Cubical Agda

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In order to define higher inductive types in practice, we need to explain the theory behind cubical Agda

Canonicity

Homotopy type theory is obtained by adding a new axiom, **univalence**, stating that the map

$$(A = B) \to (A \simeq B)$$

is an equivalence.

This can be performed axiomatically, by adding a new axiom

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but this has a defect: it breaks **canonicity**.

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Theorem
Dependent type theory (without univalence) has the canonicity property.

Proof.
We can orient definitional equality into terminating rewriting system, e.g.

$$3 + \mathsf{id}\, 2 \to 3 + 2 \to 5$$

The only terms in normal form of type $\mathbb{N}$ are natural numbers. □

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Brunerie's number

A famous example comes from Brunerie's PhD thesis [Bru16] who shows in HoTT

$$\Sigma(n : \mathbb{N}).(\pi_4(\mathsf{S}^3) \simeq \mathbb{Z}/n\mathbb{Z})$$

Corollary 3.4.5. *We have*

$$\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z},$$

where n is the absolute value of the image of $[i_2, i_2]$ by the equivalence $\pi_3(\mathbb{S}^2) \xrightarrow{\sim} \mathbb{Z}$.

This result is quite remarkable in that even though it is a constructive proof, it is not at all obvious how to actually compute this n. At the time of writing, we still haven't managed to extract its value from its definition. A complete and concise definition of this number n is presented in appendix B, for the benefit of someone wanting to implement it in a prospective proof assistant. In the rest of this thesis, we give a mathematical proof in homotopy type theory that $n = 2$.

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Cubical type theory

In 2015, Cohen, Coquand, Huber and Mörtberg [CCHM18] presented a variant of dependent type theory in which

- one can manipulate n-cubes
- one can prove the univalence axiom
- supports higher inductive types [CHM18]
- has the canonicity property [Hub19]

Cubical type theory

The current implementation of cubical Agda is based on this
it can be activated with `--cubical`.

It is described in [VMA21] as well as in Agda's reference manual.

It comes with an alternative **standard library**:
`https://github.com/agda/cubical`

Cubical type theory

The cubical type theory can be considered as an "assembly language" for HoTT:

- most of the time we don't need to understand those details in order to make proofs (we only use the cubical Agda library)
- implementation might actually change in the future while keeping most proofs valid

But it is sometimes useful to know how it works.

The interval type

We begin by adding a new type I for the **interval** with two constructors $\mathsf{i}_0 : \mathsf{I}$ and $\mathsf{i}_1 : \mathsf{I}$

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More generally,

- a term $I \to I \to A$ corresponds to a square in A,
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- etc.

This is why we are **cubical**.

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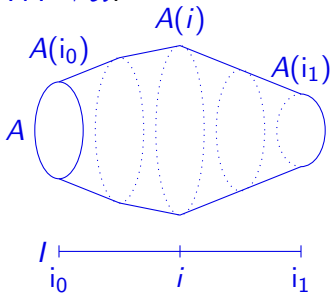
We also want to consider paths $(i : \mathsf{I}) \to A(i)$ whose type is varying.

Paths

A *varying type* is a function $A : \mathsf{I} \to \mathcal{U}$:



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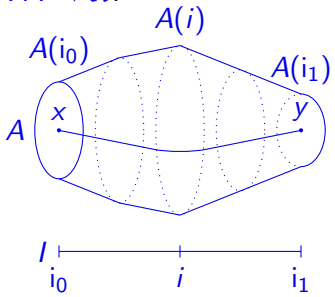

The type of *heterogeneous paths* is

$$\mathsf{PathP} : (A : \mathsf{I} \to \mathcal{U}) \to A\ \mathsf{i}_0 \to A\, i1 \to \mathcal{U}$$

where $\mathsf{PathP}\ A\, x\, y$ can be thought of as the type of functions $p : (i : \mathsf{I}) \to A\, i$ such that

$$p\ \mathsf{i}_0 \mathrel{\hat{=}} x \qquad\qquad p\ \mathsf{i}_1 \mathrel{\hat{=}} y$$

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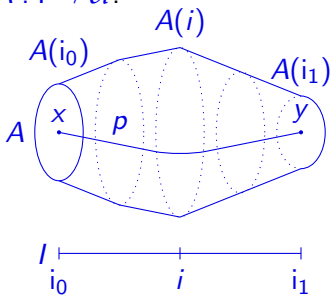

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Given $p : \mathsf{PathP}\ A\ x\ y$ and $i : \mathsf{I}$, we have

$$p\, i : A\, i \qquad p\ \mathsf{i}_0 \doteq x \qquad p\ \mathsf{i}_1 \doteq y$$

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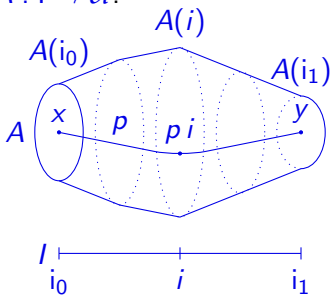

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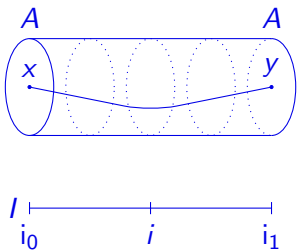

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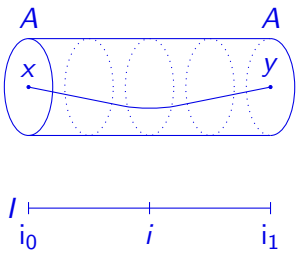

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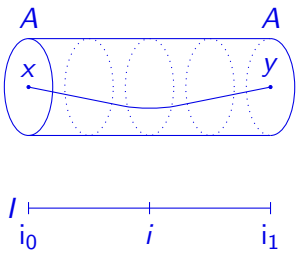

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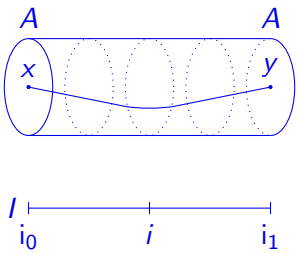

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Things which used to require J can now be proved right away. For instance,

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Function extensionality

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Those satisfy definitionally all the laws of De Morgan algebras, e.g.

$$(i \vee j) \vee k \mathrel{\hat{=}} i \vee (j \vee k) \qquad (i \vee \mathsf{i}_1) \mathrel{\hat{=}} \mathsf{i}_1 \qquad \sim \mathsf{i}_0 \mathrel{\hat{=}} \mathsf{i}_1 \qquad \dots$$

$$i \vee (j \wedge k) \mathrel{\hat{=}} (i \vee j) \wedge (i \vee k) \qquad (i \vee \mathsf{i}_0) \mathrel{\hat{=}} i \qquad \sim\!\sim i \mathrel{\hat{=}} i \qquad \dots$$

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$$i \vee (j \wedge k) \mathrel{\hat{=}} (i \vee j) \wedge (i \vee k) \qquad (i \vee \mathrm{i}_0) \mathrel{\hat{=}} i \qquad \sim\sim i \mathrel{\hat{=}} i \qquad \dots$$

i.e. all the laws of boolean algebras excepting

$$i \vee \sim i \mathrel{\hat{=}} \mathrm{i}_1 \qquad\qquad i \wedge \sim i \mathrel{\hat{=}} \mathrm{i}_0$$

Path reversal

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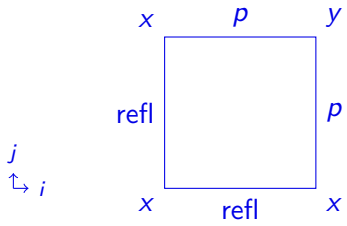

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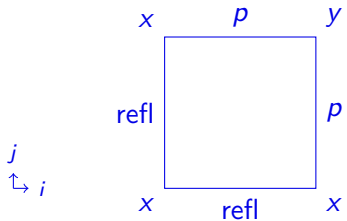

by

$$\lambda i.\lambda j.p\,(i \wedge j)$$

J

The J rule can similarly be shown for $x : A$ and $P : (y : A) \to x = y \to \mathit{Type}$ by

$$J : (r : P\ x\ \mathsf{refl})\ \{y : A\}\ (p : x = y) \to P\ y\ p$$
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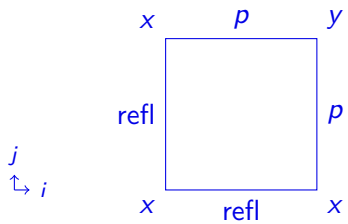

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$$J : (r : P\, x\ \mathrm{refl})\ \{y : A\}\ (p : x = y) \to P\, y\, p$$
$$r\, p \mapsto \mathrm{transport}\,(\lambda j.P\,(p\, j)\,(\lambda i.p\,(i \wedge j)))\, r$$

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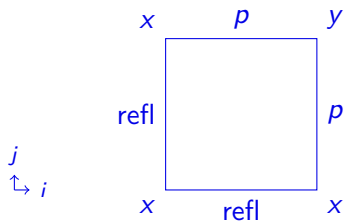


J

J does not evaluate definitionally on refl

TODO: we however have transportRefl which allows showing this propositionally

TODO: definitional transportRefl is incompatible with cubical Agda [Swa18]

Defining composition

We could define composition by J as before

...

However this does not compute nicely because transport does not on path types...

In practice

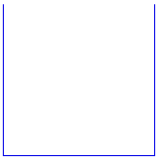
explain indices for wanted boundaries

Part I

Filling boxes

Defining composition

We deduce composition from an operation which states that for every “open cube”



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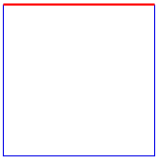

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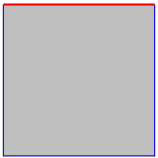

we can define

- the missing face: hcomp
- the interior cube: hfill

Partial types

We write

$$\mathsf{Partial}\ \varphi\ A$$

for the type of **partial types**: an element is a term of type A which is only defined when r is i_1, which can be thought of as a cube with missing faces.

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$$\mathsf{I} \to \mathsf{Bool}$$

are the constant functions

$$\lambda i.\,\mathsf{true} \qquad\qquad \lambda i.\,\mathsf{false}$$

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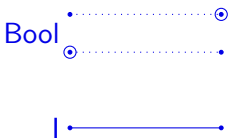
We can define a function

$$\begin{aligned} \mathsf{I} &\to \mathsf{Partial}\,(\sim i \vee i)\ \mathsf{Bool} \\ \mathsf{i}_0 &\mapsto \mathsf{false} \\ \mathsf{i}_1 &\mapsto \mathsf{true} \end{aligned}$$



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Agda checks that definitions match on their intersection so that we cannot define

$$\begin{aligned} \mathsf{I} &\to \mathsf{Bool} \\ \mathsf{i}_0 &\mapsto \mathsf{false} \\ \mathsf{i}_1 &\mapsto \mathsf{true} \\ i &\mapsto \mathsf{true} \end{aligned}$$

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The concrete syntax for pattern matching on interval arguments is

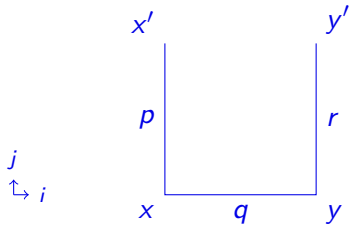
```
partialBool : (i : I) → Partial (~ i ∨ i) Bool
partialBool i (i = i0) = false
partialBool i (i = i1) = true
```

or

```
partialBool : (i : I) → Partial (~ i ∨ i) Bool
partialBool i = λ { (i = i0) → false ; (i = i1) → true }
```

Partial types

For instance, given $p : x = x'$, $q : x = y$ and $r : y = y'$, we can define



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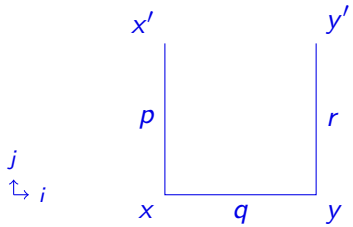

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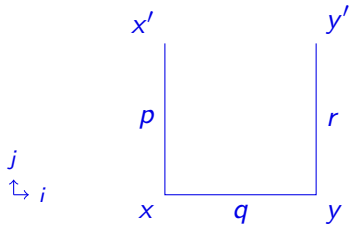


as

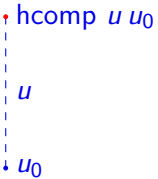
$$
\begin{array}{lll}
u : (i : \mathsf{I})\ (j : \mathsf{I}) \to \mathsf{Partial}\,(\sim i \vee \sim j \vee i)\,A & & \\
\quad \mathsf{i}_0 & j & \mapsto p\,j \\
\quad i & \mathsf{i}_0 & \mapsto q\,i \\
\quad \mathsf{i}_1 & j & \mapsto r\,j
\end{array}
$$

Homogeneous composition

The **homogeneous composition** operation is

$$\mathsf{hcomp} : \{\varphi : I\}\ (u : I \to \mathsf{Partial}\ \varphi\ A)\ (u_0 : A) \to A$$

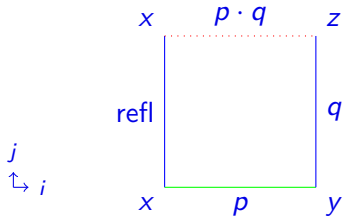
which can be pictured as



where Agda checks that u_0 agrees with u (for $i \mathrel{\hat{=}} \mathsf{i}_0$) when both are defined.

Defining composition

For instance, given $p : x = y$ and $q : y = z$, we can define their composite as



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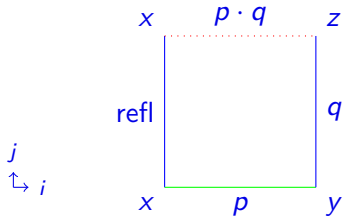

This means that we define

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u : (i : \mathsf{I})\ (j : \mathsf{I}) \to \mathsf{Partial}\,(\sim i \vee i)\,A & & \\
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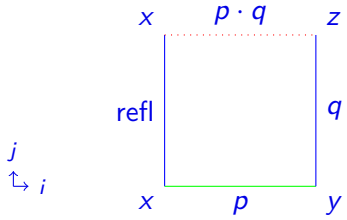

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$$\mathsf{i}_1 \quad j \quad \mapsto q\,j$$

and

$$p \cdot q \quad \hat{=} \quad \lambda i.\,\mathsf{hcomp}\,(u\,i)\,(p\,i)$$

Subtypes

We write

$$A[\varphi \mapsto u]$$

for the **subtype** of A whose elements are definitionally equal to u when φ is i_1.

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for the **subtype** of A whose elements are definitionally equal to u when φ is i_1.

It comes equipped with two operations

$$\begin{aligned} \mathsf{inS} &: \quad (u : A) \to A[\varphi \mapsto \lambda i.u] \\ \mathsf{outS} &: A[\varphi \mapsto u] \to A \end{aligned}$$

Homogeneous filler

The **homogeneous filler** is

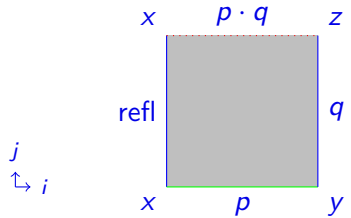
$$\mathsf{hfill} : \{\varphi : \mathsf{I}\}\ (u : (i : \mathsf{I}) \to \mathsf{Partial}\,\varphi A)\,(u_0 : A[\varphi \mapsto u\,\mathsf{i}_0])\,(i : I) \to A$$

which is

- u_0 when i is i_0
- $\mathsf{hcomp}\ u\ u_0$ when is is i_1

Homogeneous filler

For instance, if we apply hfill to the previous composition situation



we obtain a path

$$f : \mathsf{PathP}\,(\lambda j.x = q\,j)\,p\,(p \cdot q)$$
$$j\,i \mapsto \mathsf{hfill}\,(u\,i)\,(\mathsf{inS}\,(p\,i))\,j$$

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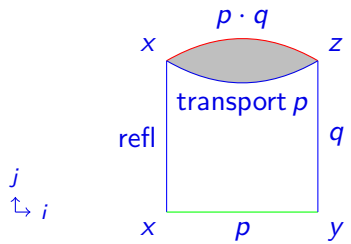

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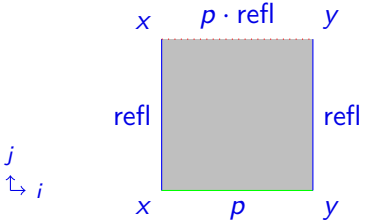
Composition is unital on the right

If we specialize to $q \hat{=} \mathsf{refl}$, we show composition is unital on the right



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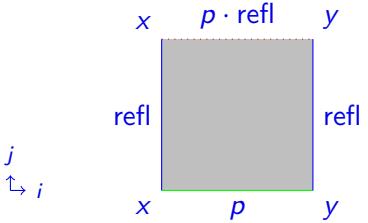

we obtain a path

$$f : p = p \cdot \mathsf{refl}$$
$$j\ i \mapsto \mathsf{hfill}\ (u\ i)\ (\mathsf{inS}\ (p\ i))\ j$$

Defining hfill

Note that hfill can be defined from hcomp:

$$\begin{array}{lll} \mathsf{hfill} : (u : (i : \mathsf{I}) \to \mathsf{Partial}\ \varphi)\ (u_0 : A[\varphi \mapsto u\ \mathsf{i}_0])\ (i : \mathsf{I}) \to A \\ \quad u \qquad\qquad\qquad\qquad\quad u_0 \qquad\qquad\qquad\quad i \qquad \mapsto \mathsf{hcomp}\ U\ (\mathsf{outS}\ u_0) \end{array}$$

where U is the function

$$\begin{array}{l} j\,(\phi = \mathsf{i}_1) \mapsto u\,(i \wedge j) \\ j\,(i = \mathsf{i}_0) \mapsto \mathsf{outS}\ u_0 \end{array}$$

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The definition of composition is biased: showing $\mathsf{refl} \cdot p = p$ is more difficult.

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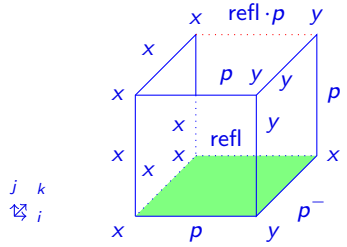

The "cube" can be defined as

$$u : (i : \mathsf{I})\ (j : \mathsf{I})\ (k : \mathsf{I}) \to \mathsf{Partial}\,(\sim i \vee i \vee \sim k)\,A$$

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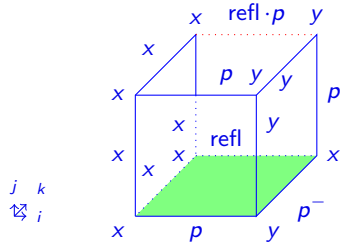

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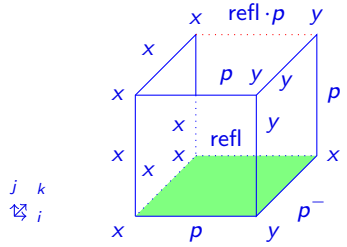

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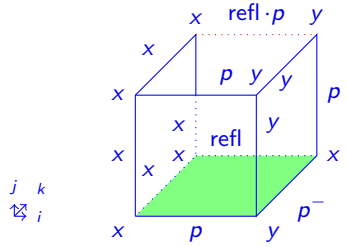

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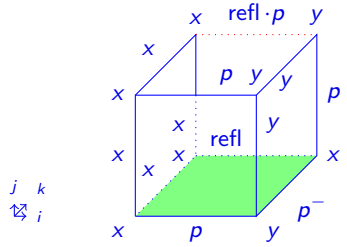

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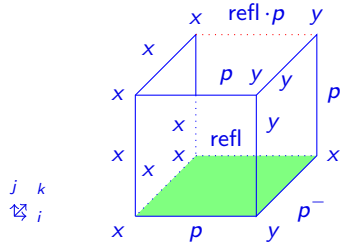

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\quad \mathsf{i}_0 & j & k & \mapsto x \\
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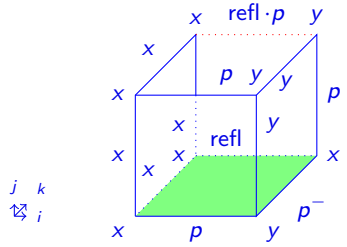

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\end{array}
$$

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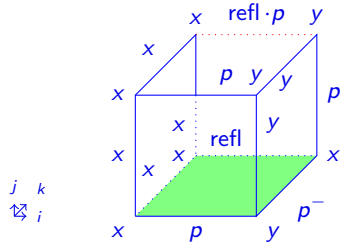

The bottom can be defined as

$$u_0 : (i : \mathbb{I})\ (k : \mathbb{I}) \to A$$
$$i \qquad k \qquad \mapsto$$

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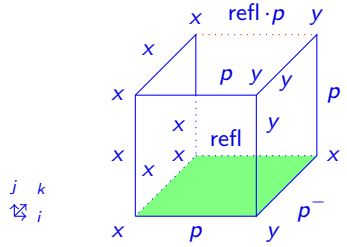

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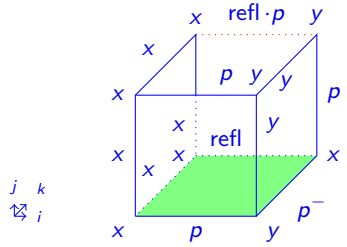


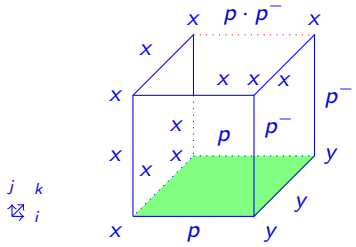
The filler is

$$f : (i : \mathbb{I})\ (j : \mathbb{I})\ (k : \mathbb{I}) \to A$$
$$i \quad j \quad k \quad \mapsto \mathsf{hfill}\,(\lambda j.u\ i\ j\ k)\,(\mathsf{inS}\,(u_0\ i\ k))$$

and we obtain the result as the top face ($j = \mathsf{i}_1$).

Composition is cancellative on the right

To show that composition is cancellative on the right, we can similarly use the "cube":



Glue types

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