Univalence

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Contractible types

We have seen that a type is contractible when it is equivalent to $\mathbf{1}$: for instance

$$\| \mathsf{Bool} \|_{-1} \quad \simeq \quad 1$$

i.e. (roughly)

However, nothing guarantees that we cannot distinguish between $\| \mathsf{Bool} \|_{-1}$ and 1.

Constructing identities

For now, we do not have a way of constructing non-trivial identities.

For instance, we have no way to show

$$\mathsf{Bool} \quad = \quad 1 \sqcup 1$$

(even though we can show that they are equivalent).

One solution for everything

UNIVALENCE

Univalence

Lemma
We have a function

$$\mathrm{idtoequiv} \quad : \quad (A = B) \to (A \simeq B)$$

Proof (non-satisfactory computationally).
We could define it by path induction by

$$\mathrm{idtoequiv}(\mathrm{refl}) \quad \hat{=} \quad \mathrm{id}$$ □

This is not entirely satisfactory, because the underlying function $\mathrm{idtoequiv}(p) : A \to B$ is not something definitionally known so that we have to do the lemmas again...

For instance: $\mathrm{idtoequiv}(p \cdot q) = \mathrm{idtoequiv}(q) \circ \mathrm{idtoequiv}(p)$.

Univalence

Lemma ([Uni13, Lemma 2.10.1])
We have a function

$$\mathrm{idtoequiv} \quad : \quad (A = B) \to (A \simeq B)$$

Proof.
We have a function

$$\mathrm{coe} : (A = B) \to (A \to B)$$

defined by $\mathrm{coe}(\mathrm{refl}) \mathrel{\hat{=}} \mathrm{id}$ or rather by $\mathrm{coe} \mathrel{\hat{=}} \mathrm{transport}\,(\lambda X.X)\,p\,x$ and we claim that $eq : (p : A = B) \to \mathrm{isEquiv}(\mathrm{coe}(p))$.

By path induction on p, we have to show $\mathrm{isEquiv}(\mathrm{id})$, which is easy.

And we define

$$\mathrm{idtoequiv}(p) \quad \hat{=} \quad (coe(p), eq(p)) \qquad \square$$

Univalence

Axiom
The **univalence axiom** states that, for $A\,B : \mathcal{U}$, the function

$$\text{idtoequiv} \quad : \quad (A = B) \to (A \simeq B)$$

is itself and equivalence.

Univalence

That $\mathsf{idtoequiv} : (A = B) \to (A \simeq B)$ is an equivalence means that we have

- an elimination rule:

$$\mathsf{idtoequiv} : (A = B) \to (A \simeq B)$$

- an introduction rule:

$$\mathsf{ua} : (A \simeq B) \to A = B$$

- a computation rule: for $f : A \simeq B$ and $x : A$,

$$\mathsf{coe}\,(\mathsf{ua}\ f)\,x = f\,x$$

  (and $\mathsf{coe}\,(\mathsf{ua}\ f)^{-1}\,x = f^{-1}\,x$)

- a uniqueness rule: for $p : A = B$

$$\mathsf{ua}\,(\mathsf{coe}\ p) = p$$

A first family of applications is the construction of equalities, by ua.

For instance, we have

$$e \quad : \quad \mathrm{Bool} \quad \simeq \quad 1 \sqcup 1$$

and therefore

$$\mathrm{ua}\ e \quad : \quad \mathrm{Bool} \quad = \quad 1 \sqcup 1$$

By transport, if we have $P(\mathrm{Bool})$ then we have $P(1 \sqcup 1)$, i.e. any property satisfied by one is satisfied by the other.

More realistically, think of two implementations of natural numbers: unary and binary.

Independence under implementation

Suppose that we have two implementations of natural numbers $\mathbb{N}_{\mathsf{unary}}$ and $\mathbb{N}_{\mathsf{binary}}$.

We will be able to show that $e : \mathbb{N}_{\mathsf{unary}} \simeq \mathbb{N}_{\mathsf{binary}}$.

And thus $\mathsf{ua}\ e : \mathbb{N}_{\mathsf{unary}} = \mathbb{N}_{\mathsf{binary}}$.

This means that any function working on one can be transported to a function working on the other: given

$$\mathsf{pred} \quad : \quad \mathbb{N}_{\mathsf{unary}} \to \mathbb{N}_{\mathsf{unary}}$$

we have

$$\mathsf{pred}' \quad \hat{=} \quad \mathsf{transport}\,(\lambda X.X \to X)\,(\mathsf{ua}\ e)\,(\mathsf{pred}) \quad : \quad \mathbb{N}_{\mathsf{binary}} \to \mathbb{N}_{\mathsf{binary}}$$

This is not magic:

$$\mathsf{pred}' \quad = \quad e \circ \mathsf{pred} \circ e^{-1}$$

Homotopy invariance

Geometrically, this means that we cannot distinguish between two homotopy equivalent types

$$A \quad \simeq \quad B$$

because we have

$$A = B$$

Otherwise said, all constructions are *invariant under homotopy equivalence*!

Univalence: universe levels

Axiom
The **univalence axiom** states that, for $A\,B : \mathcal{U}_\ell$, the function

$$\mathsf{idtoequiv} \quad : \quad (A = B) \to (A \simeq B)$$

is itself and equivalence.

We can have $A \simeq B$ for $A : \mathcal{U}_\ell$ and $B : \mathcal{U}_{\ell'}$,
but $A = B$ only makes sense at the same level.

Contractible types

We have seen earlier that a type A is contractible iff $A \simeq 1$. By univalence, we have

Lemma ([Uni13, Lemma 3.11.3])
Given a small contractible type A, we have $A = 1$.

Note: there are also large contractible types such as $\|\mathcal{U}\|_{-1}$.

We also have

Lemma
Given a small type A such that $\neg A$, we have $A = 0$.

Incompatibility with the set-theoretic interpretation

We can also now start to construct some interesting non-trivial paths!

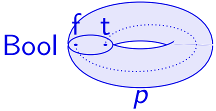
Consider the boolean negation function $\mathsf{not} : \mathsf{Bool} \to \mathsf{Bool}$.

We have $\mathsf{not} \circ \mathsf{not} = \mathsf{id}_{\mathsf{Bool}}$.

Therefore we have an equivalence $e : \mathsf{Bool} \simeq \mathsf{Bool}$: $\begin{tikzcd} \mathsf{Bool} \arrow[r, bend left, "\mathsf{not}"] & \mathsf{Bool} \arrow[l, bend left, "\mathsf{not}"] \end{tikzcd}$.

By univalence, we have a path $p : \mathsf{Bool} = \mathsf{Bool}$.

This path satisfies: $\mathsf{coe}\ p\ \mathsf{false} = \mathsf{not}\ \mathsf{false} = \mathsf{true}$ (by univalence computation).



Incompatibility with the set-theoretic interpretation

It might be secretly the case that all the types we can constructs are sets?

Proposition ([Uni13, Example 3.1.9])
The universe is not a set.

Proof.
Suppose that $\mathcal{U}$ is a set. This implies that two paths p and refl of type $\mathsf{Bool} = \mathsf{Bool}$ are equal. Therefore we have

$$\mathsf{true} = \mathsf{coe}\ p\ \mathsf{false} = \mathsf{coe}\ \mathsf{refl}\ \mathsf{false} = \mathsf{false}$$

Contradiction. $\square$

Incompatibility with classical logic

It might be secretly the case that the logic is classical?

Proposition ([Uni13, Corollary 3.2.7])
It is not the case that for every $A : \mathcal{U}$, we have

$$A \sqcup \neg A$$

Proof.
It is well known that

$$(A : \mathcal{U}) \to A \sqcup \neg A$$

implies

$$(A : \mathcal{U}) \to \neg\neg A \to A$$

(it is in fact logically equivalent) and we do not have this (see next slide). □

Note that we are parametric in A!

Incompatibility with classical logic

Proposition ([Uni13, Theorem 3.2.2])
It is not the case that for every $A : \mathcal{U}$*, we have*

$$\neg\neg A \to A$$

Proof.
Suppose that we have $F : (A : \mathcal{U}) \to \neg\neg A \to A$, in particular we have $f \mathrel{\hat{=}} F\ \mathsf{Bool} : \neg\neg\ \mathsf{Bool} \to \mathsf{Bool}$.
With $p : \mathsf{Bool} = \mathsf{Bool}$ the non-trivial path, we define $g : \mathsf{transport}\,(\lambda X. \neg\neg X \to X)\, p\, f$.

We have $g = f$ by $\mathsf{apd}\,(\lambda X. \neg\neg X \to X)\, F\, p$, i.e. for $x : \neg\neg\ \mathsf{Bool}$

$$\mathsf{not}\,(f\ x) = f\ x$$

which is impossible. $\square$

Incompatibility with classical logic

We have see that we cannot assume the principle

$$(A : \mathcal{U}) \to A \sqcup \neg A$$

however, we can assume

$$(A : \mathrm{HProp}) \to A \sqcup \neg A$$

which is the right way of formulating excluded middle.

Decidable propositions

In fact, assuming that the logic is classical amounts to assuming that all propositions are booleans.

Recall that a proposition A is **decidable** when $A \sqcup \neg A$ holds.

Proposition
Decidable propositions are booleans.

Proof (sketch).
We want to show $\Sigma(A : \mathcal{U}).(\mathrm{isProp}\, A \times (A \sqcup \neg A)) \simeq \mathrm{Bool}$. From left-to-right we pick true or false depending on if we have A or $\neg A$.
From right-to-left we pick the function picking 0 or 1 depending on the boolean.
The identity on Bool is simple, for the other side we need to use the fact that a (resp. not) inhabited proposition is contractible and thus equal to 1 (resp. 0). $\square$

Equivalences behave like identities.

In particular, we have an analogous of J called **equivalence induction**:

Proposition ([Uni13, Corollary 5.8.5])
Suppose given a property $P : \{A\, B : \mathcal{U}\} \to (A \simeq B) \to \mathcal{U}$*.*

If for every $A : \mathcal{U}$*, we have* $r_A : P\ \mathrm{id}_A$*, then for every* $e : A \simeq B$ *we have* $P\, e$*.*

Proof.
By path induction, we have $f : (p : A = B) \to P\,(\mathrm{idtoequiv}\ p)$, namely $f\ \mathrm{refl} \mathrel{\hat{=}} r_A$.

We thus have $P\,(\mathrm{idtoequiv}\,(\mathrm{ua}\ e))$ and thus $P\, e$ by the computation rule for univalence. $\square$

[Uni13] The Univalent Foundations Program.
Homotopy Type Theory: Univalent Foundations of Mathematics.
Institute for Advanced Study, 2013.
`https://homotopytypetheory.org/book`, `arXiv:1308.0729`.