The fundamental duality

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Grothendieck dualities

There are various dualities which are all take the form of a bijection between

- some maps
- the same maps turned "upside down"

There are many variants, but we can call them **Grothendieck dualities**.

Subsets and characteristic functions

Given a set B, we have a bijective correspondence between

- injective functions $A \to B$
- functions $B \to \{0, 1\}$

Namely,

$$\begin{tikzcd}
A \arrow[d] & \cdot & \cdot & & \cdot \\
B & \cdot & \cdot & \cdot & \cdot & \cdot
\end{tikzcd}$$

corresponds to

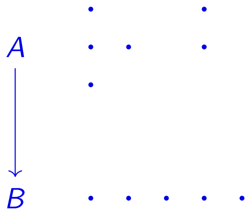
$$\begin{tikzcd}
\{0,1\} & 1 & 1 & 0 & 1 & 0 \\
B \arrow[u] & \cdot & \cdot & \cdot & \cdot & \cdot
\end{tikzcd}$$

Note: to be precise, we only recover A up to isomorphism!

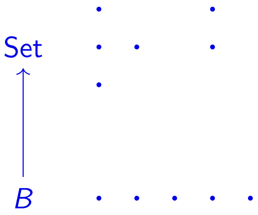
Functions and families of sets

Given a set B, we have a bijective correspondence between

- functions $A \to B$
- functions $B \to \mathrm{Set}$



corresponds to



Namely:

- to $f : A \to B$ we associate $f^{-1} : B \to \mathrm{Set}$
- to $P : B \to \mathrm{Set}$ we associate the canonical projection $f : \left(\coprod_{x \in A} P(x)\right) \to B$

Dualities

There are many variants of this duality:

- discrete fibrations $A \to B$ and presheaves $B \to \mathsf{Set}$
- fibrations $A \to B$ and pseudo-functors $B \to \mathsf{Cat}$
- ...

The duality

In homotopy type theory, we have

Theorem ([Uni13, Theorem 4.8.3])
Given a type B, we have an equivalence

$$\Sigma(A:\mathcal{U}).(A \to B) \quad \simeq \quad B \to \mathcal{U}$$

between fibrations over B and families indexed by B.

Fibers

Any function

$$f : A \to B$$

induces a fiber function

$$\mathrm{fib}\ f : B \to \mathcal{U}$$

defined by

$$\mathrm{fib}\ f\ y \quad \hat{=} \quad \Sigma(x : A).(f\ x = y)$$

Total space

Any type family

$$P : B \to \mathcal{U}$$

induces a map

$$\mathrm{total}\ P : (\Sigma(y : B).P\,y) \to B$$

which is the first projection from the *total space* of the family.

The equivalence

These two constructions form an equivalence.

Lemma ([Uni13, Lemma 4.8.1])
For a type family $P : B \to \mathcal{U}$, *the fiber of the first projection* $\mathsf{total}\ P : \Sigma B.P \to B$ *at* $y : B$ *is* $P\,y$.

Proof.
We have

$$\begin{aligned}
\mathsf{fib}\,(\mathsf{total}\ P)\,y &= \Sigma((b,x) : \Sigma B.P).(\mathsf{total}\ P\,(b,x) = y) \\
&= \Sigma((b,x) : \Sigma B.P).(b = y) \\
&= \Sigma(b : B).\Sigma(x : P\,b).(b = y) \\
&= \Sigma(b : B).(b = y) \times (P\,b) \\
&= \Sigma(\Sigma(b : B).(b = y)).(P\,b) \\
&= \Sigma(_ : 1).(P\,y) \\
&= P\,y
\end{aligned}$$

□

Lemma ([Uni13, Lemma 4.8.2])
For a function $f : A \to B$, *we have* $A = \Sigma(y : B).\, \mathsf{fib}\ f\ y$.

Proof.
We have

$$
\begin{aligned}
\Sigma(y : B).\, \mathsf{fib}\ f\ y &= \Sigma(y : B).\Sigma(x : A).(f\ x = y) \\
&= \Sigma(x : A).\Sigma(y : B).(f\ x = y) \\
&= \Sigma(x : A).1 \\
&= A
\end{aligned}
$$

$\square$

Examples

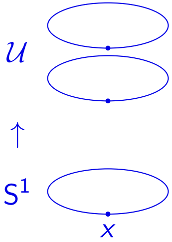
Consider the type family

$$\begin{aligned} P : \mathsf{S}^1 &\to \mathcal{U} \\ x &\mapsto \mathsf{Bool} \end{aligned}$$

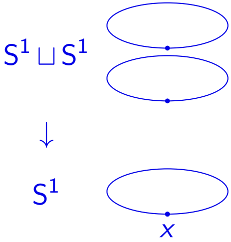
The corresponding fibration is the canonical projection

$$\mathsf{S}^1 \sqcup \mathsf{S}^1 \to \mathsf{S}^1$$

i.e.



corresponds to



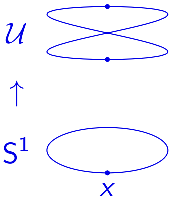
Consider the type family

$$\begin{aligned} P : \mathsf{S}^1 &\to \mathcal{U} \\ \star &\mapsto \mathsf{Bool} \\ \mathsf{loop} &\mapsto \mathsf{ua}\,\mathsf{suc} \end{aligned}$$

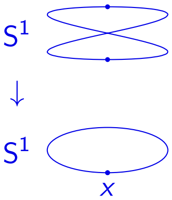
The corresponding fibration is the double speed map

$$\mathsf{S}^1 \sqcup \mathsf{S}^1$$

i.e.



corresponds to

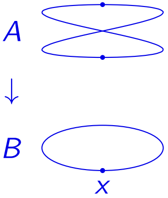


Fibration

Given a map $f : A \to B$ with B pointed, we write

$$F \hookrightarrow A \overset{f}{\twoheadrightarrow} B$$

to indicate that $F = \text{fib } f \star$ and A is the total space: this is a **fiber sequence**.



When B is connected all the fibers are the same, but the way they are glued matters.

The previous fiber sequences are

The torus

Consider the type family

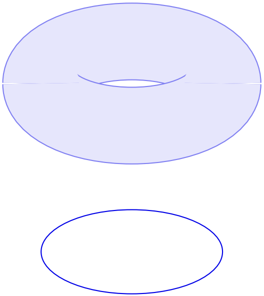
$$\begin{aligned} P : \mathsf{S}^1 &\to \mathcal{U} \\ x &\mapsto \mathsf{S}^1 \end{aligned}$$

The corresponding fibration is the projection from the torus

$$\mathsf{S}^1 \times \mathsf{S}^1 \sqcup \mathsf{S}^1$$

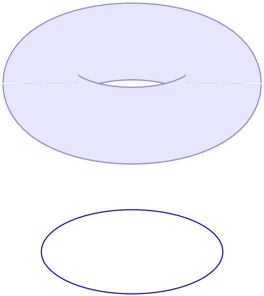
i.e.

$$\begin{array}{c} \mathcal{U} \\ \uparrow \\ \mathsf{S}^1 \end{array}$$



corresponds to

$$\begin{array}{c} \mathsf{S}^1 \times \mathsf{S}^1 \\ \downarrow \\ \mathsf{S}^1 \end{array}$$



The torus

This means that we have a fiber sequence

$$\mathsf{S}^1 \hookrightarrow \mathsf{S}^1 \times \mathsf{S}^1 \twoheadrightarrow \mathsf{S}^1$$

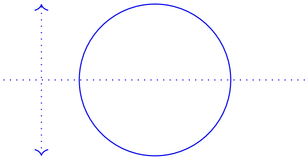
over S^1 with fibers S^1 and the torus $\mathsf{S}^1 \times \mathsf{S}^1$ as total space.

Conjugation

We have a **conjugation** operation

$$\mathsf{conj} : \mathsf{S}^1 \to \mathsf{S}^1$$

on the circle:



which is involutive

$$\mathsf{conj} \circ \mathsf{conj} = \mathsf{id}$$

and thus an equivalence.

The Klein bottle

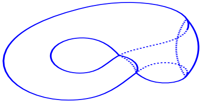
Consider the type family

$$\begin{aligned} P : \mathsf{S}^1 &\to \mathcal{U} \\ \star &\mapsto \mathsf{S}^1 \\ \mathsf{loop} &\mapsto \mathsf{ua}\ \mathsf{conj} \end{aligned}$$

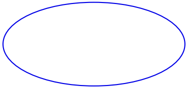
The corresponding fibration is the projection from the Klein bottle

$$\mathsf{K} \to \mathsf{S}^1$$

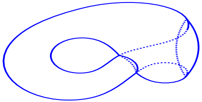
i.e.

$\mathcal{U}$ 

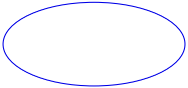
$\uparrow$

S^1 

corresponds to

K 

$\downarrow$

S^1 

The Klein bottle

We have an associated fiber sequence:

$$S^1 \hookrightarrow K \twoheadrightarrow S^1$$

Fibrations over the circle

All previous examples are particular cases of the following.

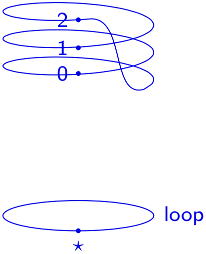
Given a type A and an equivalence $e : A \simeq A$, we can define a type family

$$\begin{aligned} P : \mathsf{S}^1 &\to \mathcal{U} \\ \star &\mapsto A \\ \mathsf{loop} &\mapsto \mathsf{ua}\ e \end{aligned}$$

The n-sheeted covering of the circle

If we start from $\mathsf{suc} : \mathbb{Z}_n \to \mathbb{Z}_n$, we obtain the n-**sheeted covering** of the circle

$$\begin{array}{c}\mathcal{U}\\ \Big\uparrow \\ \mathrm{S}^1\end{array}$$



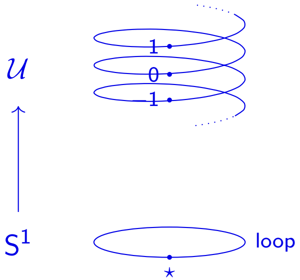
which corresponds to the fiber sequence

$$\mathbb{Z}_n \hookrightarrow \mathrm{S}^1 \twoheadrightarrow \mathrm{S}^1$$

We recover the non-trivial fibration with Bool as fibers with $n = 2$.

The universal cover of the circle

If we start from $\mathsf{suc} : \mathbb{Z} \to \mathbb{Z}$, we obtain the **universal covering** of the circle



which corresponds to the fiber sequence

$$\mathbb{Z} \hookrightarrow \mathsf{S}^1 \twoheadrightarrow \mathsf{S}^1$$

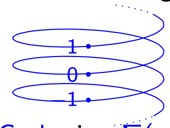
The type family is precisely the function $\mathsf{Code} : \mathsf{S}^1 \to \mathcal{U}$ and the corresponding fibration is the pointing map $1 \to \mathsf{S}^1$.

This suggests another approach for computing the loop space of the circle!

The loop space of the circle

This suggests the following alternative proof for showing $\Omega\,\mathsf{S}^1 = \mathbb{Z}$.

1. Define $\mathsf{Code} : \mathsf{S}^1 \to \mathcal{U}$ by $\mathsf{Code}\ \star \mathrel{\hat{=}} \mathbb{Z}$ and $\mathsf{ap}\ \mathsf{Code}\ \mathsf{loop} \mathrel{\hat{=}} \mathsf{ua}\ \mathsf{suc}$.
2. Define a type Line generated by points $n : \mathbb{Z}$ together with paths $p : n = n + 1$:



3. Show that it is the total space of Code, i.e. $\Sigma(x : \mathsf{S}^1).\,\mathsf{Code}\ x = \mathsf{Line}$.
4. Show that the line is contractible, i.e. $\mathsf{Line} = 1$.
5. Deduce that $\Sigma(x : \mathsf{S}^1).\,\mathsf{Code}\ x = \mathsf{Line} = 1 = \Sigma(x : \mathsf{S}^1).(\star = x)$.
6. Deduce that $\mathsf{Code}\ x = (\star = x)$ for any $x : \mathsf{S}^1$.
7. Deduce that $\Omega\,\mathsf{S}^1 = (\star = \star) = \mathsf{Code}\ \star \mathrel{\hat{=}} \mathbb{Z}$.

The first 4 points require higher inductive types (see next session). Let us detail point 6.

Duality for maps

We have seen we have a correspondence between

- type families $P : B \to \mathcal{U}$
- fibrations $f : A \to B$

This extends to morphisms between those: we have a correspondence between

- morphisms of families $P \to Q$
- morphisms of fibrations $\mathsf{total}\, P \to \mathsf{total}\, Q$

Morphisms

Given two families $P, Q : B \to \mathcal{U}$, a **morphism** between them is a map

$$F : (y : B) \to P\,y \to Q\,y$$

Given two fibrations $f : A \to B$ and $f' : A' \to B$, a **morphism** between them is an element of

$$\Sigma(g : A \to B).(f' \circ g = f)$$

$$\begin{tikzcd} A \arrow[rr, dotted, "g"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

Theorem (see labs)
There is an equivalence between the two types of morphisms and this equivalence preserves composition and identities.

Morphisms

Any morphism of families

$$F : (y : B) \to P\,y \to Q\,y$$

induces a morphism between the total spaces

$$\begin{aligned} \mathrm{total}\ F : \Sigma B.P &\to \Sigma B.Q \\ (y, x) &\mapsto (y, F\,y\,x) \end{aligned}$$

which is a morphism in the sense that it commutes with first projection.

Previous theorem implies that

Theorem ([Uni13, Theorem 4.7.7])
The morphism F is a family of equivalences if and only if total F *is an equivalence.*

Note the right-to-left direction is not obvious: the inverse might not a priori preserve the fibers! We are going to provide a direct proof of this.

Fiberwise equivalences

Theorem ([Uni13, Theorem 4.7.7])
A morphism $F : (y : B) \to P\,y \to Q\,y$, *is a family of equivalences if and only if* $\mathsf{total}\ F$ *is an equivalence.*

Proof.
We will see in next lemma that we have, for $y : B$ and $x : Q\,y$,

$$\mathsf{fib}\,(F\,y)\,x \quad = \quad \mathsf{fib}\,(\mathsf{total}\ F)\,(y, x)$$

Therefore the following are equivalent

- F is a family of equivalences,
- $F\,y$ is an equivalence for $y : B$,
- $\mathsf{fib}\,(F\,y)\,x$ is contractibl for $y : B$ and $x : Q\,y$,
- $\mathsf{fib}\,(\mathsf{total}\ F)\,(y, x)$ is contractible for $(y, x) : \Sigma B.Q$,
- $\mathsf{total}\ F$ is an equivalence. $\square$

Fiberwise equivalences

Lemma ([Uni13, Lemma 4.7.6])
For $F : (y : B) \to P\,y \to Q\,y$, $y : B$ *and* $x : Q\,y$, *we have*

$$\mathsf{fib}\,(F\,y)\,x \quad = \quad \mathsf{fib}\,(\mathsf{total}\ F)\,(y, x)$$

Proof.

$$\begin{aligned}
\mathsf{fib}\,(\mathsf{total}\ F)\,(y, x) &= \Sigma((y', x') : \Sigma B.P).\,\mathsf{total}\ F\,(y', x') = (y, x) \\
&= \Sigma((y', x') : \Sigma B.P).(y', F\,y'\,x') = (y, x) \\
&= \Sigma(y' : B).\Sigma(x' : P\,y').(y', F\,y'\,x') = (y, x) \\
&= \Sigma(y' : B).\Sigma(x' : P\,y').\Sigma(p : y' = y).F\,y'\,x' =^{Q}_{p} x \\
&= \Sigma(y' : B).\Sigma(p : y' = y).\Sigma(x' : P\,y').F\,y'\,x' =^{Q}_{p} x \\
&= \Sigma((y', p) : \Sigma(y' : B).y' = y).\Sigma(x' : P\,y').F\,y'\,x' =^{Q}_{p} x \\
&= \Sigma(x' : P\,y).F\,y\,x' = x \\
&= \mathsf{fib}\,(F\,y)\,x
\end{aligned}$$

$\square$

The fundamental theorem of identity types

[Rij25, Theorem 11.2.2]

when the Px are propositions the projection $\Sigma A.P \to A$ is an embedding

in particular the inclusion of the connected component of a point into the sapce

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