Univalence

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Contractible types

We have seen that a type is contractible when it is equivalent to 1: for instance

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i.e. (roughly)

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However, nothing guarantees that we cannot distinguish between $\| \mathsf{Bool} \|_{-1}$ and 1.

Constructing identities

For now, we do not have a way of constructing non-trivial identities.

For instance, we have no way to show

$$\mathsf{Bool} \quad = \quad 1 \sqcup 1$$

(even though we can show that they are equivalent).

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Lemma
We have a function

$$\mathsf{idtoequiv} \quad : \quad (A = B) \to (A \simeq B)$$

Proof (non-satisfactory computationally).
We could define it by path induction by

$$\mathsf{idtoequiv}(\mathsf{refl}) \quad \hat{=} \quad \mathsf{id} \qquad \square$$

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□

This is not entirely satisfactory, because the underlying function $\mathsf{idtoequiv}(p) : A \to B$ is not something definitionally known so that we have to do the lemmas again...

For instance: $\mathsf{idtoequiv}(p \cdot q) = \mathsf{idtoequiv}(q) \circ \mathsf{idtoequiv}(p)$.

Lemma ([Uni13, Lemma 2.10.1])
We have a function

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Proof.
We have a function

$$\mathsf{coe} : (A = B) \to (A \to B)$$

defined by $\mathsf{coe}(\mathsf{refl}) \hat{\equiv} \mathsf{id}$ or rather by $\mathsf{coe} \hat{\equiv} \mathsf{transport}\,(\lambda X.X)\,p\,x$

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And we define

$$\mathsf{idtoequiv}(p) \quad \hat{\equiv} \quad (coe(p), eq(p))$$

□

Univalence

Axiom
The **univalence axiom** states that, for $A\,B : \mathcal{U}$, the function

$$\mathsf{idtoequiv} \quad : \quad (A = B) \to (A \simeq B)$$

is itself and equivalence.

Univalence

That idtoequiv : $(A = B) \to (A \simeq B)$ is an equivalence means that we have

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- a uniqueness rule: for $p : A = B$

$$\mathsf{ua}\,(\mathsf{coe}\ p) = p$$

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By transport, if we have $P(\mathsf{Bool})$ then we have $P(1 \sqcup 1)$, i.e. any property satisfied by one is satisfied by the other.

More realistically, think of two implementations of natural numbers: unary and binary.

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Suppose that we have two implementations of natural numbers $\mathbb{N}_{\mathsf{unary}}$ and $\mathbb{N}_{\mathsf{binary}}$.

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This is not magic:

$$\mathsf{pred}' \quad = \quad (\mathsf{coe}\,(\mathsf{ua}\ e)) \circ \mathsf{pred} \circ (\mathsf{coe}\,(\mathsf{ua}\ e)^{-})$$

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Homotopy invariance

Geometrically, this means that we cannot distinguish between two homotopy equivalent types

$$A \quad \simeq \quad B$$

because we have

$$A = B$$

Otherwise said, all constructions are *invariant under homotopy equivalence*!

Univalence: universe levels

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is itself and equivalence.

We can have $A \simeq B$ for $A : \mathcal{U}_\ell$ and $B : \mathcal{U}_{\ell'}$,
but $A = B$ only makes sense at the same level.

Contractible types

We have seen earlier that a type A is contractible iff $A \simeq 1$. By univalence, we have

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Given a small contractible type A, we have $A = 1$.

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We also have

Lemma
Given a small type A such that $\neg A$, we have $A = 0$.

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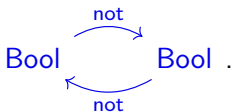
Therefore we have an equivalence $e : \mathsf{Bool} \simeq \mathsf{Bool}$: 

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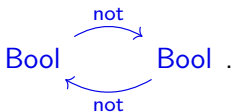

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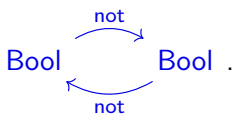

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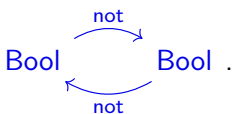
Therefore we have an equivalence $e : \mathsf{Bool} \simeq \mathsf{Bool}$: 

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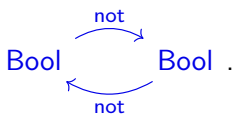
Therefore we have an equivalence $e : \mathsf{Bool} \simeq \mathsf{Bool}$: 

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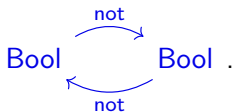
Therefore we have an equivalence $e : \mathsf{Bool} \simeq \mathsf{Bool}$: 

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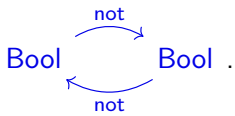
Therefore we have an equivalence $e : \mathsf{Bool} \simeq \mathsf{Bool}$: 

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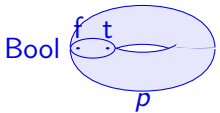

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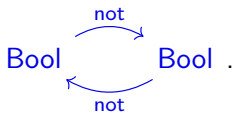
Therefore we have an equivalence $e : \mathsf{Bool} \simeq \mathsf{Bool}$: 

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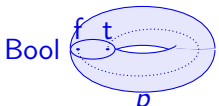

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Suppose that $\mathcal{U}$ is a set.

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Contradiction. □

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Proposition ([Uni13, Corollary 3.2.7])
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Proof.
It is well known that

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implies

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Note that we are parametric in A!

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$f \triangleq F\ \mathsf{Bool} : \neg\neg\,\mathsf{Bool} \to \mathsf{Bool}$.
With $p : \mathsf{Bool} = \mathsf{Bool}$ the non-trivial path, we define $g : \mathsf{transport}\,(\lambda X.\neg\neg X \to X)\,p\,f$.

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however, we can assume

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which is the right way of formulating excluded middle.

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From right-to-left we pick the function picking 0 or 1 depending on the boolean.
The identity on Bool is simple, for the other side we need to use the fact that a (resp. not) inhabited proposition is contractible and thus equal to 1 (resp. 0). □

Equivalence induction

Equivalences behave like identities.

In particular, we have an analogous of J called **equivalence induction**:

Proposition ([Uni13, Corollary 5.8.5])
Suppose given a property $P : \{A\, B : \mathcal{U}\} \to (A \simeq B) \to \mathcal{U}$.

If for every $A : \mathcal{U}$, *we have* $r_A : P\ \mathrm{id}_A$, *then for every* $e : A \simeq B$ *we have* $P\, e$.

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We thus have $P\,(\mathrm{idtoequiv}\,(\mathrm{ua}\ e))$ and thus $P\, e$ by the computation rule for univalence. □

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