q6. Let C be a non-separating cycle of G. Show that the signature of C is not $(0, \dots, 0)$. *Hint: You can consider a closed curve on the surface transverse to G that crosses C exactly once and homotope it onto the polygonal scheme.*

Q7. Let C be a cycle of G that contains at least one edge. Deduce from the previous questions that $w(C) \neq (0, \dots, 0)$.

For any path P of G, we define once again its **modified length** to be the pair $(\ell(P), w(P))$. We compare weights lexicographically, and this allows us to compare modified lengths lexicographically.

Q8. Show that the shortest paths for the modified length are unique. What is the algorithmic complexity of computing the weight function w?

[4] We restrict our attention to loopless graphs so that this is well-defined.

[5] Formally, the input is the planar graph formed by the superposition of the two graphs: G and the polygonal scheme.