

Equivalences II

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More equivalences

The goal of this last part about equivalences is to

- provide various equivalent definitions of equivalences
- show that being an equivalence is a proposition

Equivalences

For now, our definition of equivalence has been:

Definition

A **bi-invertible equivalence** is a map $f : A \rightarrow B$ such that

$$\text{isEquiv}(f) \quad \hat{=} \quad (\Sigma(g : B \rightarrow A).g \circ f \sim \text{id}) \times (\Sigma(g : B \rightarrow A).f \circ g \sim \text{id})$$

We recall this is better than the “bad definition”:

Definition

A **quasi-invertible** map $f : A \rightarrow B$ is such that

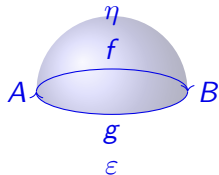
$$\text{hasQInv}(f) \quad \hat{=} \quad \Sigma(g : B \rightarrow A).((g \circ f \sim \text{id}) \times (f \circ g \sim \text{id}))$$

Half-adjoint equivalences

Consider

$$f : A \rightarrow B \qquad g : B \rightarrow A \qquad \eta : g \circ f \sim \text{id}_A \qquad \varepsilon : f \circ g \sim \text{id}_B$$

which can be pictured as



One solution: fill in the sphere!

Definition

A **half-adjoint equivalence** is a map $f : A \rightarrow B$ such that there exists

$$\text{isHAE}(f) \quad \hat{=} \quad \Sigma(g : B \rightarrow A). \Sigma(\eta : g \circ f \sim \text{id}). \Sigma(\varepsilon : f \circ g \sim \text{id}). f\eta \sim \varepsilon f$$

Contractible fibers

Given a set-theoretic function $f : A \rightarrow B$, we have that f is a bijection if and only if $f^{-1}(y)$ is a singleton for every $y : B$.

Definition

Given a map $f : A \rightarrow B$ and $y : B$, the **fiber** of f at y is

$$\text{fib } f \ y \quad \hat{=} \quad \Sigma(x : A).(f \ x = y)$$

Definition ([Uni13, Definition 4.4.1])

A map $f : A \rightarrow B$ has **contractible fibers** when

$$\text{hasCFib}(f) \quad \hat{=} \quad (y : B) \rightarrow \text{isContr}(\text{fib } f \ y)$$

We show that

$$\text{isEquiv}(f) \rightarrow \text{isHAE}(f) \rightarrow \text{hasCFib}(f) \rightarrow \text{isEquiv}(f)$$

from which we will be able to deduce that being an equivalence is a proposition and all are equivalent.

From quasi-invertibles to half-adjoint

Theorem ([Uni13, Theorem 4.2.3])

We have $\text{hasQInv}(f) \rightarrow \text{isHAE}(f)$.

Proof.

Remember that f is quasi-invertible means that we have $g : B \rightarrow A$ with $\eta : g \circ f \sim \text{id}$ and $\varepsilon : f \circ g \sim \text{id}$. An idea would be to find $\alpha : f\eta \sim \varepsilon f$, but we cannot in general!

We define ε' by $\varepsilon'_x \triangleq \varepsilon_{f g f x}^- \cdot f \eta_{g x} \cdot \varepsilon_x$ which can be pictured as



so that

$$f \eta_{g f x} \cdot \varepsilon_{f x} = \text{diagram} = \text{diagram} = \text{diagram} = \varepsilon_{f g f x} \cdot f \eta_x \quad \square$$

The equation shows a sequence of four string diagrams connected by equals signs. The first diagram represents the product $f \eta_{g f x} \cdot \varepsilon_{f x}$ and consists of a vertical line labeled 'f' at the bottom, with a 'U' shape (representing η) on top, and another 'U' shape (representing ε) on top of that. The second diagram is a simplified version where the two 'U' shapes are separated. The third diagram is another simplified version. The fourth diagram represents $\varepsilon_{f g f x} \cdot f \eta_x$ and consists of a vertical line labeled 'f' at the bottom, with a 'U' shape (representing ε) on top, and another 'U' shape (representing η) on top of that. The final result is $\varepsilon_{f g f x} \cdot f \eta_x$ followed by a square symbol indicating the proof is complete.

From quasi-invertibles to half-adjoint

Previous proof relied on two technical lemmas. First, homotopies are natural:

Lemma ([Uni13, Lemma 2.4.3])

Suppose given $f, g : A \rightarrow B$ such that $\alpha : f \sim g$, and $p : x = y$ in A . We have

$$\alpha x \cdot \text{ap } g \, p = \text{ap } f \, p \cdot \alpha y$$

Proof.

By path induction on p . □

Corollary

Given $\alpha : f \sim g$ and $\beta : f' \sim g'$, we have

$$\alpha_{f' \, x} \cdot g \beta_x = f \beta_x \cdot \alpha_{g' \, x}$$



From quasi-invertibles to half-adjoint

The second lemma is the following one:

Lemma ([Uni13, Corollary 2.4.4])

Suppose given $f : A \rightarrow A$ and $\alpha : f \sim \text{id}$. For $x : A$, we have

$$\alpha(f\,x) = f(\alpha\,x)$$

i.e.

$$\begin{array}{c} f \\ \downarrow \\ \alpha \end{array} \quad \begin{array}{c} f \\ | \\ f \end{array} = \begin{array}{c} f \\ | \\ f \end{array} \quad \begin{array}{c} f \\ \downarrow \\ \alpha \end{array}$$

Proof.

By naturality, we have

$$f\alpha_x = f\alpha_x \cdot \alpha_x \cdot \alpha_x^{-1} = \alpha_{f\,x} \cdot \alpha_x \cdot \alpha_x^{-1} = \alpha_{f\,x}$$

$$\begin{array}{c} f \\ \downarrow \\ \alpha \end{array} \quad \begin{array}{c} f \\ | \\ f \end{array} = \begin{array}{c} f \\ | \\ f \end{array} \quad \begin{array}{c} f \\ \downarrow \\ \alpha \end{array}$$

From half-adjoint to contractible fibers

Theorem ([Uni13, Theorem 4.2.6])

For $f : A \rightarrow B$, we have $\text{isHAE}(f) \rightarrow \text{hasCFib}(f)$.

Proof.

Fix $y : A$. We have to show that $\text{fib } f \ y \hat{=} \Sigma(x : A).(f \ x = y)$ is contractible.

As center of contraction, we pick $(g \ y, \varepsilon_y : f(g \ y) = y)$.

Given $(x, p) : \Sigma(x : A).(f \ x = y)$, we want to show that it is equal to the above.

We have $x = g(f \ x) = g \ y$ by $(\eta \ x)^{-1} \cdot \text{ap } g \ p$

$$\text{transport } (\lambda x. f \ x = y) ((\eta \ x)^{-1} \cdot \text{ap } g \ p) \ p$$

$$= \text{ap } f ((\eta \ x)^{-1} \cdot \text{ap } g \ p)^{-1} \cdot p$$

by transport in path types

$$= \text{ap } (f \circ g) \ p^{-1} \cdot \text{ap } f (\eta \ x) \cdot p$$

by distributivity

$$= \text{ap } f (\eta \ x) \cdot \text{ap id } p^{-1} \cdot p$$

by naturality

$$= \text{ap } f (\eta \ x)$$

$$= \varepsilon(f \ x)$$

by HAE

$$= \varepsilon \ y$$

by p $\square$

Being an equivalence is a proposition

We can finally show:

Theorem ([Uni13, Lemma 4.2.9])

For a map $f : A \rightarrow B$, being an equivalence is a proposition.

Proof.

Follows immediately from next lemma.



Being an equivalence is a proposition

Lemma

For an equivalence $f : A \simeq B$, having a left (resp. right) inverse is a proposition.

Proof.

We want to show

$$\text{isContr}(\text{fib } \phi \text{ id})$$

with

$$\phi : (B \rightarrow A) \rightarrow (A \rightarrow A)$$

$$g \mapsto g \circ f$$

Since we have shown that equivalences have contractible fibers, it is enough to show that ϕ is an equivalence, which follows from the fact that f is. □

Being an equivalence is a proposition

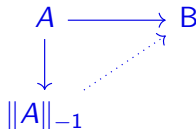
In particular, we do not have to handle all the coherences!

Corollary

Given two equivalences $e, e' : A \simeq B$. If their underlying functions $f, f' : A \rightarrow B$ are equal, then $e = e'$.

Contractible maps

note: if we apply being contractible this to equivalences associated to type constructors, we obtain unique extension



note : this is a miracle that we only need a finite quantity of data to define equivalences
!

Surjections and embeddings

A set-theoretic function $f : A \rightarrow B$ is

- *surjective* when $f^{-1}(y)$ is inhabited for every $y \in B$,
- *injective* when $f^{-1}(y)$ contains at most one element for every $y \in B$,
- *bijective* when both injective and surjective.

Surjections and embeddings

Definition

A map $f : A \rightarrow B$ is

- a **surjection** when $(y : B) \rightarrow \|\text{fib } f y\|_{-1}$,
- an **embedding** when $(y : B) \rightarrow \text{isProp}(\text{fib } f y)$.

Lemma ([Uni13, Theorem 4.6.3])

A map $f : A \rightarrow B$ is an equivalence iff both a surjection and an embedding.

Proof.

Suppose that f is bijective and fix $y : B$. We have $|(f^{-1} y, \text{refl})|_{-1} : \text{fib } f y$.

Moreover $\text{isContr}(\text{fib } f y)$ and thus $\text{isProp}(\text{fib } f y)$.

Suppose that f is both a surjection and an embedding and fix $y : B$. We have $\|\text{fib } f y\|_{-1}$ and since we are trying to prove that f is an equivalence which is a proposition, we can assume given an element of $\text{fib } f y$. We thus have $\text{fib } f y$ contractible as an inhabited proposition.

Lemma ([Uni13, Lemma 7.6.2])

A map $f : A \rightarrow B$ is an embedding if and only if the induced map

$$\text{ap } f : (x = y) \rightarrow (f\ x = f\ y)$$

is an equivalence for every $x\ y : A$.

Variants of the notions

There are variants of the notion which are not propositions.

Definition

A morphism $f : A \rightarrow B$ is **split surjection** when it satisfies

$$\prod(y : B). \sum(x : A). f\ x = y$$

This is equivalent to

$$\sum(g : B \rightarrow A). \prod(y : B). f\ (g\ y) = y$$

i.e.

$$A \xrightarrow[f]{g} B$$

This is clearly data in general, not a proposition.

Variants of the notions

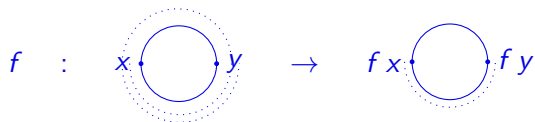
Definition

A map $f : A \rightarrow B$ is **injective** when it satisfies

$$(x\ y : A) \rightarrow (f\ x = f\ y) \rightarrow (x = y)$$

For A and B sets this is equivalent to f being an embedding.

The proofs that the identity


$$f : \begin{array}{c} \text{dotted circle} \\ \text{solid circle} \\ x \quad y \end{array} \rightarrow \begin{array}{c} \text{solid circle} \\ \text{dotted circle} \\ f\ x \quad f\ y \end{array}$$

is an embedding is $\mathbb{Z}$.

[Uni13] The Univalent Foundations Program.

Homotopy Type Theory: Univalent Foundations of Mathematics.

Institute for Advanced Study, 2013.

<https://homotopytypetheory.org/book>, arXiv:1308.0729.