Equivalences II

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École polytechnique

More equivalences

The goal of this last part about equivalences is to

- provide various equivalent definitions of equivalences
- show that being an equivalence is a proposition

Equivalences

For now, our definition of equivalence has been:

Definition
A **bi-invertible equivalence** is a map $f : A \to B$ such that

$$\mathrm{isEquiv}(f) \quad \hat{=} \quad (\Sigma(g : B \to A).g \circ f \sim \mathrm{id}) \times (\Sigma(g : B \to A).f \circ g \sim \mathrm{id})$$

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We recall this is better than the "bad definition":

Definition
A **quasi-invertible** map $f : A \to B$ is such that

$$\mathrm{hasQInv}(f) \quad \hat{=} \quad \Sigma(g : B \to A).((g \circ f \sim \mathrm{id}) \times (f \circ g \sim \mathrm{id}))$$

Half-adjoint equivalences

Consider

$$f : A \to B$$

which can be pictured as

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \end{tikzcd}$$

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$$f : A \to B \qquad g : B \to A \qquad \eta : g \circ f \sim \mathrm{id}_A$$

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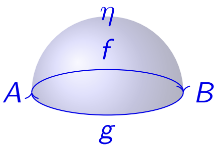

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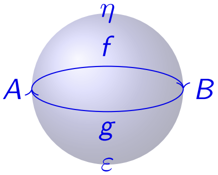

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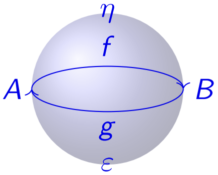

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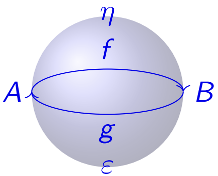

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Definition
A **half-adjoint equivalence** is a map $f : A \to B$ such that there exists

$$\mathrm{isHAE}(f) \quad \hat{=} \quad \Sigma(g : B \to A).\Sigma(\eta : g \circ f \sim \mathrm{id}).\Sigma(\varepsilon : f \circ \sim \mathrm{id}).f\eta \sim \varepsilon f$$

Contractible fibers

Given a set-theoretic function $f : A \to B$, we have that f is a bijection if and only if $f^{-1}(y)$ is a singleton for every $y : B$.

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Definition ([Uni13, Definition 4.4.1])
A map $f : A \to B$ **has contractible fibers** when

$$\mathrm{hasCFib}(f) \quad \hat{=} \quad (y : B) \to \mathrm{isContr}(\mathrm{fib}\ f\ y)$$

Roadmap

We show that

$$\mathsf{isEquiv}(f) \quad \to \quad \mathsf{isHAE}(f) \quad \to \quad \mathsf{hasCFib}(f) \quad \to \quad \mathsf{isEquiv}(f)$$

from which we will be able to deduce that being an equivalence is a proposition and all are equivalent.

From quasi-invertibles to half-adjoint

Theorem ([Uni13, Theorem 4.2.3])
We have $\mathsf{hasQInv}(f) \to \mathsf{isHAE}(f)$.

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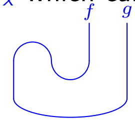

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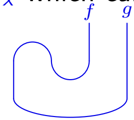
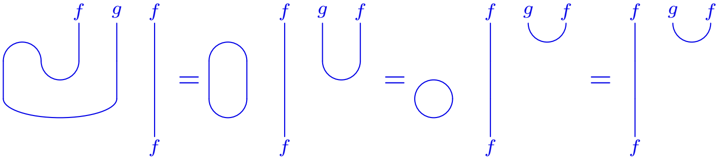


so that

$\varepsilon'_{f\,x} = \varepsilon^{-}_{f\,g\,f\,x} \cdot f\eta_{g\,x} \cdot \varepsilon_x =$



$= f\,\eta_x$

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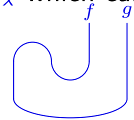

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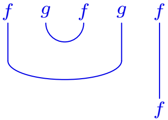
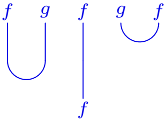
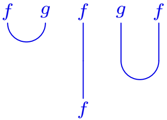
$f\eta_{g\,f\,x} \cdot \varepsilon_{f\,x} =$  $=$  $=$  $= \varepsilon_{f\,g\,f\,x} \cdot f\,\eta_x$ □

From quasi-invertibles to half-adjoint

Previous proof relied on two technical lemmas. First, homotopies are natural:

Lemma ([Uni13, Lemma 2.4.3])
Suppose given $f\ g : A \to B$ *such that* $\alpha : f \sim g$*, and* $p : x = y$ *in* A*. We have*

$$\alpha\, x \cdot \mathsf{ap}\ g\ p \quad = \quad \mathsf{ap}\ f\ p \cdot \alpha\, y$$

Proof.
By path induction on p. $\square$

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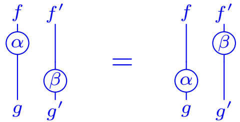
Corollary
Given $\alpha : f \sim g$ *and* $\beta : f' \sim g'$, *we have*

$$\alpha_{f'\,x} \cdot g \beta_x \quad = \quad f \beta_x \cdot \alpha_{g'\,x}$$



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The second lemma is the following one:

Lemma ([Uni13, Corollary 2.4.4])
Suppose given $f : A \to A$ *and* $\alpha : f \sim \mathrm{id}$*. For* $x : A$*, we have*

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i.e.

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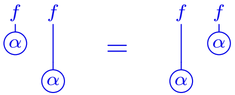

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From half-adjoint to contractible fibers

Theorem ([Uni13, Theorem 4.2.6])
For $f : A \to B$*, we have* $\mathsf{isHAE}(f) \to \mathsf{hasCFib}(f)$*.*

Proof.
Fix $y : A$. We have to show that $\mathsf{fib}\ f\ y \mathrel{\hat{=}} \Sigma(x : A).(f\ x = y)$ is contractible.

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\end{aligned}$$

Being an equivalence is a proposition

We can finally show:

Theorem ([Uni13, Lemma 4.2.9])
For a map $f : A \to B$, being an equivalence is a proposition.

Proof.
Follows immediately from next lemma. $\square$

Being an equivalence is a proposition

Lemma
For an equivalence $f : A \simeq B$, having a left (resp. right) inverse is a proposition.

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Lemma

For an equivalence $f : A \simeq B$, having a left (resp. right) inverse is a proposition.

Proof.

We want to show

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Since we have shown that equivalences have contractible fibers, it is enough to show that ϕ is an equivalence, which follows from the fact that f is. $\square$

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In particular, we do not have to handle all the coherences!

Corollary

Given two equivalences $e\, e' : A \simeq B$. *If their underlying functions* $f\, f' : A \to B$ *are equal, then* $e = e'$.

Contractible maps

note: if we apply being contractible this to equivalences associated to type constructors, we obtain unique extension

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ \|A\|_{-1} \arrow[ur, dotted] & \end{tikzcd}$$

note : this is a miracle that we only need a finite quantity of data to define equivalences !

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- *bijective* when both injective and surjective.

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A map $f : A \to B$ is

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Embeddings

Lemma ([Uni13, Lemma 7.6.2])
A map $f : A \to B$ is an embedding if and only if the induced map

$$\mathsf{ap}\, f \quad : \quad (x = y) \quad \to \quad (f\, x = f\, y)$$

is an equivalence for every $x\, y : A$.

There are variants of the notion which are not propositions.

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This is clearly data in general, not a proposition.

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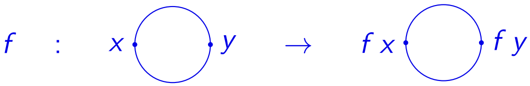

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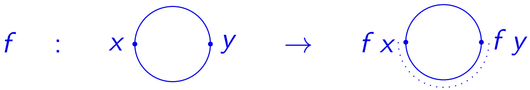

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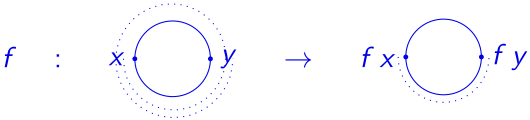

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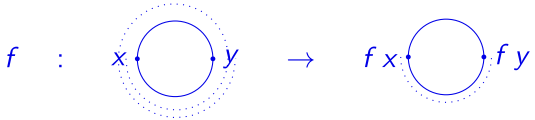

is an embedding is $\mathbb{Z}$.

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