Multi-triangulations as complexes of star polygons

Vincent Pilaud (École Normale Supérieure)
& **Francisco Santos** (Universidad de Cantabria)

Brussels, March 2008

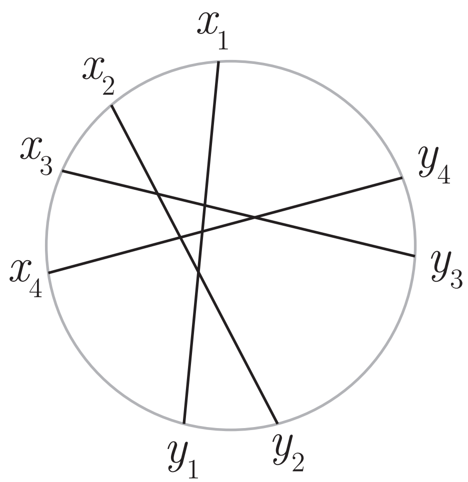
Definitions

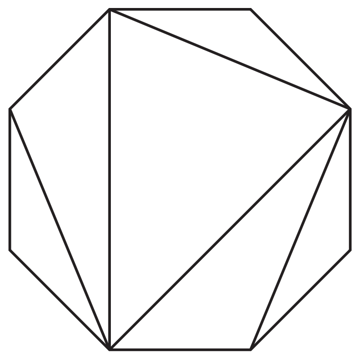
Multi-triangulations

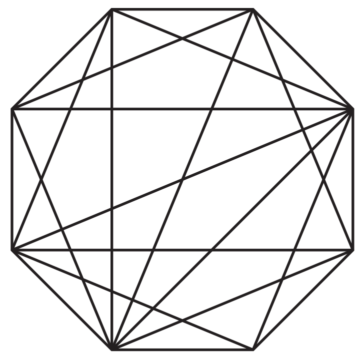
Let k and n be two integers with $n \geq 2k+1$.
Let V_n be the set of vertices of a convex n-gon..
Let E_n be the set of the edges of the complete graph on V_n.

Two edges $[a,b]$ and $[c,d]$ cross if the corresponding open segments $]a,b[$ and $]c,d[$ intersect.
An ℓ-crossing is a subset of E_n of ℓ mutually intersecting edges.

A k-triangulation of the n-gon is a maximal subset of E_n without $(k+1)$-crossing.





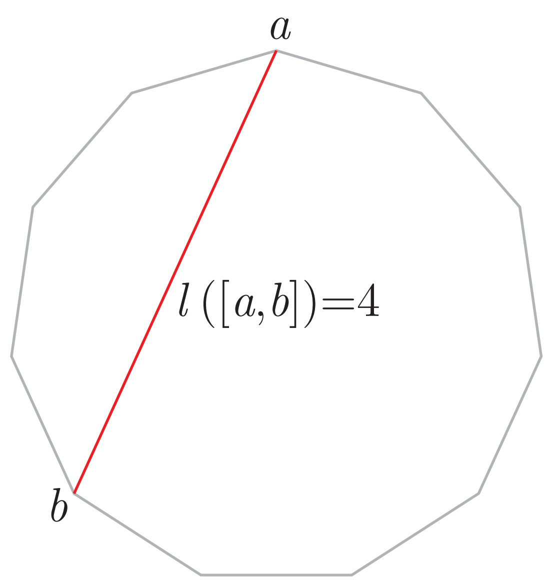


Remarks & examples

The length of an edge $[a, b]$ is

$$\ell([a, b]) = \min(|\llbracket a, b\llbracket|, |\llbracket b, a\llbracket|).$$

The only edges that may appear in a $(k + 1)$-crossing are those of length $> k$.



Remarks & examples

The length of an edge $[a,b]$ is

$$\ell([a,b]) = \min(|[\![a,b[\![|, |[\![b,a[\![|).$$

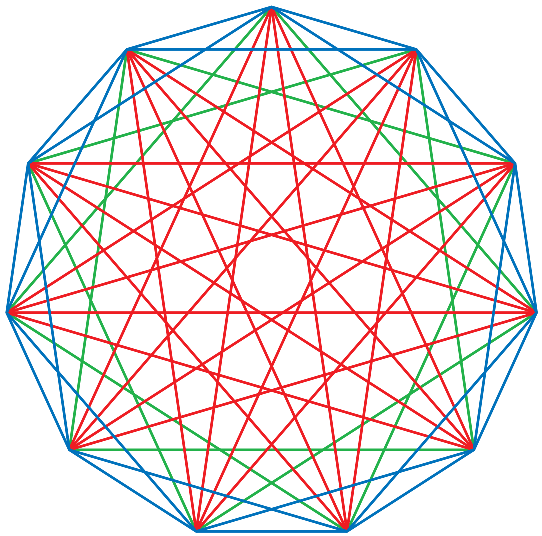
The only edges that may appear in a $(k+1)$-crossing are those of length $> k$.

We say that $[a,b]$ is a

(i) k-relevant edge if $\ell(\{a,b\}) > k$;

(ii) k-boundary edge if $\ell(\{a,b\}) = k$;

(iii) k-irrelevant edge if $\ell(\{a,b\}) < k$.



Remarks & examples

The length of an edge $[a, b]$ is

$$\ell([a,b]) = \min(|[\![a,b[\![|, |[\![b,a[\![|).$$

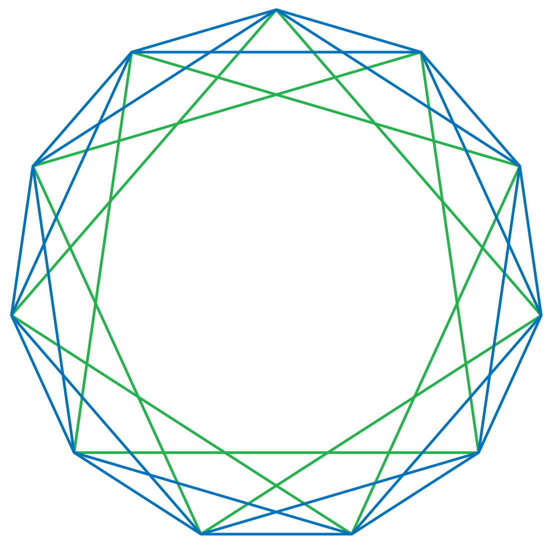
The only edges that may appear in a $(k+1)$-crossing are those of length $> k$.

We say that $[a, b]$ is a

(i) k-relevant edge if $\ell(\{a,b\}) > k$;

(ii) k-boundary edge if $\ell(\{a,b\}) = k$;

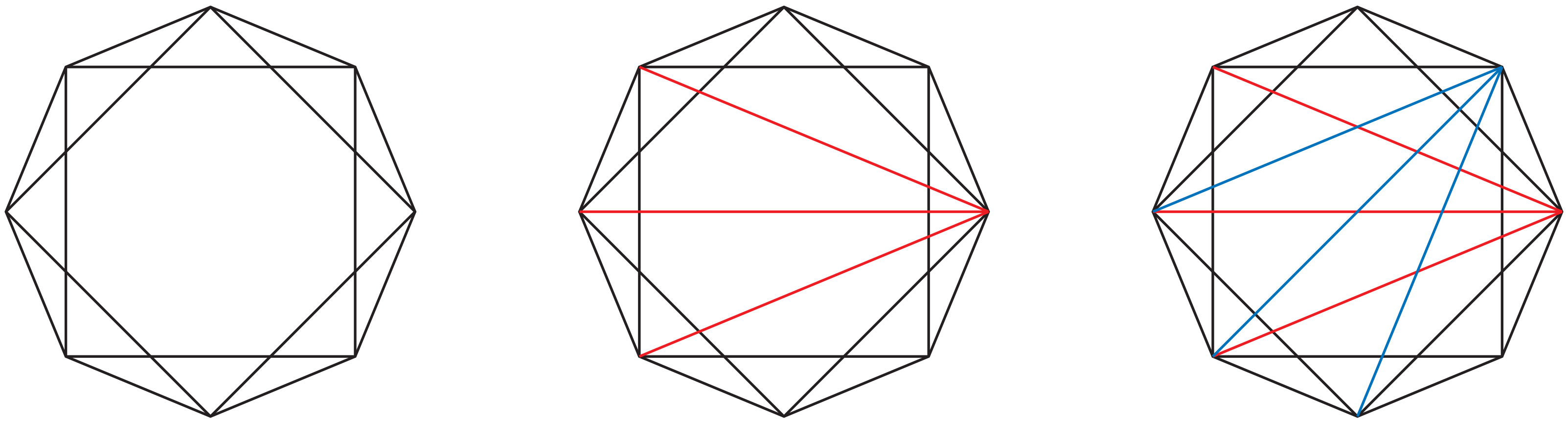
(iii) k-irrelevant edge if $\ell(\{a,b\}) < k$.



Any k-triangulation of the n-gon contains all the k-irrelevant and the k-boundary edges of E_n.

Remarks & examples

A general construction



$n = 2k + 1$

The complete graph K_{2k+1} is the unique k-triangulation of the $(2k+1)$-gon.

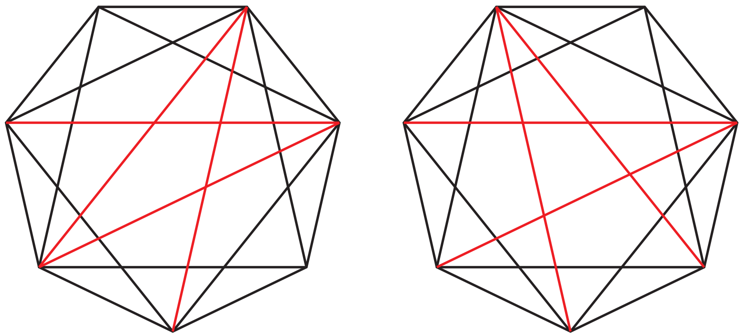
$n = 2k + 2$

All k-triangulations of the $(2k+2)$-gon are obtained by suppression of a long diagonal of the complete graph K_{2k+2}.

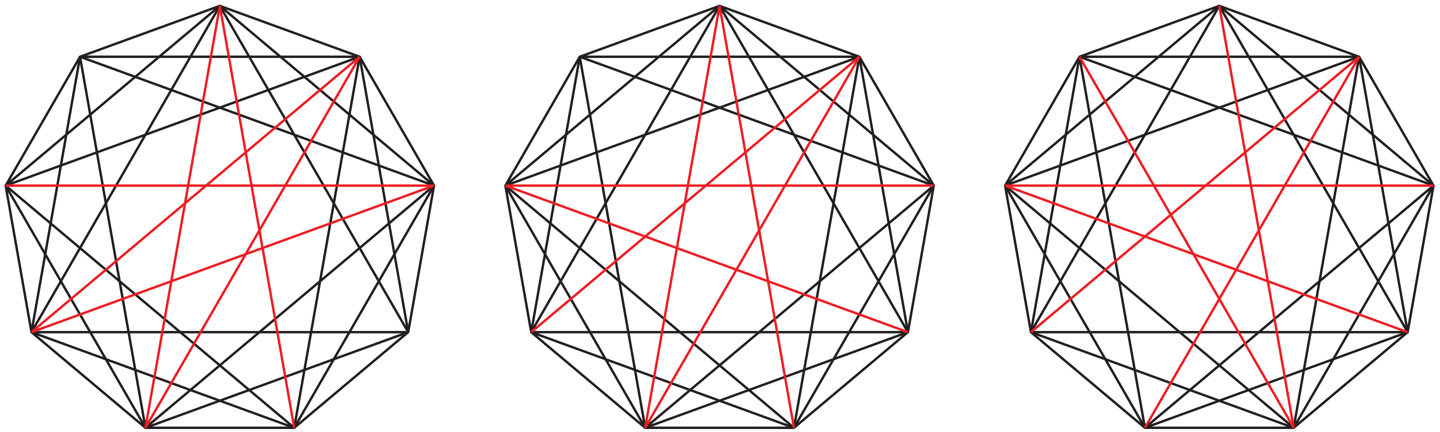
Remarks & examples

$n = 2k + 3$

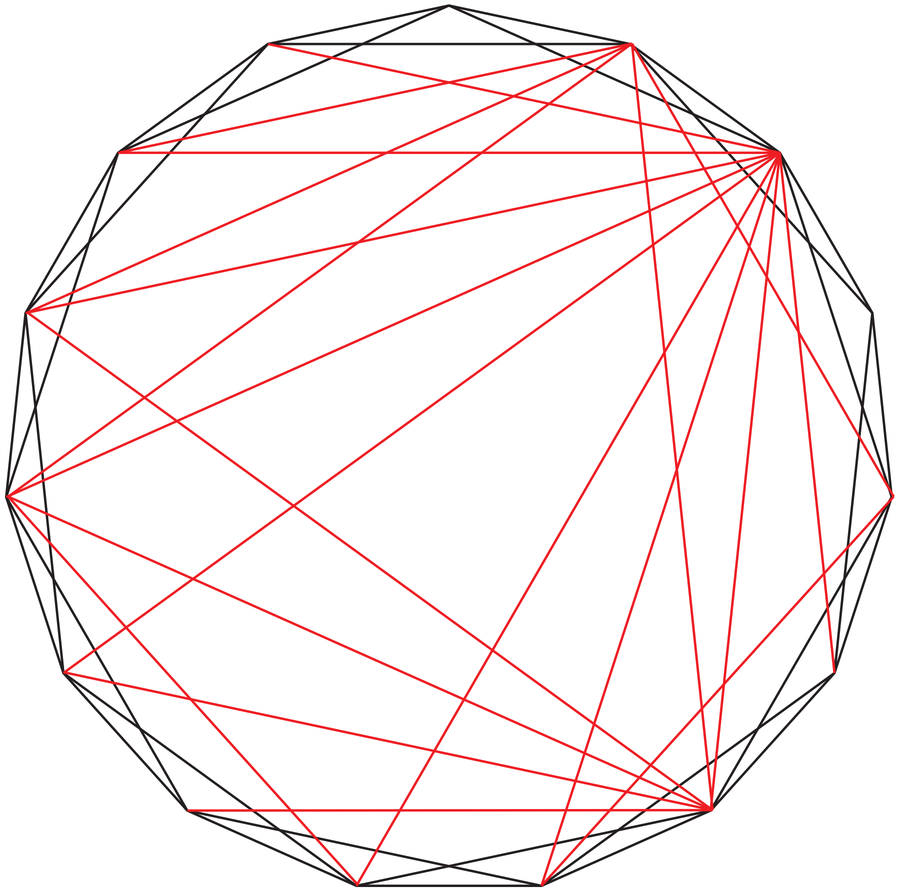
There are 14 2-triangulations of the heptagon :



There are 30 3-triangulations of the nonagon :



Remarks & examples



Already known results

Théorème.

1. A k-triangulation of the n-gon contains $k(2n-2k-1)$ edges. [Nak], [DKM]
2. Any relevant edge can be flipped and the graph of flips is connected. [Nak], [Jon]
3. There exists a deletion/insertion operation that transforms a k-triangulation of the $(n+1)$-gon into a k-triangution of the n-gon and reciprocaly. [Nak], [Jon]
4. The k-triangulations of the n-gon are counted by a Catalan determinant : $\det(C_{n-i-j})_{i,j\leq k}$. [Jon]
5. If $n \geq 2k+3$, any k-triangulation of the n-gon has at least $2k$ ears. [Nak]

v. Capoyleas & j. Pach, A Turán-type theorem on chords of a convex polygon, 1992

t. Nakamigawa, A generalization of diagonal flips in a convex polygon, 2000

a. Dress, j. Koolen & v. Moulton, On line arrangements in the hyperbolic plane, 2002

j. Jonsson, Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005

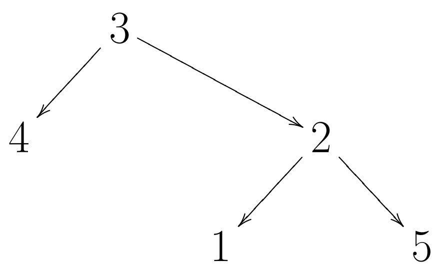
Already known results

Théorème.

1. A k-triangulation of the n-gon contains $k(2n-2k-1)$ edges. [Nak], [DKM]
2. Any relevant edge can be flipped and the graph of flips is connected. [Nak], [Jon]
3. There exists a deletion/insertion operation that transforms a k-triangulation of the $(n+1)$-gon into a k-triangution of the n-gon and reciprocaly. [Nak], [Jon]
4. The k-triangulations of the n-gon are counted by a Catalan determinant : $\det(C_{n-i-j})_{i,j\leq k}$. [Jon]
5. If $n \geq 2k+3$, any k-triangulation of the n-gon has at least $2k$ ears. [Nak]

Two remarks.

– undirect proofs :



– generalisation of triangles ?

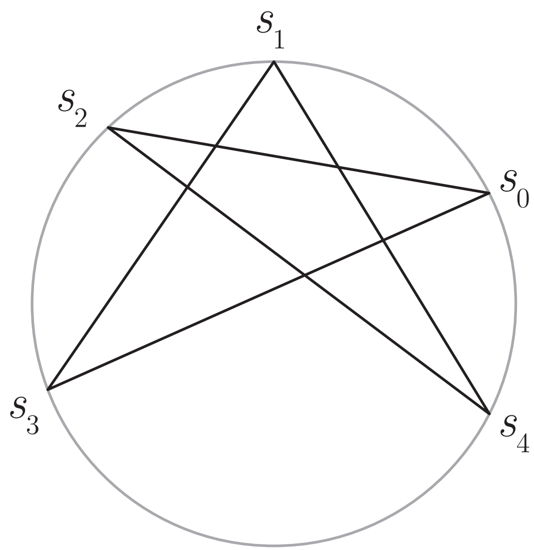
k-STARS

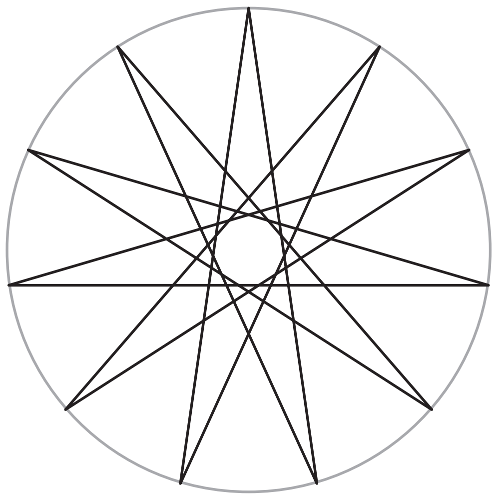
Let $s_0, \dots, s_{2k}$ be $2k+1$ points of the unit circle in counterclockwise order.

We say that the polygon

- whose vertices are $s_0, \dots, s_{2k}$,
- and whose edges are $[s_0, s_k], [s_1, s_{1+k}], \dots, [s_k, s_{2k}], [s_{k+1}, s_0], \dots, [s_{2k}, s_{k-1}]$

is a k-star.





Angles

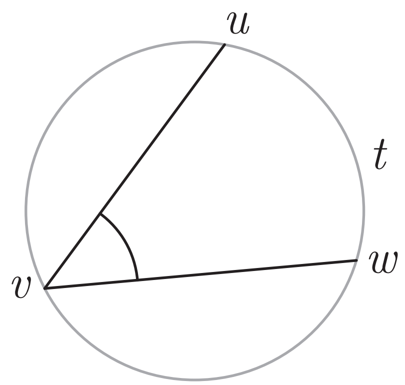
An angle of a subset F of E_n is a couple

$$\angle(u,v,w) = ([u,v],[v,w])$$

of edges of F such that

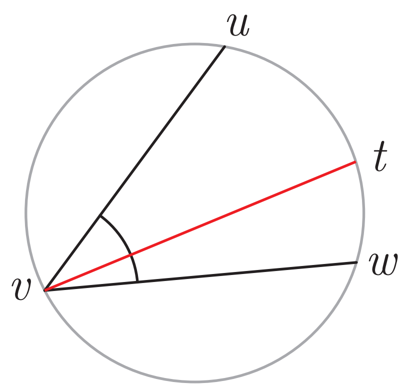
– $u \prec v \prec w$ (for the counterclockwise order),

– for all $t \in [\![w,u]\!]$, the edge $\{v,t\}$ is not in F.



v is the vertex of the angle $\angle(u,v,w) = (\{u,v\},\{v,w\})$.

For all $t \in [\![w,u]\!]$, the edge $\{v,t\}$ is a bisector of $\angle(u,v,w)$.



An angle $\angle(u,v,w)$ is k-relevant if its edges are both either k-relevant, or k-boundary.

Results

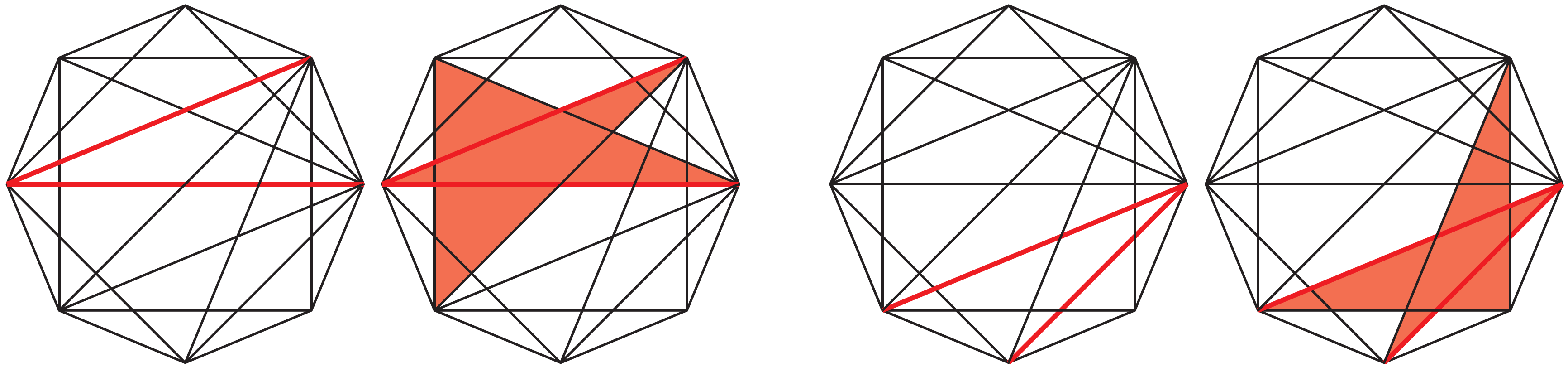
k-TRIANGULATIONS = COMPLEXES DE k-STARS

THEOREM.

Let T be a k-triangulation.

Any angle of a k-star of T is a k-relevant angle of T.

Reciprocaly, any k-relevant angle of T is contained in a k-star of T.

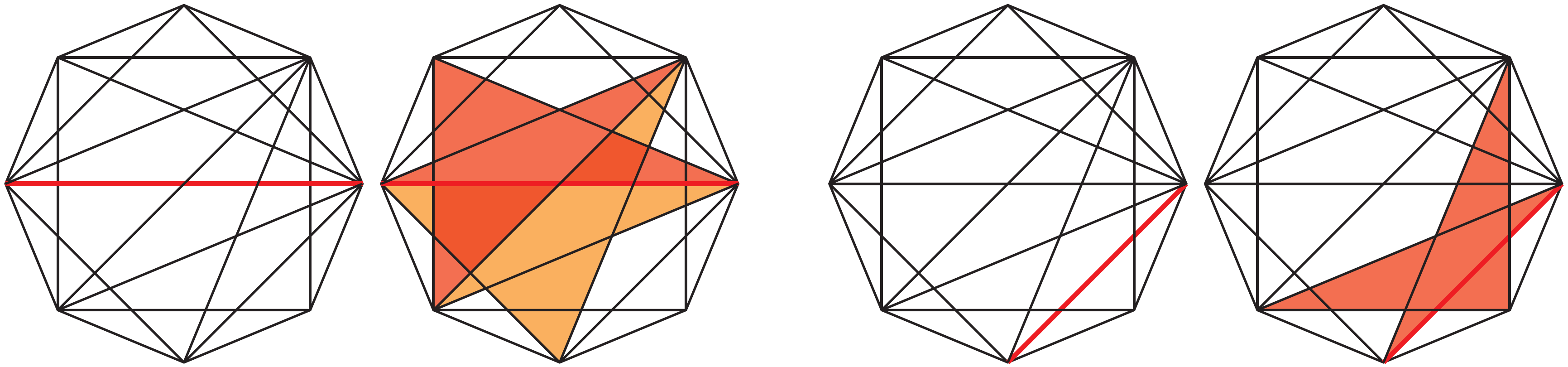


k-TRIANGULATIONS = COMPLEXES DE k-STARS

COROLLARY.

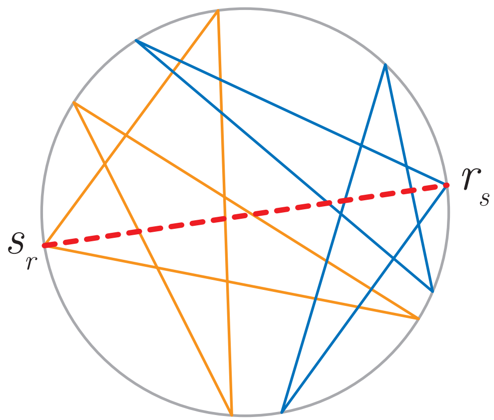
Let e be an edge of a k-triangulation T. Then

1. if e is a k-relevant edge, it belongs to exactly two k-stars of T,
2. if e is a k-boundary edge, it belongs to exactly one k-star of T,
3. if e is a k-irrelevant edge, it does not belong to any k-star of T.



Common bisector

Theorem.
Every pair of k-stars of a k-triangulation have a unique common bisector.



Proposition.
Let T be a k-triangulation. Any edge which is not in T is the common bisector of a unique pair of k-stars of T.

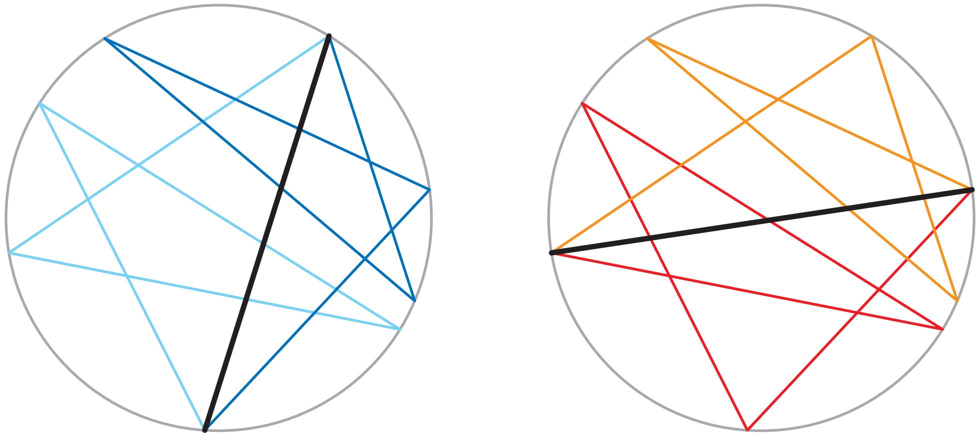
Corollary.
Any k-triangulation of the n-gon contains exactly $n-2k$ k-stars and thus $k(2n-2k-1)$ edges.

Flips

Theorem.

Let T be a k-triangulation of the n-gon. Let e be an edge of T. Let R and S be the two k-stars of T containing e. Let f be the common bisector of R and S.

Then T and $T\triangle\{e,f\}$ are the only two k-triangulations of the n-gon containing $T \setminus \{e\}$.



The k-triangulation $T\triangle\{e,f\}$ is obtained by flipping the edge e in the k-triangulation f.

Flips

Let $G_{n,k}$ be the graph of flips of the set of k-triangulations of the n-gon.

Theorem.

The graph $G_{n,k}$ is connected, regular of degree $k(n-2k-1)$, and its diameter is at most $2k(n-2k-1)$.

Remark.

(i) if $n > 8k^3 + 4k^2$, the bound on the diameter can be improved to be $2nk - (8k^2 + 2k)$. [Nak]

(ii) for $k = 1$, this bound is optimal.

d.d. Sleator, r.e. Tarjan & w.p. Thurston,
Rotation distance, triangulations and hyperbolic geometry, 1988

For $k > 1$ and $n > 4k$, we only know that the diameter is at least $k(n-2k-1)$.

k-ears & k-colorable k-triangulations

Let assume here that $n > 2k + 3$.
A k-ear is an edge of length $k + 1$.
We say that a k-star is internal if it does not contain any k-boundary edge.

PROPOSITION.

The number of k-ears of a k-triangulation T equals the number of internal k-stars plus $2k$.
In particular, T contains at least $2k$ k-ears.

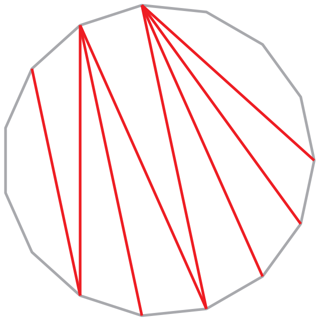
k-EARS & k-COLORABLE k-TRIANGULATIONS

We say that a k-triangulation is k-colorable if there exists a coloration with k color of its k-relevant edges such that there is no monochromatic crossing.

A k-accordion of E_n is a set

$$Z = \{[a_i, b_i] \mid 1 \leq i \leq n - 2k - 1\}$$

of $n - 2k - 1$ edges such that
- $b_1 = a_1 + k + 1$
- $[a_i, b_i] \in \{[a_{i-1}, b_{i-1} + 1], [a_{i-1} - 1, b_{i-1}]\}$, for all i.



PROPOSITION.

Let T be a k-triangulation, with $k > 1$. The following assertions are equivalent

(i) T is k-colorable ;

(ii) T contains exactly $2k$ k-ears ;

(iii) T has no internal k-star ;

(iv) the set of k-relevant edges of T is the disjoint union of k k-accordions.

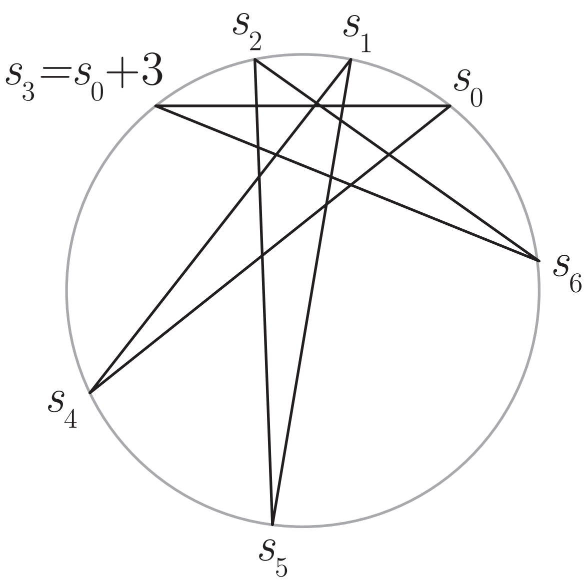
Flattening a k-star/inflatting a k-crossing

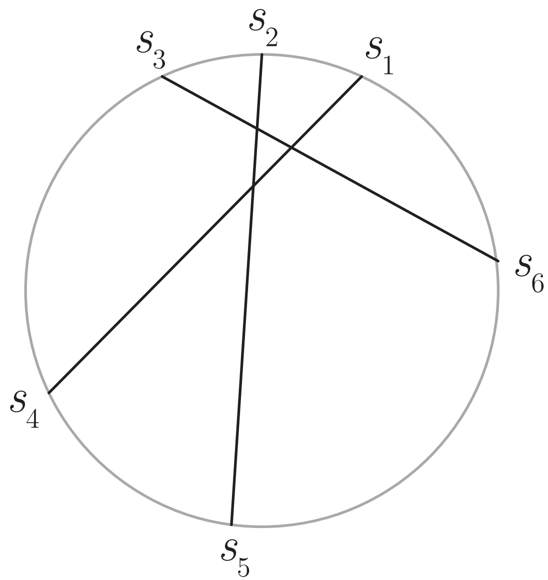
Theorem.

There is a bijection between

(i) the set of k-triangulations of the $(n+1)$-gon with a marked boundary edge, and

(ii) the set of k-triangulations of the n-gone with a marked k-crossing with k consecutives vertices.





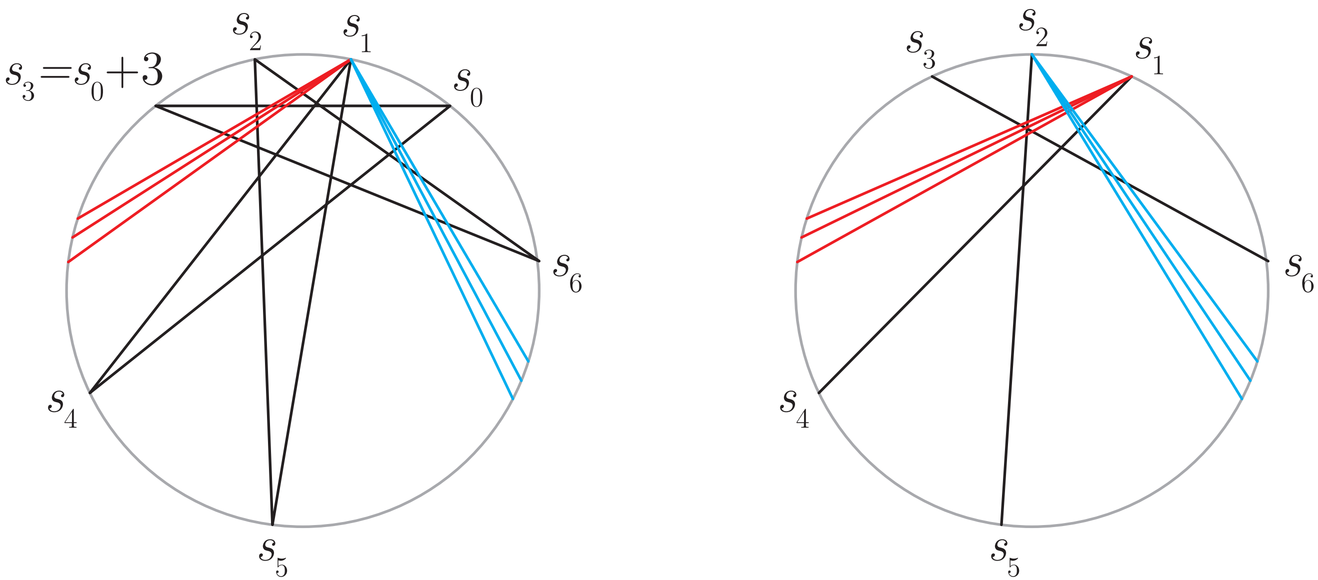
Flattening a k-star/inflatting a k-crossing

Theorem.

There is a bijection between

(i) the set of k-triangulations of the $(n+1)$-gon with a marked boundary edge, and

(ii) the set of k-triangulations of the n-gone with a marked k-crossing with k consecutives vertices.



Further topics and open questions

Multi-Dyck Paths

Theorem. [Jon]

The number of k-triangulations of the n-gon is

$$\det(C_{n-i-j})_{1\le i,j\le k} = \left| \begin{pmatrix} C_{n-2} & C_{n-3} & \dots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \dots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \dots & C_{n-2k+1} & C_{n-2k} \end{pmatrix} \right|, \qquad \text{where} \quad C_m = \frac{1}{m+1}\binom{2m}{m}.$$

Multi-Dyck Paths

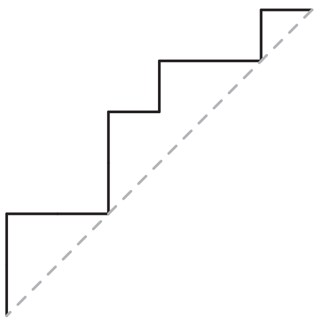
Theorem. [Jon]

The number of k-triangulations of the n-gon is

$$\det(C_{n-i-j})_{1\le i,j\le k} = \left|\begin{pmatrix} C_{n-2} & C_{n-3} & \dots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \dots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \dots & C_{n-2k+1} & C_{n-2k} \end{pmatrix}\right|, \qquad \text{where} \quad C_m = \frac{1}{m+1}\binom{2m}{m}.$$

A Dyck path of semi-length ℓ is a lattice path using north steps $N = (0,1)$ and east steps $E = (1,0)$ starting from $(0,0)$ and ending at (ℓ,ℓ), and such that it never goes below the diagonal $y = x$.

The set of Dyck paths of semi-length $n-2$ is in bijection with the set of triangulations of the n-gon.



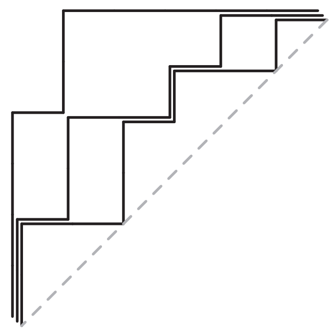
Multi-Dyck Paths

Theorem. [Jon]

The number of k-triangulations of the n-gon is

$$\det(C_{n-i-j})_{1\le i,j\le k} = \left| \begin{pmatrix} C_{n-2} & C_{n-3} & \dots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \dots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \dots & C_{n-2k+1} & C_{n-2k} \end{pmatrix} \right|, \qquad \text{where} \quad C_m = \frac{1}{m+1}\binom{2m}{m}.$$

A Dyck path of semi-length ℓ is a lattice path using north steps $N = (0,1)$ and east steps $E = (1,0)$ starting from $(0,0)$ and ending at (ℓ,ℓ), and such that it never goes below the diagonal $y = x$.



A k-Dyck path of semi-length ℓ is a k-tuple $(d_1, \dots, d_k)$ of Dyck paths of semi-length ℓ such that each d_i never goes above d_{i-1}, for $2 \le i \le k$.

Multi-Dyck Paths

Theorem. [Jon]

The number of k-triangulations of the n-gon is

$$\det(C_{n-i-j})_{1\le i,j\le k} = \left| \begin{pmatrix} C_{n-2} & C_{n-3} & \dots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \dots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \dots & C_{n-2k+1} & C_{n-2k} \end{pmatrix} \right|, \qquad \text{where} \quad C_m = \frac{1}{m+1}\binom{2m}{m}.$$

Theorem.

The number of k-Dyck paths of semi-length $n-2k$ is $\det(C_{n-i-j})_{1\le i,j\le k}$.

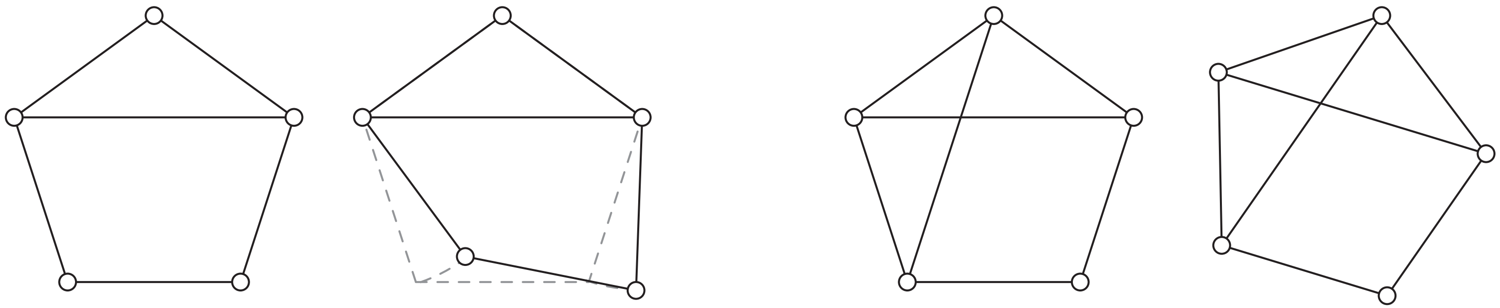
m. Desainte-Catherine & g. Viennot,
Enumeration of certain Youg tableaux with bounded height, 1986

We have explicit bijections only when $k=1$ and $k=2$.

s. Elizalde, A bijection between 2-triangulations and pairs of non-crossing Dyck paths, 2006

Rigidity

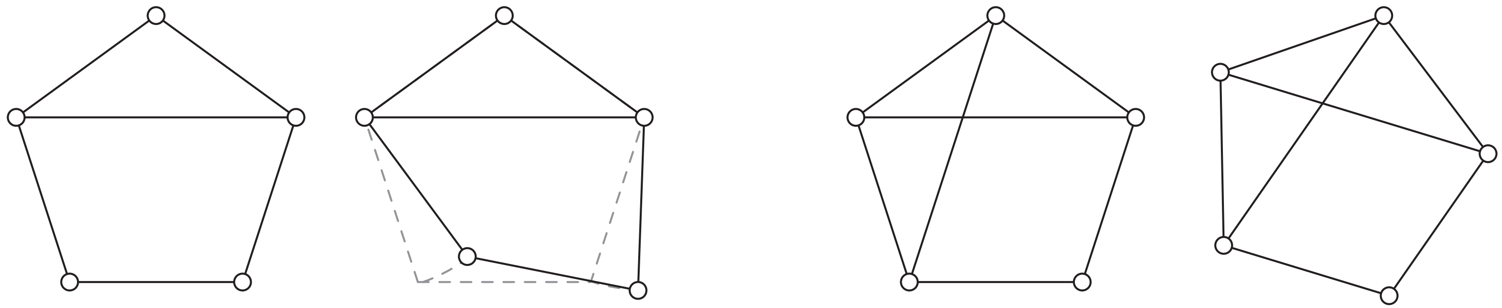
A graph $G = (V, E)$, embedded in $\mathbb{R}^d$, is said to be rigid if any continuous movement of its vertices that preserves all edges lengths is an isometry of $\mathbb{R}^d$.



A triangulation is a minimally rigid graph of the plane.

Rigidity

A graph $G = (V, E)$, embedded in $\mathbb{R}^d$, is said to be rigid if any continuous movement of its vertices that preserves all edges lengths is an isometry of $\mathbb{R}^d$.



A triangulation is a minimally rigid graph of the plane.

Conjecture.

A k-triangulation is a minimally rigid graph in dimension $2k$.

Two remarks.

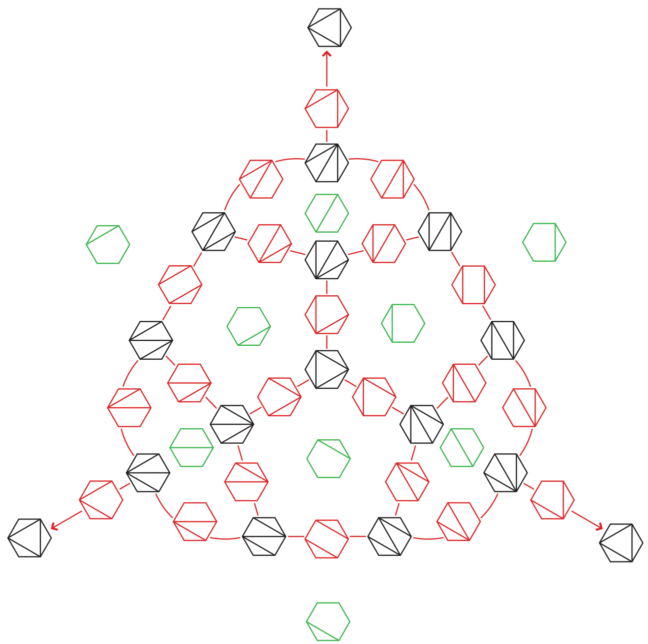
– k-triangulations have $2k$-Laman property.
– we have a proof for $k = 2$.

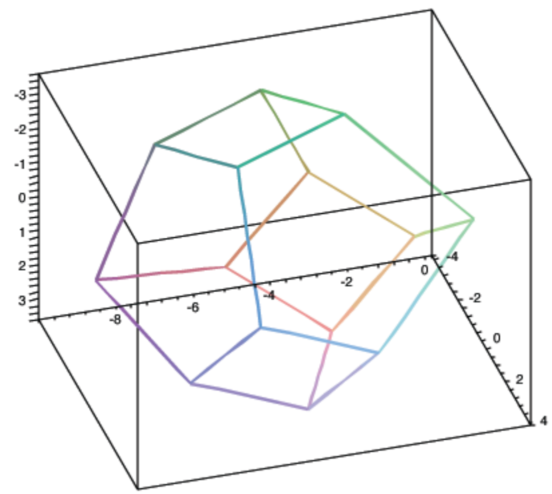
Multi-associahedron

Let $\Delta_{n,k}$ be the complex of all subsets of k-relevant edges of E_n that do not contain any $(k+1)$-crossing.

When $k = 1$, this complex is known to be the boundary complex of the associahedron.

C. Lee, The associahedron and triangulations of an n-gon, 1989





Multi-associahedron

Let $\Delta_{n,k}$ be the complex of all subsets of k-relevant edges of E_n that do not contain any $(k+1)$-crossing.

When $k = 1$, this complex is known to be the boundary complex of the associahedron.

c. Lee, The associahedron and triangulations of an n-gon, 1989

When $k \geq 2$, we only know that $\Delta_{n,k}$ is topologically a sphere . [Jon]

Conjecture.
There exists a simple polytope of dimension $k(n-2k-1)$ with boundary complex $\Delta_{n,k}$.

Remark. area of stars and rigidity can help.

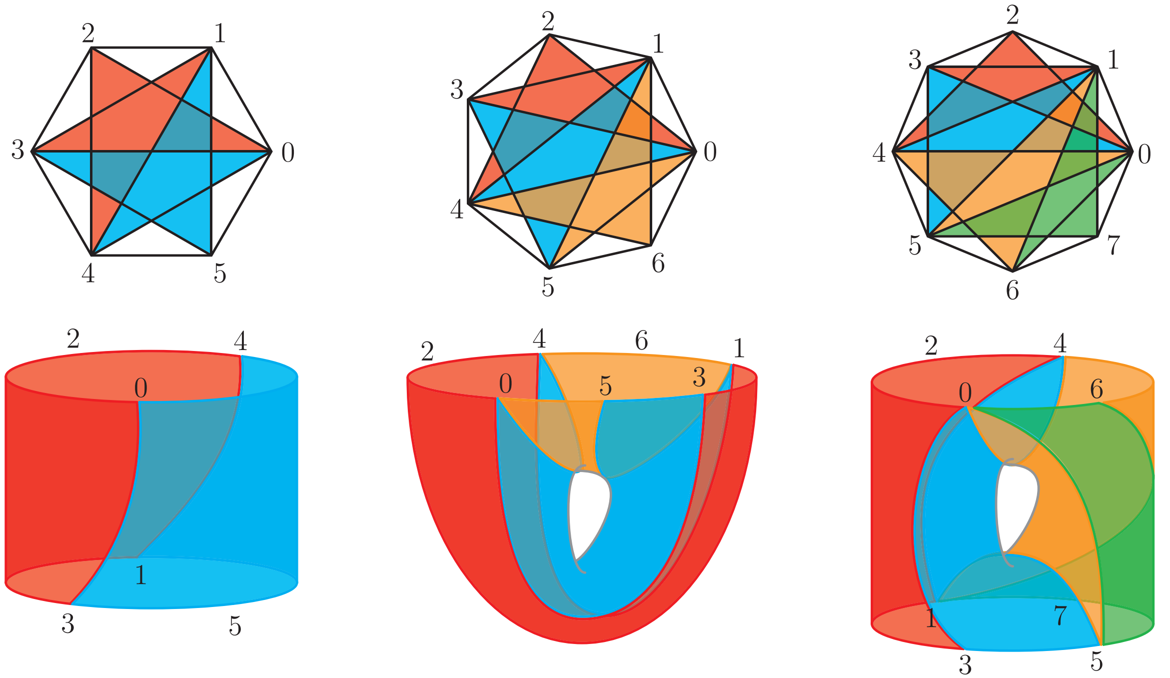
l. Billera, p. Filliman & B. Sturmfels,
Constructions and complexity of secondary polytopes, 1990

g. Rote, f. Santos & I. Streinu,
Expansive motions and the polytope of pointed pseudo-triangulaitons, 2003

Surfaces

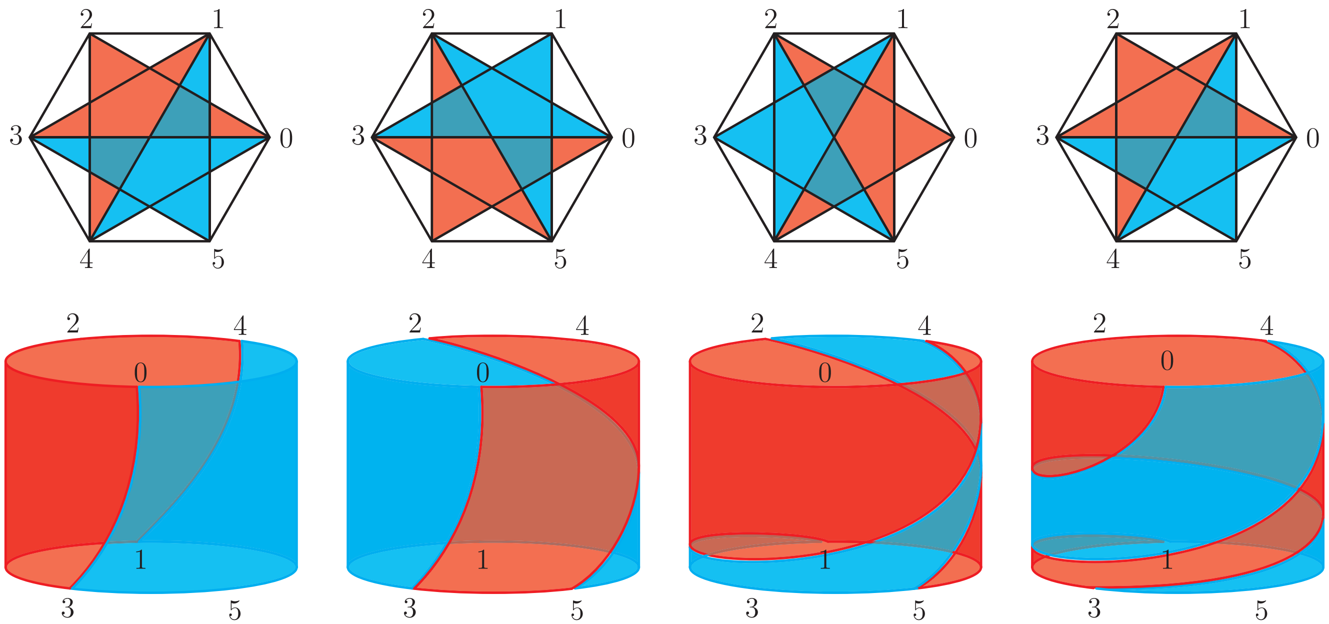
Let T be a k-triangulation of the n-gon.

The polygonal complex $\mathcal{C}(T)$ associated to T is a polygonal decomposition of an orientable surface with boundary $\mathcal{S}_{n,k}$.



The genus of $\mathcal{S}_{n,k}$ is $g_{n,k} = \frac{1}{2}(2 - f + e - v - b) = \frac{1}{2}(2 - n + k + kn - 2k^2 - \gcd(n,k))$.

Surfaces



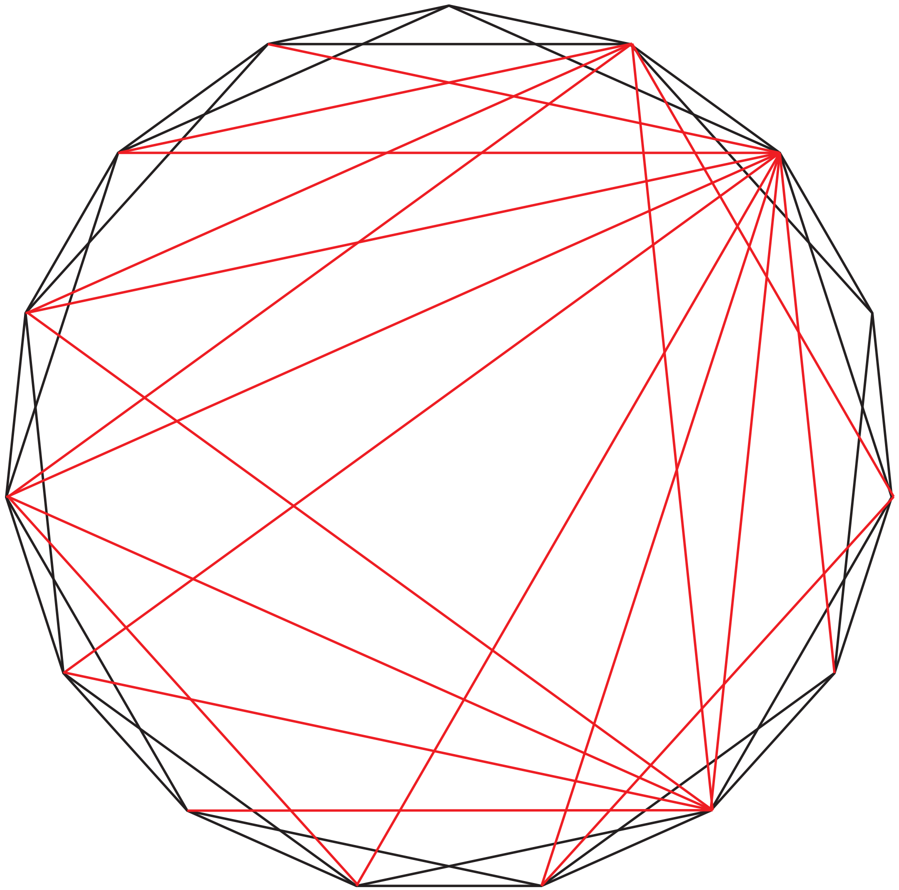
Flips define a morphism between

(i) the fundamental group $\pi_{n,k}$ of the graph of flips $G_{n,k}$ (*i.e.* the set of loops in $G_{n,k}$, up to homotopy), and

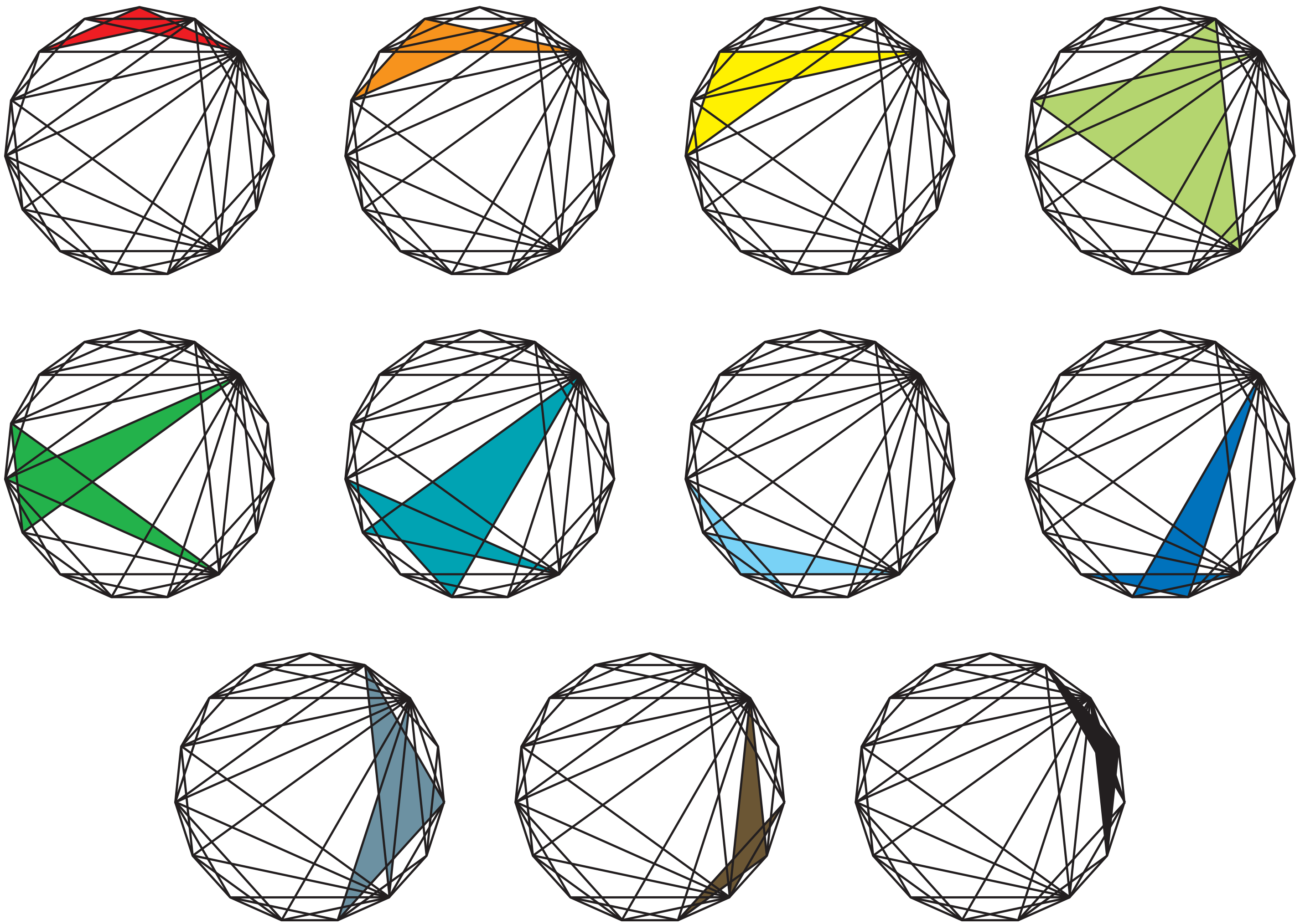
(ii) the mapping class group $\mathcal{M}_{n,k}$ of the surface $\mathcal{S}_{n,k}$ (*i.e.* the set of diffeomorphisms of the surface $\mathcal{S}_{n,k}$ into itself that preserve the orientation and that fixe the boundary of $\mathcal{S}_{n,k}$, up to isotopy).

Conclusion

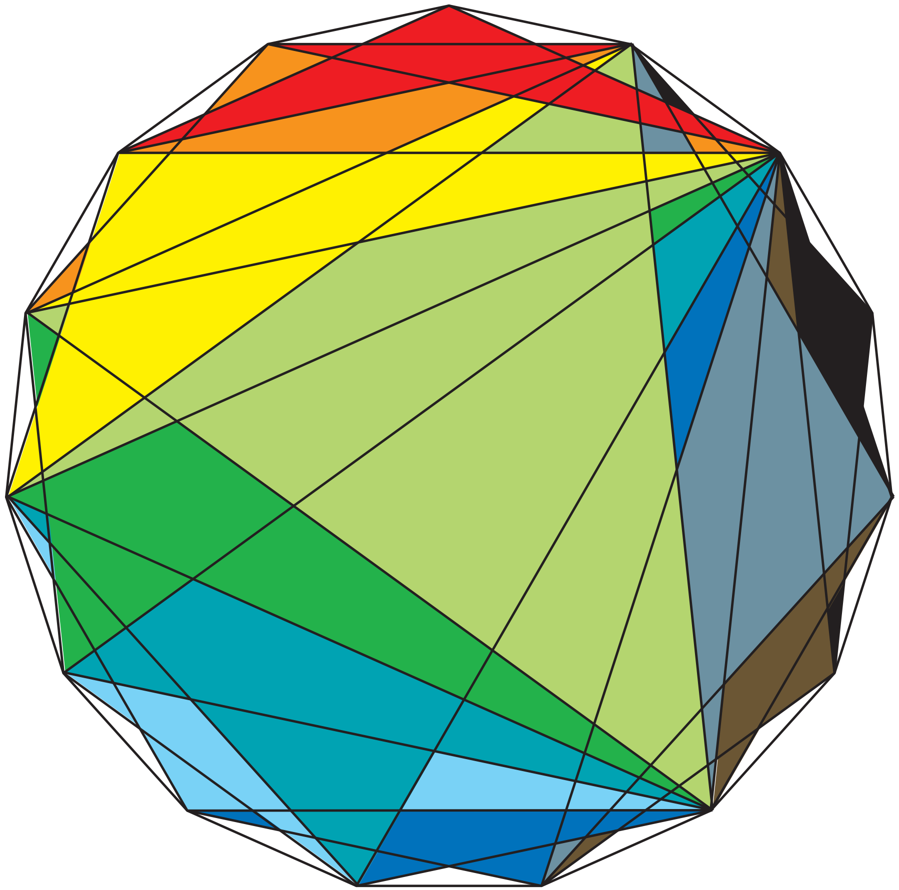
Conclusion



Conclusion



Conclusion



Multi-triangulations as complexes of star polygons

Vincent Pilaud & Francisco Santos

arXiv : 0706.3121v2