The fundamental duality

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Grothendieck dualities

There are various dualities which are all take the form of a bijection between

- some maps
- the same maps turned "upside down"

There are many variants, but we can call them **Grothendieck dualities**.

Subsets and characteristic functions

Given a set B, we have a bijective correspondence between

- injective functions $A \to B$
- functions $B \to \{0, 1\}$

Namely,

$$\begin{array}{cccccc} A & \cdot & \cdot & & \cdot & \\ \downarrow & & & & & \\ B & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \text{corresponds to} \qquad \begin{array}{cccccc} \{0,1\} & 1 & 1 & 0 & 1 & 0 \\ \uparrow & & & & & \\ B & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

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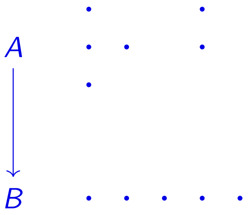
Note: to be precise, we only recover A up to isomorphism!

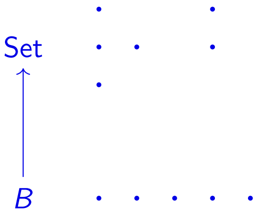
Functions and families of sets

Given a set B, we have a bijective correspondence between

- functions $A \to B$
- functions $B \to \mathrm{Set}$



corresponds to



Again, "up to isomorphism" in the source.

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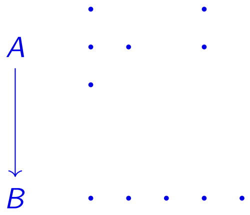

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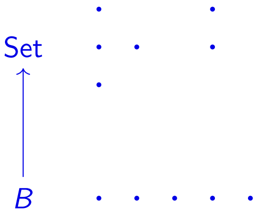

Namely:

- to $f : A \to B$ we associate $f^{-1} : B \to \mathrm{Set}$
- to $P : B \to \mathrm{Set}$ we associate the canonical projection $f : \left(\bigsqcup_{x \in A} P(x)\right) \to B$

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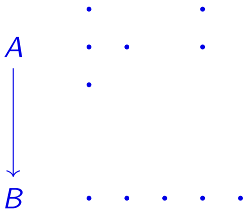

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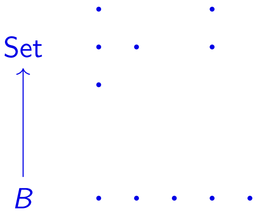

The fact that we have a correspondence means that

- for $f : A \to B$, we have $A \cong \Sigma(y : B).f^{-1}(y)$
- for $P : B \to \mathsf{Set}$, the fiber of the projection $f : \Sigma(y : B).P(y) \to B$ at b is $P(b)$

Dualities

There are many variants of this duality:

- discrete fibrations $A \to B$ and presheaves $B \to \mathsf{Set}$
- fibrations $A \to B$ and pseudo-functors $B \to \mathsf{Cat}$
- ...

The duality

In homotopy type theory, we have

Theorem ([Uni13, Theorem 4.8.3])
Given a type B, we have an equivalence

$$\Sigma(A : \mathcal{U}).(A \to B) \quad \simeq \quad B \to \mathcal{U}$$

between fibrations over B and families indexed by B.

Fibers

Any function

$$f : A \to B$$

induces

Fibers

Any function

$$f : A \to B$$

induces a fiber function

$$\mathrm{fib}\ f : B \to \mathcal{U}$$

defined by

$$\mathrm{fib}\ f\ y \quad \hat{=} \quad \Sigma(x : A).(f\ x = y)$$

Total space

Any type family

$$P : B \to \mathcal{U}$$

induces a map

Total space

Any type family

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induces a map

$$\mathrm{total}\ P : (\Sigma(y : B).P\,y) \to B$$

which is the first projection from the *total space* of the family.

The equivalence

These two constructions form an equivalence.

The equivalence

Lemma ([Uni13, Lemma 4.8.1])
For a type family $P : B \to \mathcal{U}$*, the fiber of the first projection* $\mathrm{total}\ P : \Sigma B.P \to B$ *at* $y : B$ *is* $P\,y$*.*

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Proof.
We have

$$\mathsf{fib}\,(\mathsf{total}\ P)\,y$$

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$$\mathsf{fib}\,(\mathsf{total}\ P)\,y = \Sigma((b,x) : \Sigma B.P).(\mathsf{total}\ P\,(b,x) = y)$$

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&= \Sigma(b : B).\Sigma(x : P\,b).(b = y) \\
&= \Sigma(b : B).(b = y) \times (P\,b)
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For a function $f : A \to B$*, we have* $A = \Sigma(y : B).\,\mathsf{fib}\ f\ y$.

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$\square$

Examples

Consider the type family

$$\begin{aligned} P : \mathsf{S}^1 &\to \mathcal{U} \\ x &\mapsto \mathsf{Bool} \end{aligned}$$

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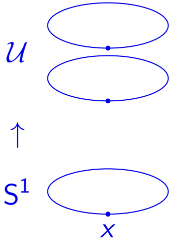
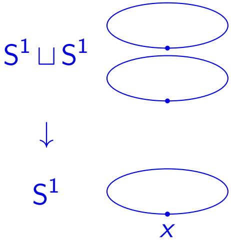
The corresponding fibration is the canonical projection

$$\mathsf{S}^1 \sqcup \mathsf{S}^1 \to \mathsf{S}^1$$

i.e.



corresponds to



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The corresponding fibration is the double speed map

$$\mathsf{S}^1 \sqcup \mathsf{S}^1$$

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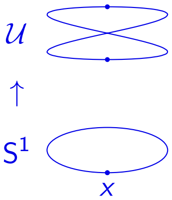

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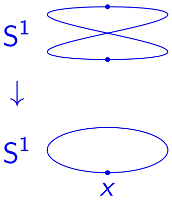
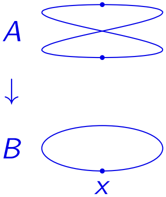


Fibration

Given a map $f : A \to B$ with B pointed, we write

$$F \hookrightarrow A \overset{f}{\twoheadrightarrow} B$$

to indicate that $F = \mathrm{fib}\ f\ \star$ and A is the total space: this is a **fiber sequence**.



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The previous fiber sequences are

$$\mathrm{Bool} \hookrightarrow \mathrm{S}^1 \sqcup \mathrm{S}^1 \xrightarrow{f} \mathrm{S}^1 \qquad \text{and} \qquad \mathrm{Bool} \hookrightarrow \mathrm{S}^1 \xrightarrow{f} \mathrm{S}^1$$

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The torus

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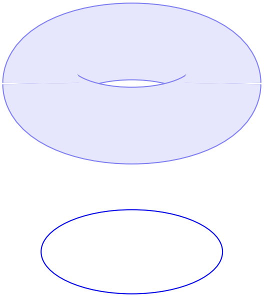
The corresponding fibration is the projection from the torus

$$\mathsf{S}^1 \times \mathsf{S}^1 \sqcup \mathsf{S}^1$$

i.e.

$$\begin{array}{c} \mathcal{U} \\ \uparrow \\ \mathsf{S}^1 \end{array}$$



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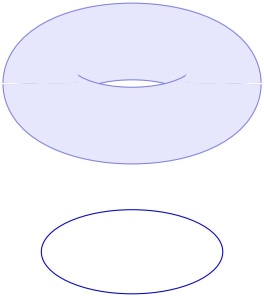
$$\begin{array}{c} \mathsf{S}^1 \times \mathsf{S}^1 \\ \downarrow \\ \mathsf{S}^1 \end{array}$$



This means that we have a fiber sequence

$$\mathsf{S}^1 \hookrightarrow \mathsf{S}^1 \times \mathsf{S}^1 \twoheadrightarrow \mathsf{S}^1$$

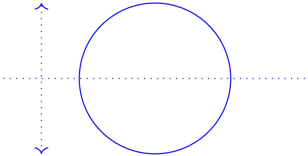
over S^1 with fibers S^1 and the torus $\mathsf{S}^1 \times \mathsf{S}^1$ as total space.

Conjugation

We have a **conjugation** operation

$$\mathsf{conj} : \mathsf{S}^1 \to \mathsf{S}^1$$

on the circle:



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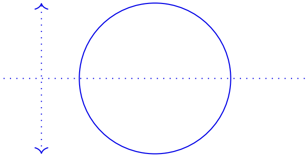

which is involutive

$$\mathsf{conj} \circ \mathsf{conj} = \mathsf{id}$$

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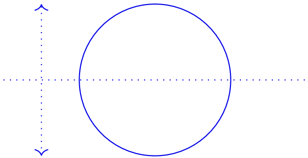

which is involutive

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and thus an equivalence.

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The corresponding fibration is

The Klein bottle

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The corresponding fibration is the projection from the Klein bottle

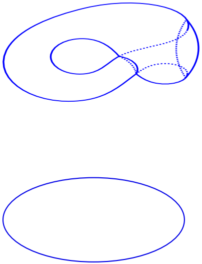
$$\mathsf{K} \to \mathsf{S}^1$$

i.e.

$\mathcal{U}$

$\uparrow$

S^1

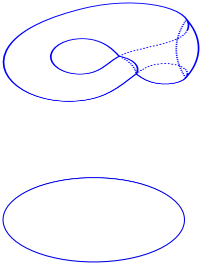


corresponds to

K

$\downarrow$

S^1



The Klein bottle

We have an associated fiber sequence:

$$S^1 \hookrightarrow K \twoheadrightarrow S^1$$

Fibrations over the circle

All previous examples are particular cases of the following.

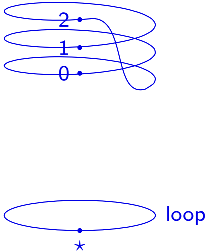
Given a type A and an equivalence $e : A \simeq A$, we can define a type family

$$\begin{aligned} P : \mathsf{S}^1 &\to \mathcal{U} \\ \star &\mapsto A \\ \mathsf{loop} &\mapsto \mathsf{ua}\ e \end{aligned}$$

The n-sheeted covering of the circle

If we start from $\mathsf{suc} : \mathbb{Z}_n \to \mathbb{Z}_n$, we obtain the **$n$-sheeted covering** of the circle

$$\begin{tikzcd} \mathcal{U} \\ \mathsf{S}^1 \arrow[u] \end{tikzcd}$$



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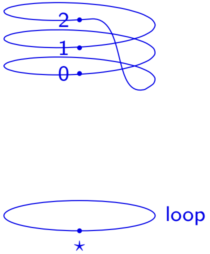

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We recover the non-trivial fibration with Bool as fibers with $n = 2$.

The universal cover of the circle

If we start from $\mathsf{suc} : \mathbb{Z} \to \mathbb{Z}$, we obtain the **universal covering** of the circle

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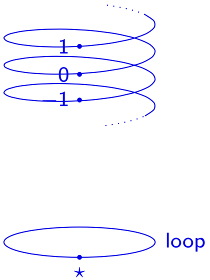

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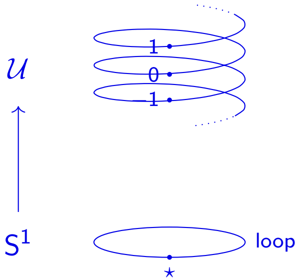

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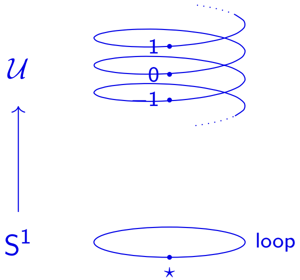

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The type family is precisely the function $\mathsf{Code} : \mathsf{S}^1 \to \mathcal{U}$

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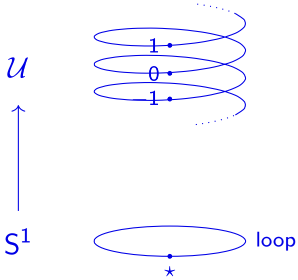

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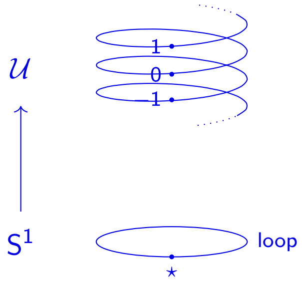

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This suggests another approach for computing the loop space of the circle!

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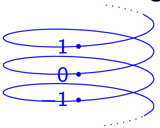

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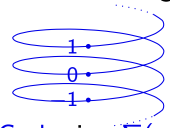

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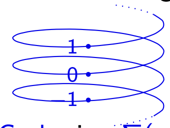

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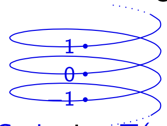

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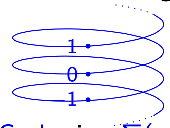

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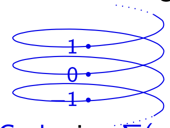

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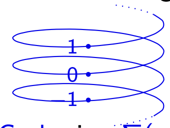

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The first 4 points require higher inductive types (see next session). Let us detail point 6.

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Theorem (see labs)
There is an equivalence between the two types of morphisms and this equivalence preserves composition and identities.

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$\square$

The fundamental theorem of identity types

[Rij25, Theorem 11.2.2]

when the Px are propositions the projection $\Sigma A.P \to A$ is an embedding

in particular the inclusion of the connected component of a point into the sapce

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