Superellipsoids: a generalization of the interval, zonotope and ellipsoid domains

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14th of july 2011

The Egg of Columbus

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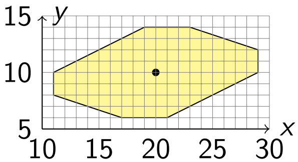
Introduction

Context

- Static analysis of (numerical) programs, by abstract interpretation
- In order to infer *range* of program variables, *generate tests*, *prove functional properties* (FLUCTUAT, both f.p. and real numbers) etc.

Contents

- Recap on the (functional) *zonotopic* abstract domain
- Extension to *ellipsoids*, with the same kind of parametrization but a change of norm
- Some preliminary results



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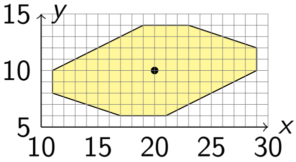

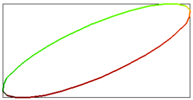

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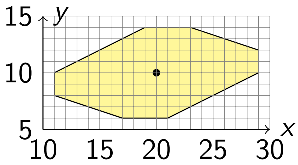

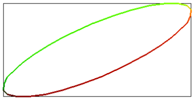

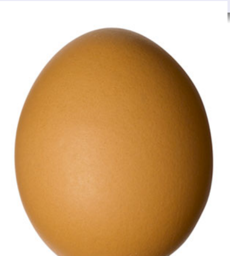

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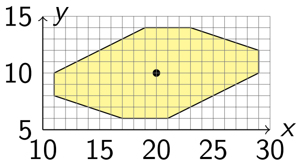
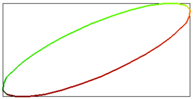
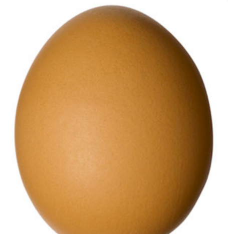






Abstraction based on Affine Arithmetic (Stolfi 93)

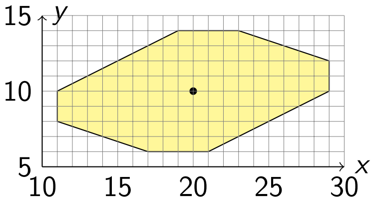
- A variable x is represented by an affine form $\hat{x}$:

$$\hat{x} = x_0 + x_1\varepsilon_1 + \ldots + x_n\varepsilon_n,$$

  where $x_i \in \mathbb{R}$ and the ε_i are independent symbolic variables with unknown value in $[-1, 1]$.

- Sharing ε_i between variables expresses *implicit dependency*: concretization as a zonotope

$$\begin{aligned} x &= 20 - 4\varepsilon_1 + 2\varepsilon_3 + 3\varepsilon_4 \\ y &= 10 - 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4 \end{aligned}$$



Affine arithmetic : arithmetic operations

- *Assignment* of a of a variable x whose value is given in a range $[a, b]$ introduces a noise symbol ε_i :

$$\hat{x} = \frac{(a+b)}{2} + \frac{(b-a)}{2}\,\varepsilon_i.$$

- *Addition* is computed componentwise (no new noise symbol):

$$\hat{x} + \hat{y} = (\alpha_0^x + \alpha_0^y) + (\alpha_1^x + \alpha_1^y)\varepsilon_1 + \ldots + (\alpha_n^x + \alpha_n^y)\varepsilon_n$$

- *Non linear operations* : approximate linear form (Taylor expansion), new noise term for the approximation error. Example (gross over-approx!):

$$\hat{x}\hat{y} \;\; = \;\; \alpha_0^x\alpha_0^y + \sum_{i=1}^{n}\left(\alpha_i^x\alpha_0^y + \alpha_i^y\alpha_0^x\right)\varepsilon_i + \left(\sum_{i,j>0}^{n}|\alpha_i^x\alpha_j^y|\right)\varepsilon_{n+1}.$$

- Efficient join operator (SAS 2006, and extensions for meet operators - CAV 2010)

Order relations

Standard "geometric" order on zonotopes

- Necessary for proving the analysis correct, testing the convergence of the analysis (lfp through Kleene iteration in particular).
- Given $A \in \mathcal{M}(n+1, p)$,

$$
\begin{array}{rcl}
x & = & 20 - 4\varepsilon_1 + 2\varepsilon_3 + 3\varepsilon_4 \\
y & = & 10 - 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4
\end{array}
\qquad
A = \begin{pmatrix} 20 & 10 \\ -4 & -2 \\ 0 & 1 \\ 2 & 0 \\ 3 & -1 \end{pmatrix}
$$

Order relations

Standard "geometric" order on zonotopes

- Given $A \in \mathcal{M}(n+1,p)$, $\forall t \in \mathbb{R}^p$ ($\|e\|_1 = \sum_{i=0}^n |e_i|$, ℓ_1 norm):

$$\sup_{y \in \gamma(A)} \langle t, y \rangle = \|At\|_1$$

- Hence $X \subseteq Y$ iff for all $t \in \mathbb{R}^p$, $\|Xt\|_1 \leq \|Yt\|_1$

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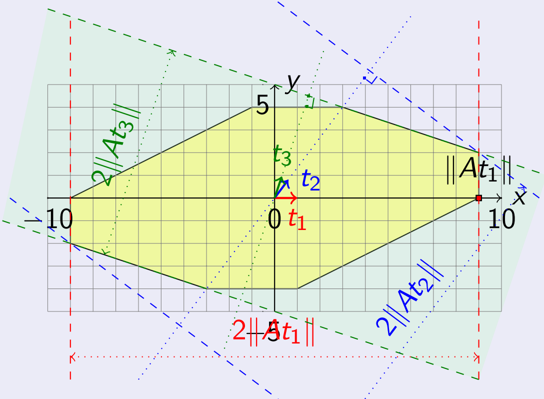


Functional order

Standard formulation

- Let $x : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a function that we wish to abstract, and $x_1, \ldots, x_p$ its p components.
- Instead of abstracting the set of values that $x_1, \ldots, x_p$ can take, given the possible input values, we introduce slack variables $e_1, \ldots, e_n$ which represent the initial values of the n input variables of function x,
- and we abstract the set of values that $e_1, \ldots, e_n, x_1, \ldots, x_p$ can take, conjointly
- The geometric order on this *augmented* zonotope is the functional order

$\rightarrow$ Many distinct parameterizations for the same functional
$\rightarrow$ Non-economic and non-necessary algorithmic representation!

Order relations

Canonical form of the functional order

- Separate out noise symbols coming from the inputs:

  central noise symbols$\rightarrow$matrix C

  with noise symbols not directly related to the inputs

  perturbation noise symbols$\rightarrow$matrix P

- Let two affine sets over p variables and n input noise symbols X and Y, we say that $X \leq Y$ iff

$$\sup_{u \in \mathbb{R}^p} \left(\|(C^Y - C^X)u\| + \|P^X u\| - \|P^Y u\| \right) \leq 0$$

The two formulations are equivalent!

Ellipsoids

Usual definition (constraints)

- In dimension 3, in suitable (x, y, z) coordinates:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

  (in fact, we want all of the interior, so better write ≤ 1)

- In general, in dimension n:

$$(x - v)^T A(x - v) \leq 1$$

  where A is symmetric positive definite

- Typical *quadratic Lyapunov functions* useful for proving stability/convergence of numerous systems (see also the quadratic template domain of Adjé et al., Feret's ellipsoidal domain, Kurzhanski's ellipsoidal calculus and work by Cousot, and by Féron)

Ellipsoids

Our definition (parameterization)

- Affine transformation on the n-dimensional disc

$$D^{n-1} = \{\varepsilon_1^2 + \ldots + \varepsilon_n^2 \leq 1\}$$

- Hence, just like zonotopes:

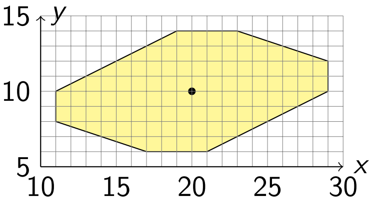
$$\hat{x} = x_0 + x_1\varepsilon_1 + \ldots + x_n\varepsilon_n,$$

but with $\|\varepsilon\|_2 \leq 1$ (and not $\|\varepsilon\|_\infty \leq 1$)

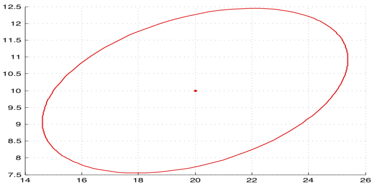
Equivalent to the former one:
if $\hat{X} = X_0 + M\varepsilon$, take $A = M^t M$, $v = X_0$...

Example

$$
\begin{aligned}
x &= 20 - 4\varepsilon_1 + 2\varepsilon_3 + 3\varepsilon_4 \\
y &= 10 - 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4
\end{aligned}
$$



$\|\varepsilon\|_\infty \leq 1$



$\|\varepsilon\|_2 \leq 1$

Going functional

Separate out the set of symbols, again...

- We define an ellipsoidal set X by a pair of matrices $(C^X, P^X) \in \mathcal{M}(n+1,p) \times \mathcal{M}(m,p)$
- The ellipsoidal form

$$\pi_k(X) = c^X_{0k} + \sum_{i=1}^{n} c^X_{ik}\varepsilon_i + \sum_{j=1}^{m} p^X_{jk}\eta_j$$

  where the ε_i are the central noise symbols and the η_j the perturbation or union noise symbols, describes the kth variable of X
- The noise symbols satisfy $\|\epsilon, \eta\|_2 \leq 1$

Order

Geometric order

$X \leq Y$ iff

$$\forall u \in \mathbb{R}^p,\ \|Xu\|_2 \leq \|Yu\|_2 \ .$$

Functional order

We say that $X \leq Y$ iff

$$\forall u \in \mathbb{R}^p,\ \|(C^Y - C^X)u\|_2 \leq \|P^Y u\|_2 - \|P^X u\|_2 \ .$$

Once again, this can be proved to be the right order for comparing functional abstractions!

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Remarks

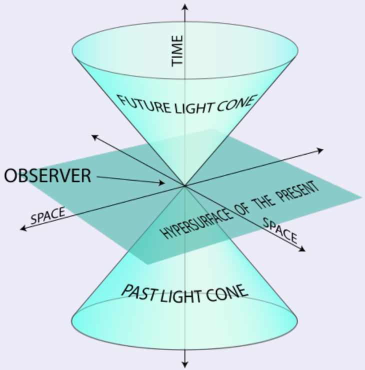
This is the Lorentz cone of special relativity!

More than just a mere remark:

- Our order is the Lorentz order (many theoretical tools available)
- Practical tools available! Second-Order Cone Programming in particular:

$$\min_{\|A_i x + b_i\|_2 \leq c_i^T x + d_i, Fx = g} f^T x$$

(subsumes quadratic constraints - hence concretisation in particular)



Arithmetic operations - first steps

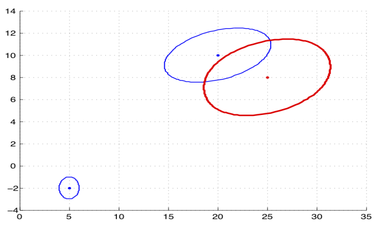
As before

Similar calculus as before:

$$\hat{x} + \hat{y} = (\alpha_0^x + \alpha_0^y) + (\alpha_1^x + \alpha_1^y)\varepsilon_1 + \ldots + (\alpha_n^x + \alpha_n^y)\varepsilon_n$$

Not as before

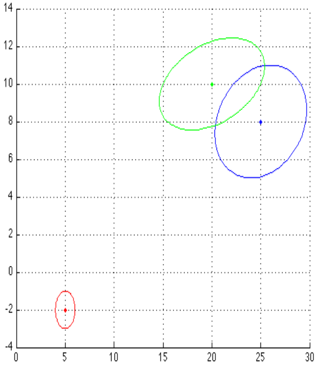
- Ellipsoids not closed under Minkowski sum (red=sum of blue):



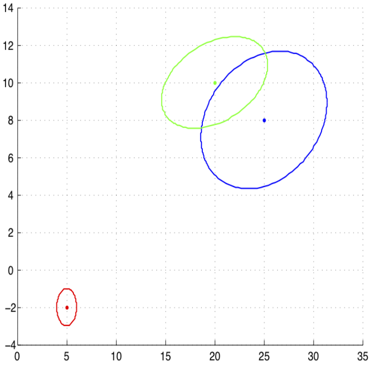
- Ellipsoidal calculus: find *smallest* ellipsoid containing the Minkowski sum of two ellipsoids, see for instance "Calculus Rules for Combinations of Ellipsoids and Applications" (A. Seeger)

Examples

$$
\begin{aligned}
x &= 20 - 4\varepsilon_1 + 2\varepsilon_3 + 3\varepsilon_4 \\
y &= 10 - 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4
\end{aligned}
\quad + \quad
\begin{aligned}
x &= 5 + \varepsilon_1 \\
y &= -2 + \varepsilon_2
\end{aligned}
$$



(linear sum)



(one of the ellipsoidal external sums, forgets dependencies)

Towards super-ellipsoids

All this can be generalized...

- Consider the transformation by an affine map of the n-disc for norm ℓ_p ($p \geq 1$):

$$\|\varepsilon\|_p = \left(\sum_{i=1}^{n} |\varepsilon|^p \right)^{\frac{1}{p}}$$

- Variables are represented as $\hat{x} = x_0 + x_1\varepsilon_1 + \ldots + x_n\varepsilon_n$, with $\|\varepsilon\|_p \leq 1$ (not $\|\varepsilon\|_\infty \leq 1$)

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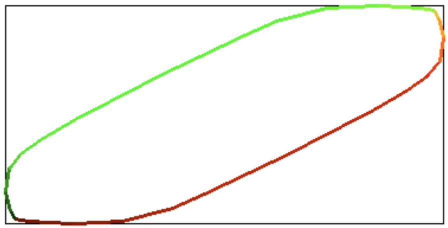
Degree 4 super-ellipsoid



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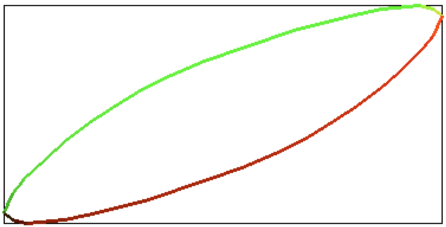
Degree $\frac{3}{2}$ super-ellipsoid



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More classically

In dimension 3, this is generally defined as

$$\left(\left|\frac{x}{a}\right|^r + \left|\frac{y}{b}\right|^r\right)^{\frac{t}{r}} + \left|\frac{z}{c}\right|^t \leq 1$$

(but here, we consider only $t = r$)

Order relation

Use ℓ_p/ℓ_q duality, or Hölder's inequality $\|fg\|_1 \leq \|f\|_p \|g\|_q$ with $\frac{1}{p} + \frac{1}{q} = 1$

Geometric order for p-superellipsoids ($q = \frac{p}{p-1}$)

$X \subseteq Y$ if and only if for all $t \in \mathbb{R}^p$

$$\|Xt\|_q \leq \|Yt\|_q$$

Functional order

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Ellipsoids

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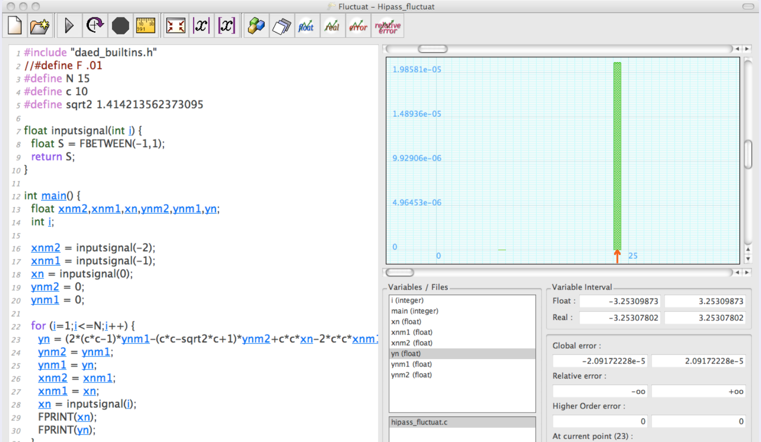
- Still has to be experimented... In particular in our analyzer FLUCTUAT (with extensions of this to floating-point/error semantics)



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Superellipsoids

- More a formal game for now
- Useful?