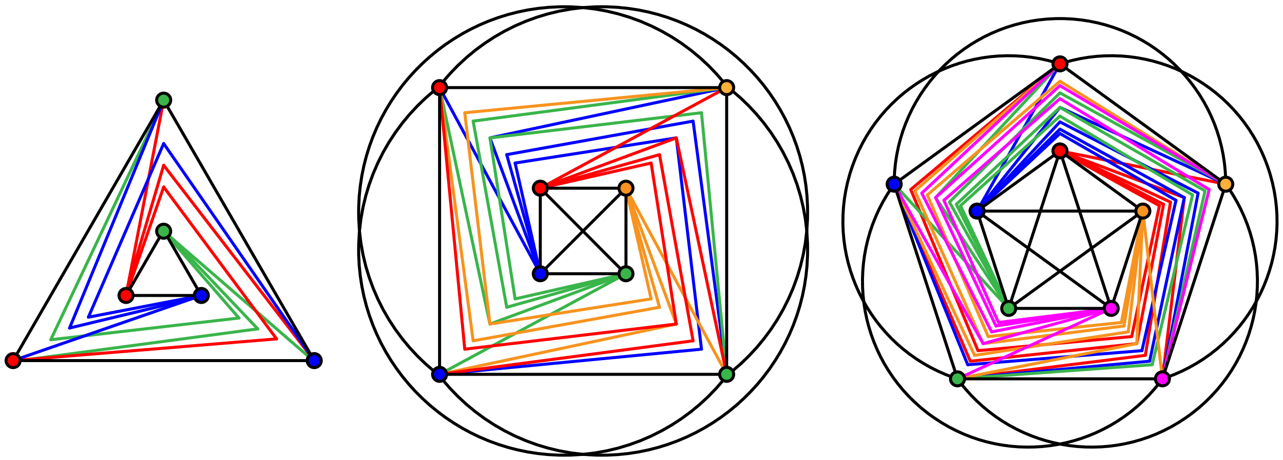
Planar & geometric graphs



V. PILAUD

MPRI 2-38-1. Algorithms and combinatorics for geometric graphs

Friday September 16th, 2022

slides available at: http://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/MPRI-2-38-1-VP-1.pdf
Course notes available at: https://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/notesCoursMPRI22.pdf

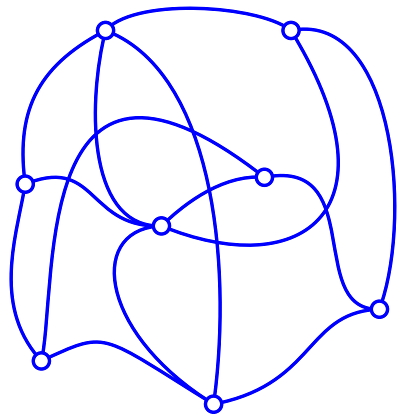
GRAPH DRAWINGS & EMBEDDINGS

DEF. drawing of $G = (V, E)$ in the plane $\mathbb{R}^2$ =

- an injective map $\phi_V : V \to \mathbb{R}^2$
- a continuous map $\phi_e : [0,1] \to \mathbb{R}^2$ for each $e \in R$

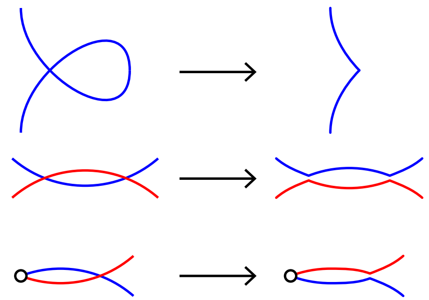
such that

- $\phi_e(0) = u$ and $\phi_e(1) = v$ for any edge $e = (u, v)$,
- $\phi_e(]0,1[) \cap \phi_V(V) = \varnothing$ for edge e.



DEF. topological drawing if

- no edge has a self-intersection,
- two edges with a common endpoint do not cross,
- two edges cross at most once.



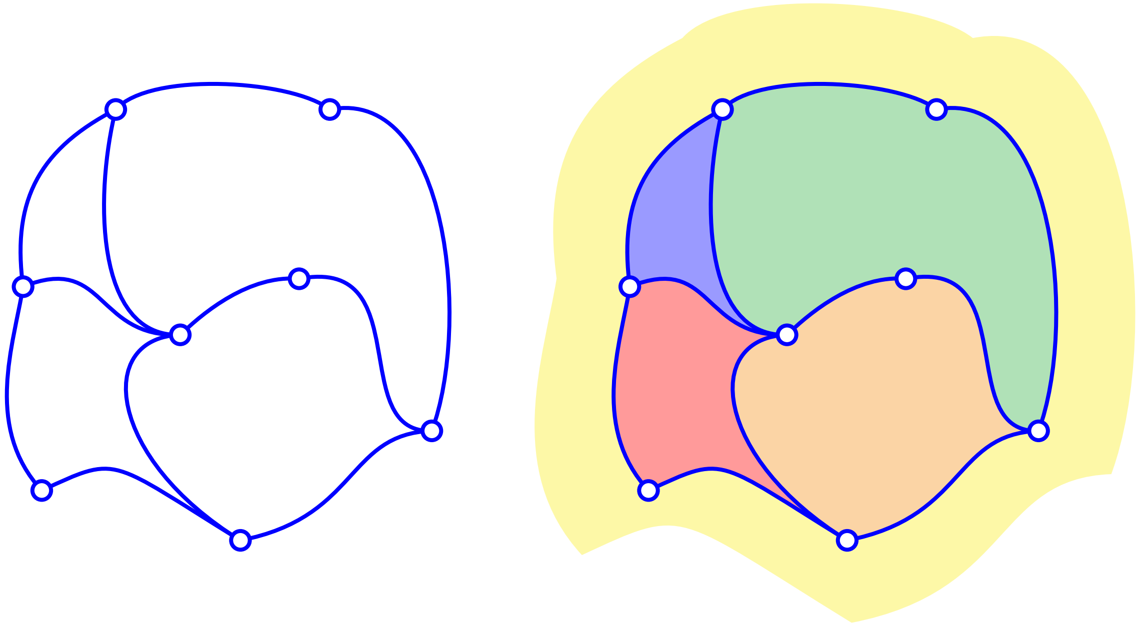
DEF. embedding if $\phi_e(x) \neq \phi_{e'}(x')$ for $(e, x) \neq (e', x')$ with $e, e' \in E$ and $x, x' \in]0,1[$.

PLANAR GRAPHS

PLANAR GRAPHS

DEF. planar graph = admits an embedding in the plane $\mathbb{R}^2$.

DEF. faces = connected components of the complement of an embedding.



Planar graphs are very special:

- combinatorially (few edges, Euler relation, 4-colorable, ...),
- algorithmically (use planar structure to design more efficient algorithms).

EULER'S FORMULA

THM. (Euler's formula)

$$v - e + f = 2$$

for any connected planar graph with v vertices, e edges, and f faces.

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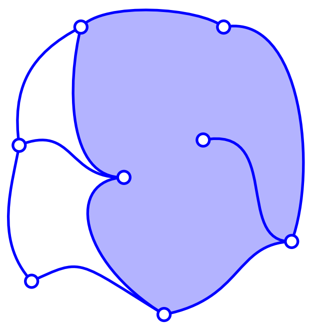
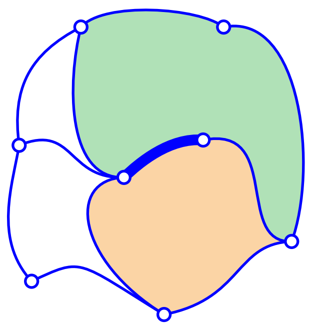
proof 1: Induction on e

- valid for a tree,
- adding an edge separates a face into two (Jordan's theorem),
- hence $v - e + f$ is constant.





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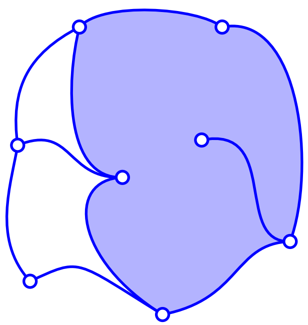
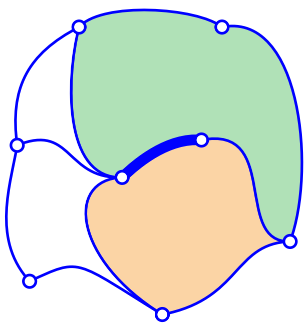
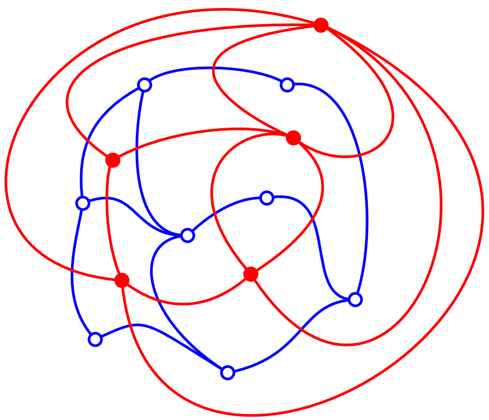
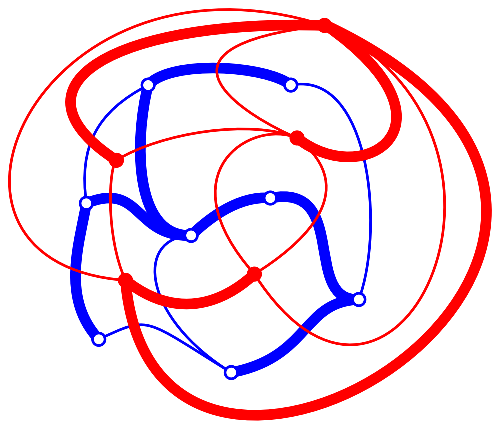
<u>proof 2:</u> pair of spanning trees

- G and $G^\star$ dual planar graphs,
- T and $T^\star$ dual spanning trees,
- $e_T = v_T - 1 = v - 1$,
- $e_{T^\star} = v_{T^\star} - 1 = f - 1$,
- $e = e_T + e_{T^\star} = v - 1 + f - 1 = v + f - 2$.

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CORO. For a simple planar graph with $v \geq 3$,

- $e \leq 3v - 6$,
- $e \leq 2v - 4$ if no triangular face,
- $e \leq (v-2)(k+1)/(k-1)$ if no face has degree $\leq k$.

proof:

$$\sum_{e \in E} \# \{f \in F \mid e \in f\} = \# \{(e,f) \in E \times F \mid e \in f\} = \sum_{f \in F} \# \{e \in E \mid e \in f\}$$

$$2e = \qquad \geq (k+1)f$$

$$= (k+1)e - (k+1)(v-2)$$

hence

$$(k-1)e \leq (k+1)(v-2).$$

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CORO. The minimum degree of a planar graph is at most 5.

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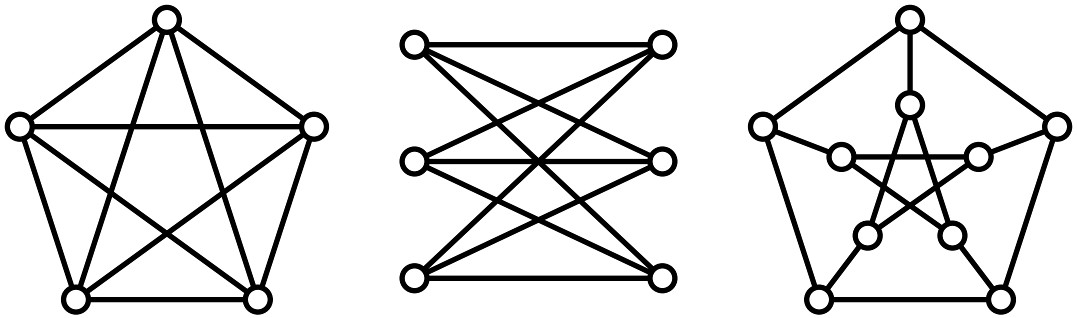
CORO. The complete graph K_5, the complete bipartite graph $K_{3,3}$, and the Petersen graph are not planar.



KURATOWSKI'S THEOREM

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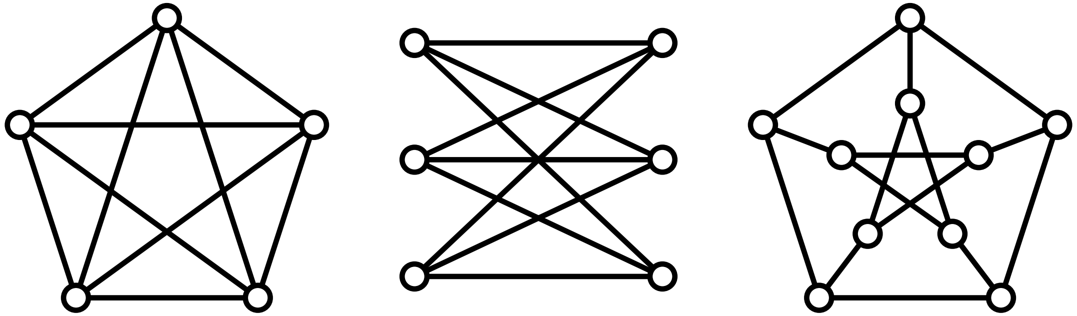
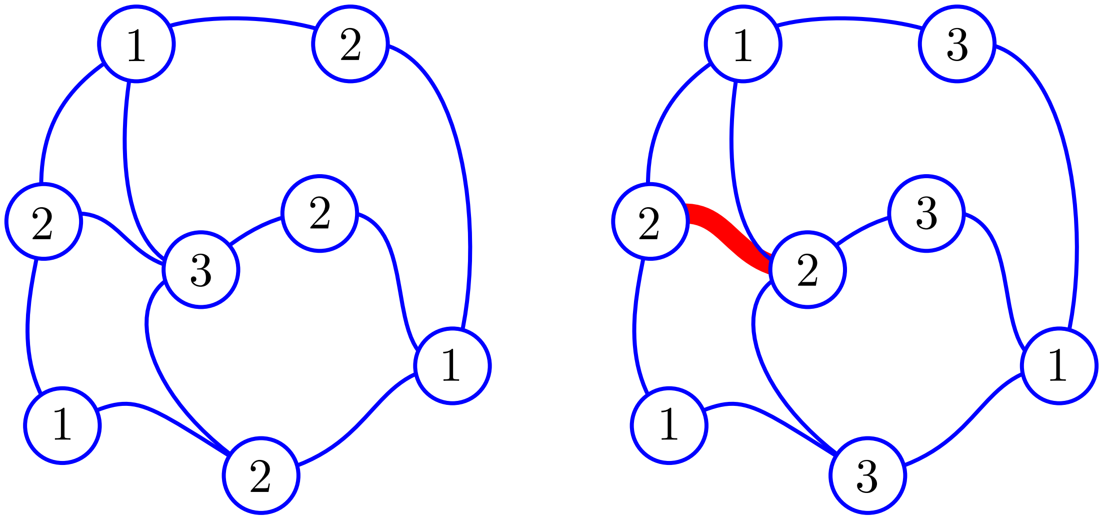


THM. (Kuratowski's theorem)
A graph is planar if and only if it contains no subdivision of K_5 and $K_{3,3}$.

COLORABILITY OF PLANAR GRAPHS

DEF. a graph is k-colorable if there is a coloring of its vertices by k colors such that no edge is monochromatic.



EXO. Show that any planar graph is 6-colorable.

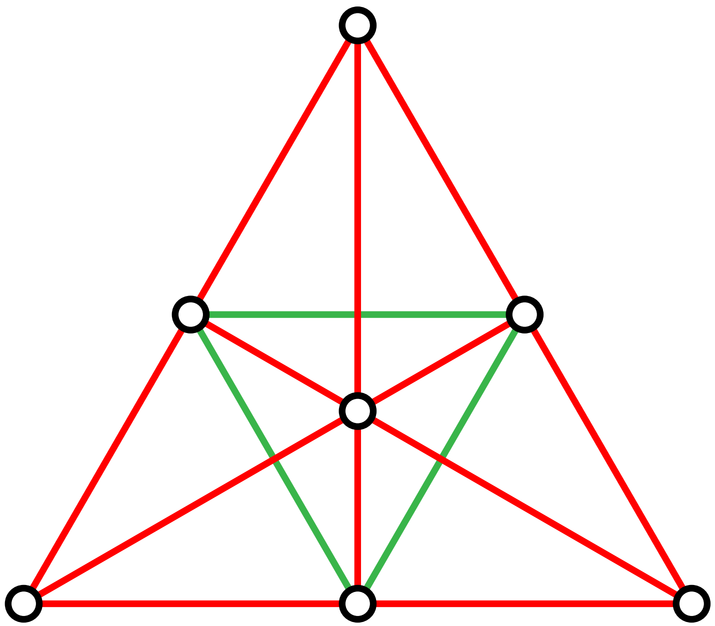
EXO. Show that any planar graph is 5-colorable.

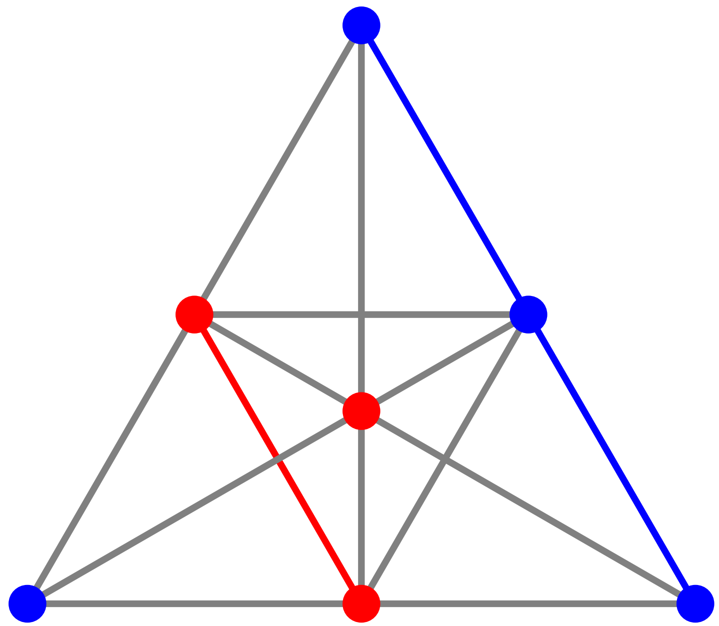
EXO. Show that any planar graph is 4-colorable.

SOME APPLICATIONS TO DISCRETE GEOMETRY

THM. (Sylvester–Gallai thm) If $n \geq 3$ points in the plane are not all on a line, there always exists a line containing exactly two points.

THM. There is always a monochromatic line in a bicolored planar point configuration.



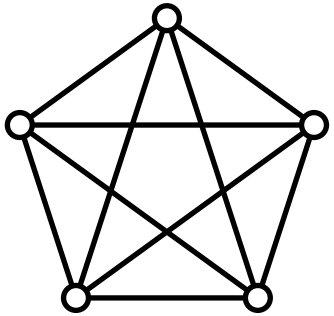


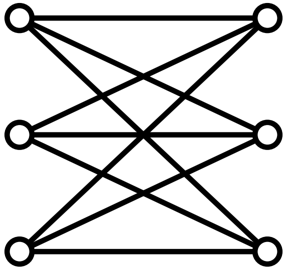
TOPOLOGICAL GRAPHS

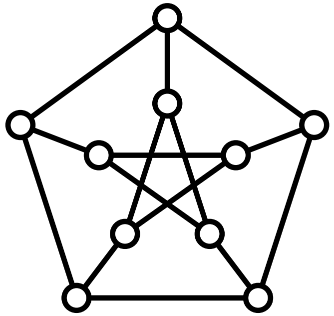
CROSSING NUMBER

DEF. crossing number of G = minimal number of crossings in a drawing of G.

QU. What is the crossing number of these graphs?







CROSSING NUMBER OF COMPLETE GRAPH

QU. What is the crossing number of the complete graph?

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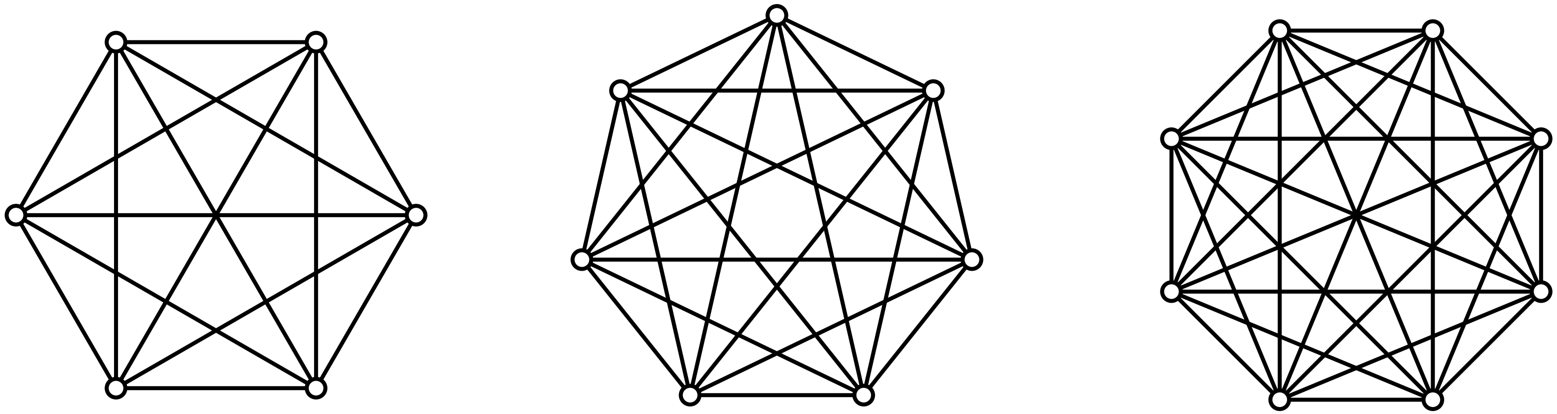
circle:

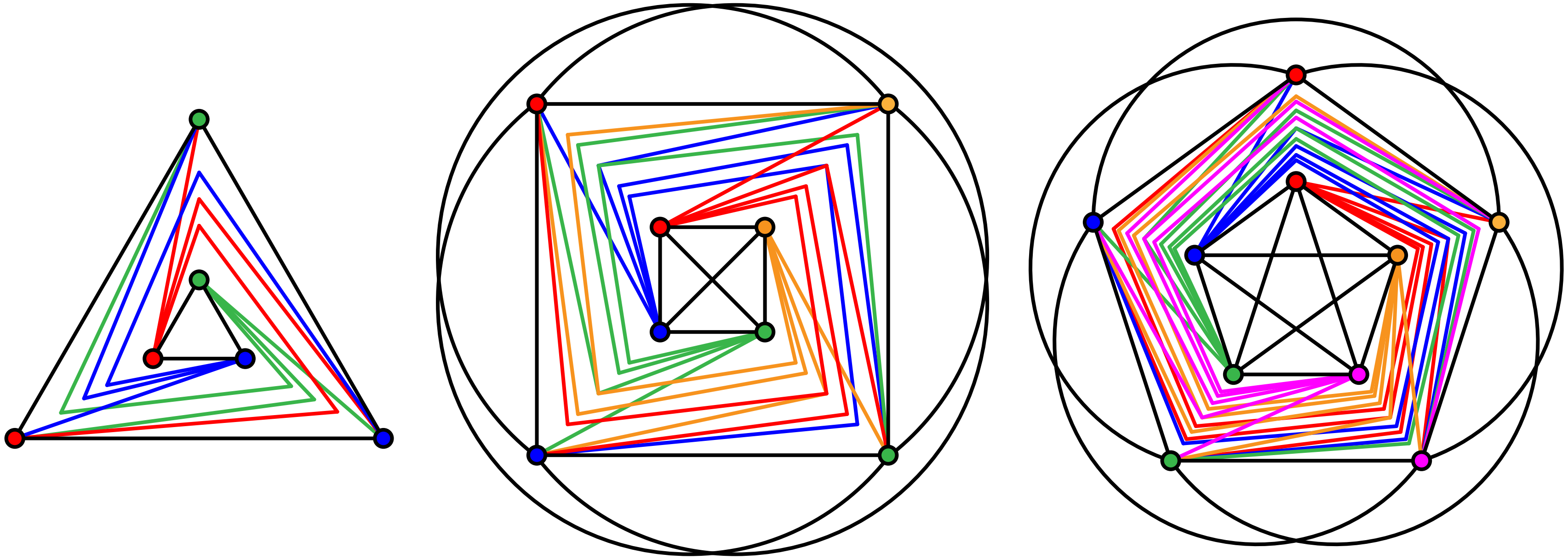


$$\mathrm{cr}(K_n) \leq \binom{n}{4}.$$

CROSSING NUMBER OF COMPLETE GRAPH

QU. What is the crossing number of the complete graph?

double circle:



$$n = 2\nu \qquad \mathrm{cr}(K_n) \leq \frac{\nu(\nu-1)^2(\nu-2)}{4} \simeq \frac{n^4}{64} \simeq \frac{3}{8}\binom{n}{4}.$$

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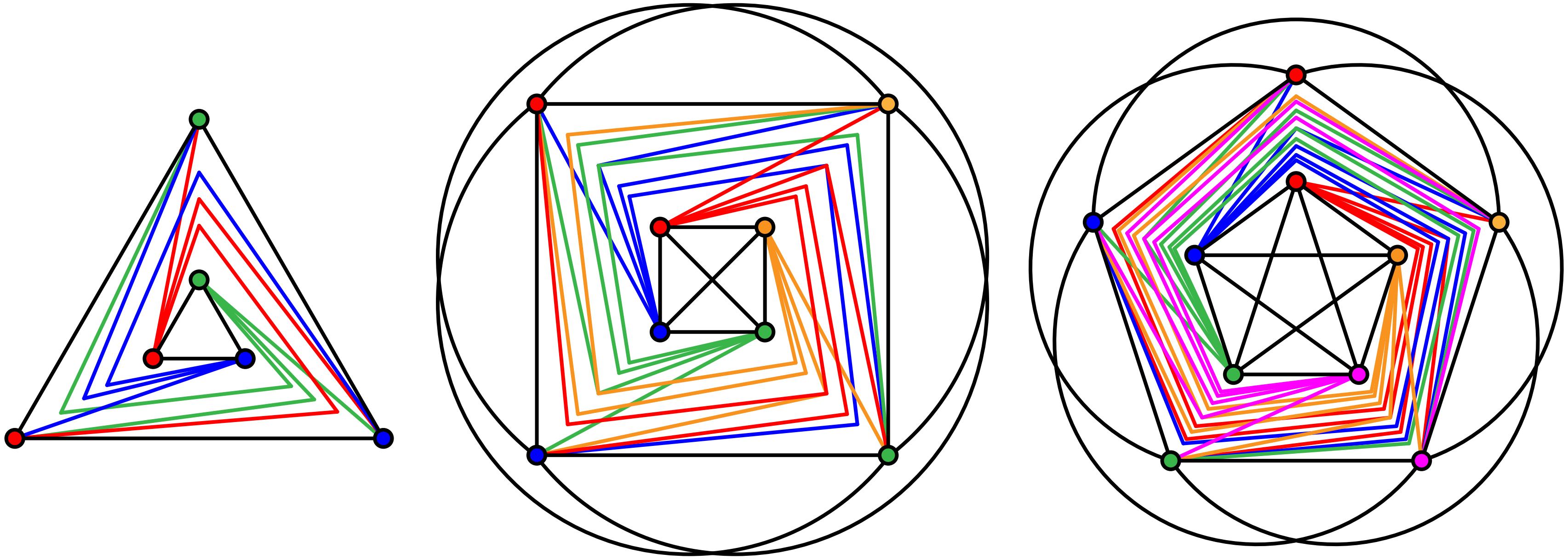

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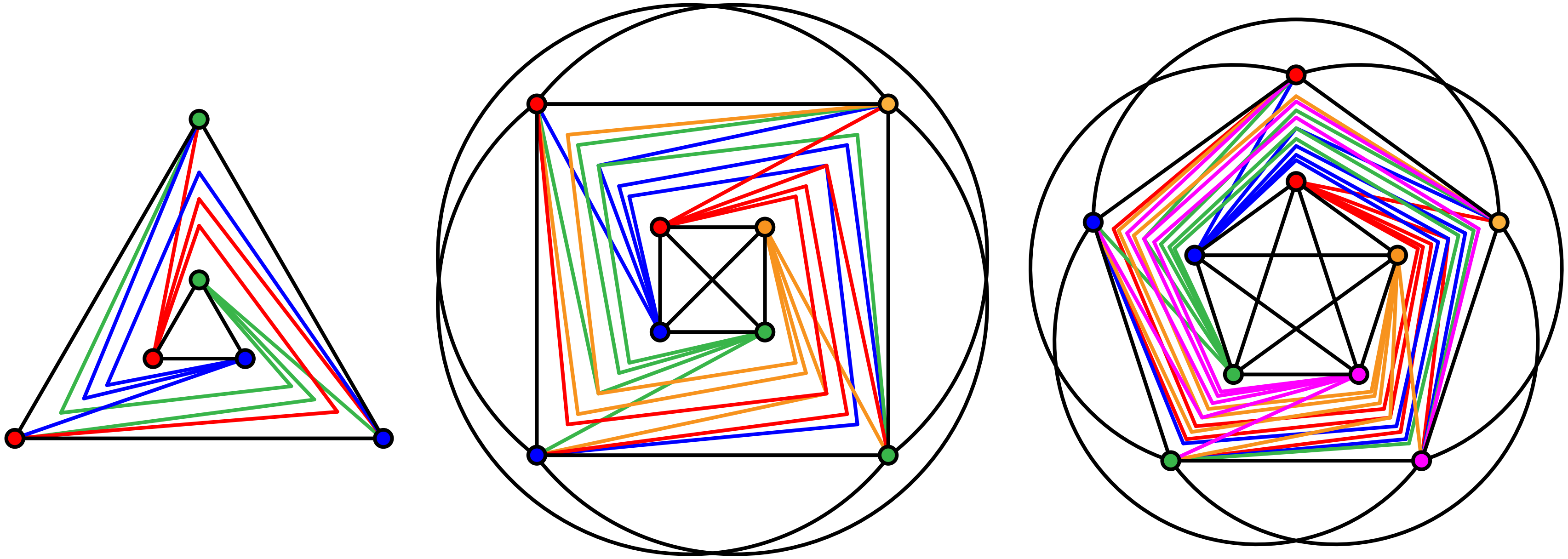
hints: E_i = edges joining a_i to b_j with $j \neq i$

- $\#E_i \cap E_j = (|i-j|-1)|i-j|/2 + (\nu - |i-j| - 1)(\nu - |i-j|)/2$,
- \# crossings on $E_i = \nu(\nu-1)^2(\nu-2)/3$.

CROSSING NUMBER OF COMPLETE GRAPH

QU. What is the crossing number of the complete graph?

double circle:



$$\mathrm{cr}(K_n) \le \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor \simeq \frac{n^4}{64} \simeq \frac{3}{8}\binom{n}{4}.$$

CONJ. (Guy '60) $$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

CROSSING LEMMA

LEM. For a graph G with v vertices and e edges, $\mathrm{cr}(G) \geq e - 3v + 6$.

THM. For a graph G with v vertices and $e \geq 4v$ edges, $\mathrm{cr}(G) \geq e^3/64v^2$.

rem: this bound is tight up to a constant as

$$v(K_n) = n \qquad e(K_n) = \binom{n}{2} \quad \text{and} \quad \mathrm{cr}(K_n) \leq \frac{3}{8}\binom{n}{4} \simeq \frac{e^3}{8v^2}.$$

CROSSING LEMMA

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proof: Consider an optimal drawing of G, and a random induced subgraph H of G obtained by picking independently each vertex of G with probability p.

The expected number of vertices, edges, and crossings of H are

$$\mathbb{E}(v(H)) = p \cdot v(G), \quad \mathbb{E}(e(H)) = p^2 \cdot e(G), \quad \text{and} \quad \mathbb{E}(\mathrm{cr}(H)) = p^4 \cdot \mathrm{cr}(G).$$

Hence

$$p^4 \cdot \mathrm{cr}(G) \geq p^2 \cdot e(G) - 3 \cdot p \cdot v(G) + 6 \geq p^2 e - 3pv.$$

Fix probability $p = 4v/e$ (thus the assumption $e \geq 4v$). Then

$$\mathrm{cr}(G) \geq e/p^2 - 3v/p^3 = e^3/64v^2.$$

APPLICATIONS TO INCIDENCE PROBLEMS

THM. (Szemerédi–Trotter) The maximum number

$$I(p,\ell) := \max_{\substack{\#\boldsymbol{P}=p \\ \#\boldsymbol{L}=\ell}} \#\{(\boldsymbol{p},\boldsymbol{l}) \mid \boldsymbol{p} \in \boldsymbol{P},\ \boldsymbol{l} \in \boldsymbol{L},\ \boldsymbol{p} \in \boldsymbol{l}\}$$

of incidences between p points and ℓ lines in the plane is bounded by

$$I(p,\ell) \leq 3.2\, p^{2/3}\ell^{2/3} + 4p + 2\ell.$$

THM. (Unit distances in the plane) The maximum number

$$U(p) := \max_{\#\boldsymbol{P}=p} \#\left\{(\boldsymbol{p},\boldsymbol{q}) \in \boldsymbol{P}^2 \;\middle|\; \|p-q\| = 1\right\}$$

of unit distances between p points in the plane is bounded by

$$U(p) \leq 4\, p^{4/3}.$$

GEOMETRIC GRAPHS

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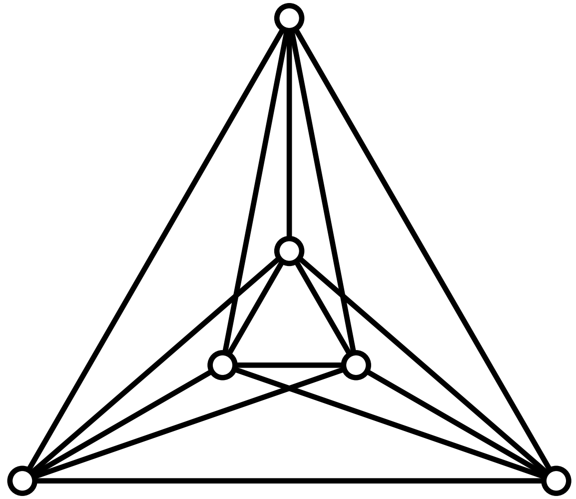
DEF. geometric drawing = vertices are points in $\mathbb{R}^2$ and edges are straight segments.



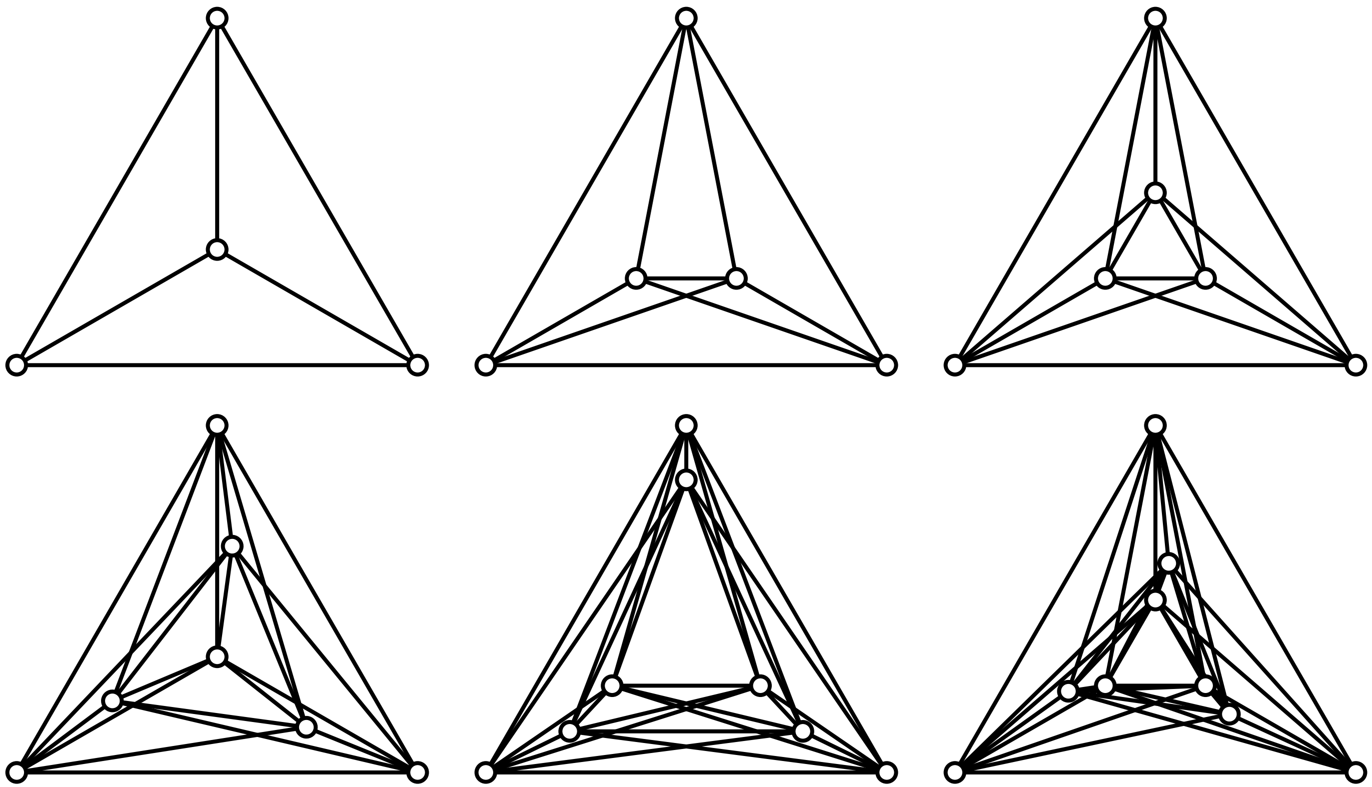
rem: sometimes, "geometric graphs" is used for "graphs defined by geometric means" like:

- graphs of polytopes,
- intersection graphs (intervals, disks, etc),
- visibility graph between objects,
- incidence graphs (point – line incidences),
- etc

RECTILINEAR CROSSING NUMBER

DEF. rectilinear crossing number of G = minimal number of crossings in a geometric drawing of G.

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\overline{\mathrm{cr}}(K_n)$	0	1	3	9	19	36	62	102	153	229	324	447	603	798	1029	1318



RECTILINEAR CROSSING NUMBER OF COMPLETE GRAPH

REM. Since any 5 points determine a convex quadrilateral,

$$\frac{1}{5}\binom{n}{4} \leq \overline{\mathrm{cr}}(K_n) \leq \binom{n}{4}.$$

We will prove that

$$\overline{\mathrm{cr}}(K_n) \geq \frac{1}{4}\binom{n}{4}.$$

Using more sophisticated arguments, one can show that:

PROP.

$$\overline{\mathrm{cr}}(K_n) \geq \left(\frac{3}{8}+\varepsilon\right)\binom{n}{4}.$$

CORO. For n large enough, $\mathrm{cr}(K_n) < \overline{\mathrm{cr}}(K_n)$.

CONCLUSION & REFERENCES

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Planar graphs are very special:

- combinatorially (few edges, Euler relation, 4-colorable, ...),
- algorithmically (use planar structure to design more efficient algorithms).

topological graphs $\neq$ geometric graphs

for instance, different asymptotic crossing numbers

References:

- Stefan Felsner. *Geometric graphs and arrangements*. *Advanced Lectures in Mathematics*. Friedr. Vieweg & Sohn, Wiesbaden, 2004.
- Richard K. Guy. A combinatorial problem. Nabla, 7:68–72, 1960.
- László Lovász, Katalin Vesztergombi, Uli Wagner, and Emo Welzl. Convex quadrilaterals and k-sets. In Towards a theory of geometric graphs, volume 342 of Contemp. Math., pages 139–148. Amer. Math. Soc., Providence, RI, 2004.