Equivalences

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Constructing identities

For now, we cannot construct all the useful identities we would need.

In fact, nothing forces (yet) equality to be non-trivial and thus useful:
for instance, we have a set-theoretic model.

In order to fix this, we will add the univalence axiom which roughly states:

equivalent types are equal.

We first need to make precise what we mean by equivalence.

Contractible types

The types

and

respectively correspond to

- one point x
- two points a and b which are equal

We therefore expect them to be identified, i.e. equal.

However the following space should not be identified to them

because there are 2 (actually, $\mathbb{Z}$) identifications of a and b.

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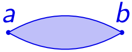
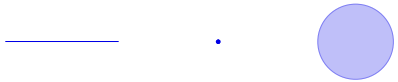
because there are 2 (actually, $\mathbb{Z}$) identifications of a and b. But 

is trivial.

Equivalences

When do we expect two types to be equivalent?

- contractible types:



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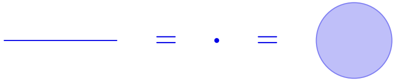

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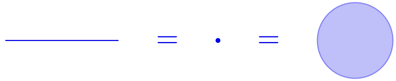

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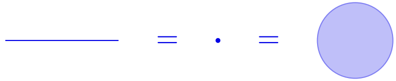

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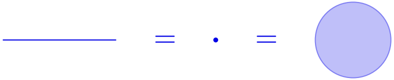

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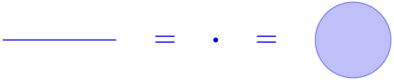

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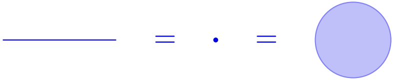

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- sets: when isomorphic
- ...

Equivalences

We expect that the following notion is suitable:

Definition
A map $f : A \to B$ is an **isomorphism** when there exists $g : B \to A$ such that

$$g \circ f = \mathrm{id}_A \qquad \text{and} \qquad f \circ g = \mathrm{id}_B\,.$$

Excepting that we need to make sure that details are right:

- we have to make sure that we can compare functions in the right way (we don't want to assume funext)
- we have to make sure that we can say "exists", i.e. being a quasi-inverse is a proposition.

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This is not the case here.

Equivalences

For the first point (comparison of functions), suppose that we want to show that $\mathrm{id} : A \to A$ is an isomorphism with id as inverse.

We need to be able to show

$$\mathrm{id} \circ \mathrm{id} = \mathrm{id}$$

which we cannot without function extensionality.

Homotopy

For the first problem there is an easy fix. Instead of requiring

$$g \circ f = \mathrm{id}$$

we require

$$(x : A) \to g\,(f\,x) = x$$

what we will write

$$g \circ f \sim \mathrm{id}$$

Bi-invertible maps

For the second problem there is also an easy fix:

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For the second problem there is also an easy fix:

Definition
A map $f : A \to B$ is **quasi-invertible** when there exists $g : B \to A$ such that

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For the second problem there is also an easy fix:

Definition
A map $f : A \to B$ is **bi-invertible** when there exists $g : B \to A$ and $g' : B \to A$ such that

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Let us understand why this is the case.

Part I

Equivalences in category theory

TODO: the fundamental groupoid is characterized by the fundamental group (when path connected)

Categories

Recall that a **category** C consists of

- a set of objects $x \in C$
- sets of morphisms $C(x, y)$ for $x, y \in C$
- composition operations

$$x \xrightarrow{f} y \xrightarrow{g} z \qquad \rightsquigarrow \qquad x \xrightarrow{g \circ f} z$$

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- identities
- satisfying axioms

A **groupoid** is a category in which every morphism is invertible.

Typical example: the fundamental groupoid of a space.

Functors between categories

A functor $F : C \to D$ between categories consists in

- a function $F : C \to D$ from the objects of C to those of D,
- for every $x, y : C$, a function $C(x, y) \to D(Fx, Fy)$
- in a way which respects composition and identities

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A functor $F : C \to D$ is an **equivalence** when there exists $G : D \to C$ such that $G \circ F \sim \mathrm{Id}$ and $F \circ G \sim \mathrm{Id}_D$.

Lemma

Given invertible morphisms $f : x' \to x$ and $h : y \to y'$ in a category C, we have $C(x, y) \cong C(x', y)$:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend left=60, dotted] \arrow[rr, bend right=60, dotted] & x' \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "C(x',y)"] & y \end{tikzcd}$$

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Proof.
We have maps

$$
\begin{aligned}
C(x', y) &\to C(x, y) \\
g &\mapsto g \circ f
\end{aligned}
\qquad\qquad
\begin{aligned}
C(x, y) &\to C(x', y) \\
g &\mapsto g \circ f^{-1}
\end{aligned}
$$

$\square$

A groupoid is **connected** when there is a morphism $f_{x,y} : x \to y$ between any two objects x, y.

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Proposition

Given a groupoid C and an object x, we have a group $C(x,x)$.

When C is connected, we have $\phi_{y,z} : C(y,z) \xrightarrow{\sim} C(x,x)$ and composition corresponds to multiplication: for $f \in C(y,z)$ and $g : C(z,w)$, we have $\phi(g \circ f) = \phi g \circ \phi f$.

In particular $C(y,y) \cong C(x,x)$ as a group.

Part II

Equivalences in topology

Let's investigate the situation on the topological side.

Topological spaces are difficult to study and classify, which should instead study some equivalence classes of such spaces.

Identity

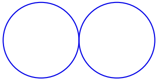
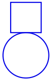
The strict notion of equality is clearly too fine: for instance, we want to identify the spaces

$$\overset{x}{\bullet} \qquad \overset{y}{\bullet} \qquad\qquad \text{and} \qquad\qquad \overset{a}{\bullet} \qquad \overset{b}{\bullet}$$

Homeomorphisms

An **homeomorphism** $f : A \to B$ is a continuous map admitting a continuous inverse.

For instance,

 $\cong$ 

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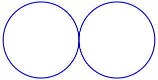
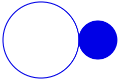
 $\not\cong$ 

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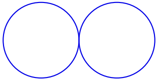

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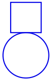

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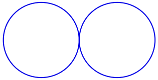

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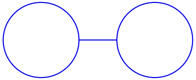

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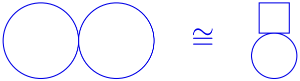

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This notion is too strong!

Homotopy

A **homotopy** between two maps $f, g : A \to B$ is a continuous map

$$\alpha : \mathsf{I} \to A \to B$$

with $\mathsf{I} = [0, 1]$, such that

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We can also see this as a continuous map

$$\alpha : A \to I \to B$$

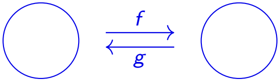
which means that for every $x \in A$ we should have paths

$$f\, x = g\, x$$

Homotopy equivalences

A **homotopy equivalence** $f : A \to B$ is a continuous map such that there is a continuous map $g : B \to A$ with $g \circ f \sim \mathrm{id}$ and $f \circ g \sim \mathrm{id}$.

For instance



Weak and homotopy equivalences in topology

fundamental group (quotient by homotopy = path between paths)

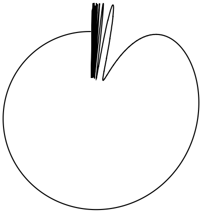
higher groups

weak homotopy equivalences

existence of a whe is reflexive and transitive, but not symmetric in general

homotopy equivalences

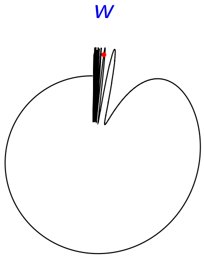
homotopy equivalence in topology



Consider the **extended topologist's sine curve** aka the **Warsaw circle** W, which consists of the following points in $\mathbb{C}$:

$$W = \left\{ \left(1 + \frac{1}{2}\sin\left(\frac{1}{\theta}\right)\right) e^{\mathrm{i}\,\theta} \mid \theta \in]0, 2\pi] \right\}$$

The Warsaw circle



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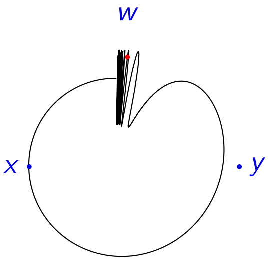

Given two points x and y, there is no *continuous* path from x to y going through w.

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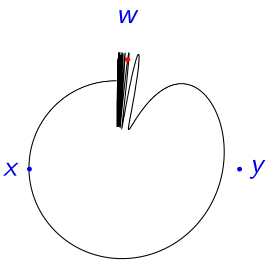

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As a consequence, there is no non-trivial map $p : \mathsf{S}^1 \to W$.

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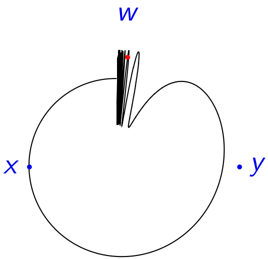

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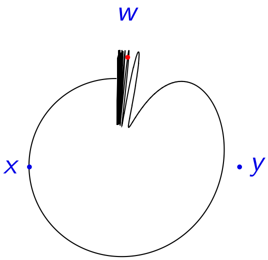
Therefore $\pi_1(W) = 1$.



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More generally $\pi_k(W) = 1$, and W is thus weakly equivalent to 1.

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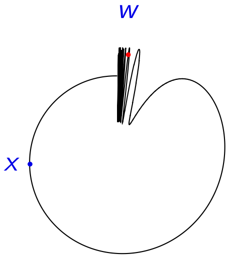

Consider the map $f : 1 \to W$ pointing at x. We have seen that this is a weak homotopy equivalence.

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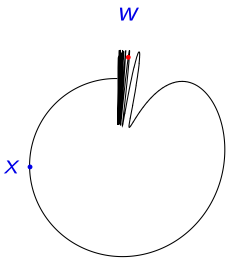

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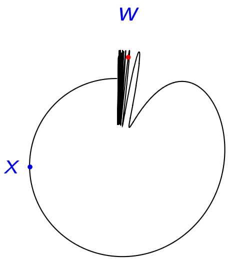

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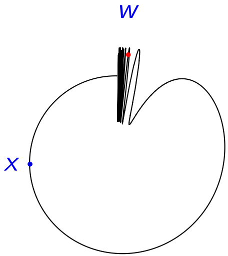

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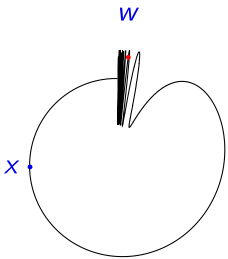

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Lemma
Every homotopy equivalence is a weak homotopy equivalence.

Proof.
We have

$$\begin{tikzcd}
\pi_n(A,a) \arrow[rr, "\sim"] \arrow[dr] & & \pi_n(A,g(f(a))) \arrow[dr] & \\
& \pi_n(A,f(a)) \arrow[ur] \arrow[rr, "\sim"'] & & \pi_n(A,f(g(f(a))))
\end{tikzcd}$$

By the 2-out-of-6 property, all the diagonal maps are equivalences (the first one in particular). □

Definition

S^1, S^2, Mobius, torus, Klein

Theorem ([Whi49a, Whi49b])
A map $f : A \to B$ between CW-complexes A and B is a homotopy equivalence iff it is a weak homotopy equivalence.

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Theorem (CW approximation)
For every space A there exists a space $\tilde{A}$ and a weak homotopy equivalence $\tilde{A} \to A$.

Part III

Equivalences

Homotopies

A **homotopy** between two functions $f\, g : (x : A) \to B(x)$ is a function of type

$$(f \sim g) \quad \hat{=} \quad (x : A) \to f\, x = g\, x$$

Homotopies

Lemma ([Uni13, Lemma 2.4.2])
Homotopy is an equivalence relation: we have

- *reflexivity:* $f \sim f$
- *symmetry:* $f \sim g \to g \sim f$
- *transitivity:* $f \sim g \to g \sim h \to f \sim h$

for $f\ g\ h : (x : A) \to B(x)$.

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We can consider:

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$$\lambda x.\,\mathsf{refl} \quad : \quad (x : A) \to f\,x = f\,x$$

and

$$\lambda p.\lambda x.(p\,x)^{-} \quad : \quad ((x : A) \to f\,x = g\,x) \to (x : A) \to g\,x = f\,x$$

Homotopies

Lemma ([Uni13, Lemma 2.4.2])
Homotopy is an equivalence relation: we have

- *reflexivity:* $f \sim f$
- *symmetry:* $f \sim g \to g \sim f$
- *transitivity:* $f \sim g \to g \sim h \to f \sim h$

for $f\ g\ h : (x : A) \to B(x)$.

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$$\lambda p.\lambda q.\lambda x \to p\,x \cdot q\,x \quad : \quad ((x : A) \to f\,x = g\,x) \to ((x : A) \to g\,x = h\,x) \to (x : A) \to f\,x =$$

Homotopies: a 2-category

We have a **2-category** categories, functors and natural transformations:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

Similarly, we have something like a 2-category of types, functions and homotopies:

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

with $\alpha : f \sim g$.

Homotopies: a 2-category

The fact that $\sim$ is an "equivalence relation" can be pictured as

$$\text{reflexivity:} \qquad A \xrightarrow{f} B \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\mathrm{id}\Downarrow"] & B \end{tikzcd}$$

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$$\text{transitivity:} \quad \begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\beta\circ\alpha\Downarrow"] & B \end{tikzcd}$$

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symmetry: $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Uparrow\alpha^{-1}"] & B \end{tikzcd}$

Homotopies: whiskering

Proposition
As in a 2-category, we also have ***whiskering*** *compositions:*

- *left whiskering:*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\alpha\Downarrow"] & C \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, bend left, "g\circ f"] \arrow[r, bend right, "g'\circ f"'] \arrow[r, phantom, "\alpha_f \Downarrow"] & C \end{tikzcd}$$

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Proof.
Define $\alpha_f\, x \triangleq$ and $g\, \alpha\, x \triangleq$. □

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Homotopies satisfy the following **naturality** property:

Lemma ([Uni13, Lemma 2.4.3])
Given functions $f\ g : A \to B$, *homotopy* $\alpha : f \sim g$ *and equality* $p : x = y$ *in* A, *we have*

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Proof.
By path induction on p, it is enough to show

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Quasi-invertible maps

This suggests that we consider the following notion of equivalence:

Definition
For a map $f : A \to B$, a **quasi-inverse** is $g : B \to A$ such that

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$$\mathrm{hasQInv}(f) \quad \triangleq \quad \Sigma(g : B \to A).(g \circ f \sim \mathrm{id}\, A) \times (f \circ g \sim \mathrm{id}_B)$$

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The reason why this is not called an equivalence is that this <u>not</u> the right notion.

Being quasi-invertible is not a property: we might have two non-equal inverses g and g'!

We cannot say “is quasi-invertible”!

Quasi-invertible maps are not right

Consider

$$f : A \to B$$

which can be pictured as

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \end{tikzcd}$$

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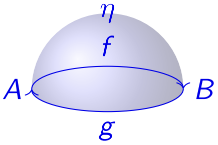

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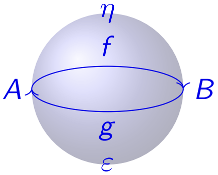

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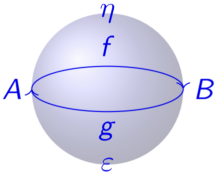

One solution: fill in the sphere!

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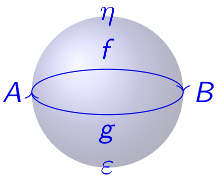

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Definition
A **half-adjoint equivalence** is a map $f : A \to B$ such that there exists

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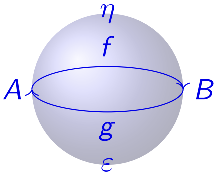

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We will first see a simpler solution.

Bi-invertible equivalences

Definition
A map $f : A \to B$ is a **bi-invertible equivalence** where there are $g : B \to A$ and $h : B \to A$ such that

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We write

$$\mathrm{isEquiv}(f) \quad \hat{=} \quad \Sigma(g : B \to A).(g \circ f \sim \mathrm{id}_A) \times \Sigma(h : B \to A).(f \circ h \sim \mathrm{id}_B)$$

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Theorem
Being a bi-invertible equivalence is a property: $\mathrm{isProp}(\mathrm{isEquiv}(f))$.

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Being a bi-invertible equivalence is a property: $\mathrm{isProp}(\mathrm{isEquiv}(f))$*.*

Proof.
Wait some more! $\square$

Consider

$$f : A \to B$$

which can be pictured as

$$\xrightarrow{\quad f \quad}$$

Bi-invertible equivalences

Consider

$$f : A \to B \qquad g : B \to A$$

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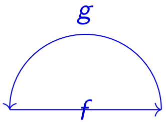

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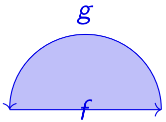

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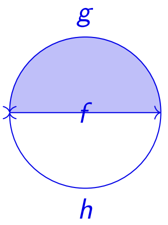

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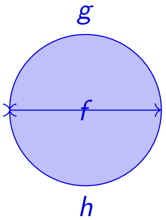

The data is contractible!

Quasi- vs bi-invertibles

Oddly, we have that

Proposition
Having a quasi-inverse and being bi-invertible are equivalent:

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Note: the pair of maps does not itself form an equivalence...

From quasi-invertibles to equivalences

A map

$$f : A \to B$$

which is quasi-invertible by

$$(g, \eta, \varepsilon)$$

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Interlude
In a group, a left inverse of an element a is equal to any right inverse.

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Proof.
Suppose given a, b and c such that $ba = 1$ and $ac = 1$. We want to show $b = c$.

We have

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We have

$$1 = ba$$

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From equivalences to quasi-invertibles

Conversely, suppose given a bi-invertible equivalence $(f, g, \eta, h, \varepsilon)$.

We can build a quasi-invertible

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Conversely, suppose given a bi-invertible equivalence $(f, g, \eta, h, \varepsilon)$.

We can build a quasi-invertible

$$(f, g, \eta, g, \varepsilon')$$

where we still have to define

$$\varepsilon' : f \circ g \sim \mathrm{id}$$

from

$$\eta : g \circ f \sim \mathrm{id} \qquad\qquad \varepsilon : f \circ h \sim \mathrm{id}$$

From equivalences to quasi-invertibles

The left inverse g has to be the same as the right inverse h:

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and we can therefore define ε' as the composite

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$$g = gfh = h$$

and we can therefore define ε' as the composite

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From equivalences to quasi-invertibles

The left inverse g has to be homotopic to the right inverse h:

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More explicitly:

$$\varepsilon' x \quad \hat{=} \quad \mathrm{ap}\,(f \circ g)((\varepsilon\, x)^{-}) \cdot \mathrm{ap}\, f\,(\eta\,(h\, x)) \cdot \varepsilon\, x$$

Equivalences

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$$A \simeq B \quad \hat{=} \quad \Sigma(f : A \to B).\, \mathsf{isEquiv}(f)$$

Part IV

First properties of equivalences

Proposition ([Uni13, Lemma 2.4.12])
Being equivalent is an equivalence relation:

- $A \simeq A$,
- $A \simeq B$ *implies* $B \simeq A$,
- $A \simeq B$ *and* $B \simeq C$ *implies* $A \simeq C$.

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Proof.
The identity $\mathrm{id} : A \to A$ is an equivalence because it admits id as inverse.

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A function $f : A \to B$ is bi-invertible, thus quasi-invertible with inverse $f^{-1} : B \to A$, thus f^{-1} is quasi-invertible, thus f^{-1} is an equivalence.

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Proof.
Given $f : A \to B$ and $g : B \to C$, we have that $gf : A \to C$ admits $f^{-1}g^{-1} : C \to A$ as inverse. $\square$

Equivalences between propositions

Lemma ([Uni13, Lemma 3.3.3])
Two propositions A and B are equivalent if and only if they are logically equivalent:

$$(A \leftrightarrow B) \qquad \leftrightarrow \qquad (A \simeq B)$$

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For $x : A$, we have $g(f(x)) = x$ because A is a proposition and thus $g \circ f \sim \mathrm{id}$.

Similarly, $f \circ g \sim \mathrm{id}$. □

Equivalences between propositions

We can even upgrade previous lemma:

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Given two propositions A and B, we have

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Proof.
We have that

$$(A \leftrightarrow B) \quad \hat{=} \quad (A \to B) \times (B \to A)$$

is a proposition and

$$(A \simeq B) \quad \hat{=} \quad \Sigma(f : A \to B).\,\mathrm{isEquiv}(f)$$

is a proposition. □

Lemma ([Uni13, Lemma 3.11.3])
We have

$$\mathrm{isContr}(A) \quad \simeq \quad (A \simeq 1)$$

Equivalences between contractible types

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We have

$$\mathsf{isContr}(A) \quad \simeq \quad (A \simeq 1)$$

Proof.
We have that $\mathsf{isContr}(A)$ is a proposition as well as

$$(A \simeq 1) \quad \hat{=} \quad \Sigma(f : A \to 1).\, \mathsf{isEquiv}(f)$$

and therefore it is enough to show that

$$(A \simeq 1) \leftrightarrow \Sigma(f : A \to 1).\, \mathsf{isEquiv}(f)$$

Equivalences between contractible types

Lemma ([Uni13, Lemma 3.11.3])
We have

$$\mathrm{isContr}(A) \quad \simeq \quad (A \simeq 1)$$

Proof.
Suppose that A is contractible. The we can build an equivalence

$$f : A \leftrightarrow 1 : g$$

where f is the terminal arrow, and g sends $\star$ to the center of contraction a of A.

We then have $g \circ f(x) = x$ and $f \circ g(y) = y$ because both A and 1 are propositions.

Lemma ([Uni13, Lemma 3.11.3])
We have

$$\mathrm{isContr}(A) \quad \simeq \quad (A \simeq 1)$$

Proof.
Suppose given $f : A \simeq 1$. We chose $a \hat{=} f^{-1}(\star)$ as center of contraction. Given $x : A$, we have

$$x = f^{-1}(f(x)) = f^{-1}(\star) = a$$

□

Standard equivalences

Lemma
Coproduct and product satisfy the expected equivalences:

$$0 \sqcup A \simeq A \simeq A \sqcup 0 \qquad (A \sqcup B) \sqcup C \simeq A \sqcup (B \sqcup B) \qquad A \sqcup B \simeq B \sqcup A$$

$$1 \times A \simeq A \simeq A \times 1 \qquad (A \times B) \times C \simeq A \times (B \times C) \qquad A \times B \simeq B \times A$$

$$0 \times A \simeq 0 \simeq A \times 0 \qquad A \times (B \sqcup C) \simeq (A \times B) \sqcup (A \times C)$$

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$$1 \times A \simeq A \simeq A \times 1 \qquad (A \times B) \times C \simeq A \times (B \times C) \qquad A \times B \simeq B \times A$$

$$0 \times A \simeq 0 \simeq A \times 0 \qquad A \times (B \sqcup C) \simeq (A \times B) \sqcup (A \times C)$$

Lemma ([Uni13, Section 2.15])
Arrow types satisfy the expected equivalences:

$$A \to 1 \simeq 1 \qquad X \to A \times B \simeq (X \to A) \times (X \to B)$$

$$0 \to X \simeq 1 \qquad A \sqcup B \to X \simeq (A \to X) \times (B \to X)$$

$$1 \to C \simeq C \qquad (A \times B) \to C \simeq A \to (B \to C)$$

The universal property of products

Let's focus on $(X \to A \times B) \simeq (X \to A) \times (X \to B)$: we have

$$\begin{aligned} \phi : (X \to A \times B) &\to (X \to A) \times (X \to B) \\ f &\mapsto \end{aligned}$$

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satisfying

$$\begin{aligned} \psi \circ \phi(f) &= \psi(\lambda x.\, \mathsf{fst}(f\,x), \lambda x.\, \mathsf{snd}(f\,x)) = \lambda x. (\mathsf{fst}(f\,x), \mathsf{snd}(f\,x)) = \lambda x. f\,x \\ \phi \circ \psi(f, g) &= \phi(\lambda x. (f\,x, g\,x)) = (\lambda x.\, \mathsf{fst}(f\,x, g\,x), \lambda x.\, \mathsf{snd}(f\,x, g\,x)) = (\lambda x. f\,x, \lambda x. g\,x) = (f, \end{aligned}$$

The universal property of products

The equivalence

$$(X \to A \times B) \simeq (X \to A) \times (X \to B)$$

can be understood as encoding the universal property of products:

$$\begin{tikzcd}
 & X \arrow[d, dotted] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
 & A \times B \arrow[dl, "\mathsf{fst}"] \arrow[dr, "\mathsf{snd}"'] & \\
A & & B
\end{tikzcd}$$

The universal property of coproducts

....

TODO: for coproducts : introduction / elimination / computation / uniqueness

this only encodes non-dependent variant and does not inform us about definitional equalities

Lemma ([Uni13, Section 2.15])
We also have the expected variants for dependent types:

- *given* $B\,C : A \to \mathcal{U}$*:*

$$(x : A) \to B\,x \times C\,x \quad \simeq \quad ((x : A) \to B\,x) \times ((x : A) \to C\,x)$$

Variants for dependent types

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$$(x : A \times B) \to C\,x \quad \simeq \quad (a : A) \to (b : B) \to C\,(a, b)$$

- *given* $B : A \to \mathcal{U}$ *and* $C : (\Sigma A.B) \to \mathcal{U}$*:*

$$(x : \Sigma A.B) \to C\,x \quad \simeq \quad (a : A) \to (b : B\,a) \to C\,(a, b)$$

Skolemizing

In model theory, there is a *Skolemization* operation, which allows removing existential quantification for prenex formulas.

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The idea is that we can replace a formula

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by a formula

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for some function f.

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for some function f.

For instance, we can replace

$$\forall x.\exists y.y \times x = 1 = x \times y$$

by

$$\forall x.x^{-1} \times x = 1 = x \times x^{-1}$$

The type-theoretic axiom of choice

A dependent variant of $A \times (B \sqcup C) \simeq (A \times B) \sqcup (A \times C)$ is

Proposition ([Uni13, Theorem 2.15.7])
Given $A : X \to \mathcal{U}$ *and* $B : (x : X) \to A\,x \to \mathcal{U}$, *we have*

$$\Pi(x : X).\Sigma(a : A\,x).B\,x\,a \quad \simeq \quad \Sigma(f : (x : X) \to A\,x).\Pi(x : X).B\,x\,(f\,x)$$

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This is sometimes called the **type-theoretic axiom of choice**.

The usual axiom of choice is rather

$$\Pi(x : X).\|\Sigma(a : A\,x).B\,x\,a\|_{-1} \quad \to \quad \|\Sigma(f : (x : X) \to A\,x).\Pi(x : X).B\,x\,(f\,x)\|_{-1}$$

Equivalence is a congruence

Lemma
Equivalence is a congruence with usual type formers: for $A \simeq A$ *and* $B \simeq B'$

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and similarly for $\sqcup$ *and* $\to$.

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and similarly for $\sqcup$ and $\to$.

Note: this is quite painful / by hand. With univalence, will become automatic!

We can also characterize path types in constructed types.

Paths in products types

Recall that, for $x : A$ and $x' : B$, we have a function

$$(x = x') \to (\mathsf{fst}\ x = \mathsf{fst}\ x') \times (\mathsf{snd}\ x = \mathsf{snd}\ x')$$

given by $p \mapsto$

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We need to define a function

$$\mathrm{pair}^{=} : (\mathrm{fst}\ x = \mathrm{fst}\ x') \times (\mathrm{snd}\ x = \mathrm{snd}\ x') \to x = x'$$

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(by product induction we assume that $x \triangleq (a, b)$ and $x' \triangleq (a', b')$) which we can do by path induction on both components.

The fact that they are mutually inverse is (again) done by path induction. $\square$

Paths in products types

This equivalence means that we have

- an elimination rule

Paths in products types

This equivalence means that we have

- an elimination rule
- an introduction rule

Paths in products types

This equivalence means that we have

- an elimination rule
- an introduction rule
- a computation rule

Paths in products types

This equivalence means that we have

- an elimination rule
- an introduction rule
- a computation rule
- a uniqueness rule

Paths in Σ-types

Given $A : \mathcal{U}$ and $B : A \to \mathcal{U}$, and $(x, y)\,(x', y') : \Sigma A.B$, recall that we have a function

$$((x, y) = (x', y')) \to \Sigma(p : x = x').(y =^B_p y')$$

and a function

$$\mathsf{pair}^= : \Sigma(p : x = x').(y =^B_p y') \to ((x, y) = (x', y'))$$

Paths in Σ-types

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and a function

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Theorem ([Uni13, Theorem 2.7.2])
We have

$$(x, y) = (x', y') \quad \simeq \quad \Sigma(p : x = x').(y =^B_p y')$$

[Uni13, Section 2.14] [Uni13, Section 9.8]

SIP

Part V

Being an equivalence is a proposition

Let's try to prove that being an equivalence is a proposition.

Being an equivalence is a proposition

Theorem ([Uni13, Lemma 4.2.9])
For a map $f : A \to B$, being an equivalence is a proposition.

Proof (first draft).
Our goal is to show that

$$\mathsf{isEquiv}(f) \quad \hat{=} \quad \Sigma(g : B \to A).(g \circ f \sim \mathsf{id}) \times \Sigma(g : B \to A).(f \circ g \sim \mathsf{id})$$

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A first plan is to show that $\Sigma(g : B \to A).(g \circ f \sim \mathrm{id})$ is a proposition (and similarly for the other component) and conclude because propositions are closed under product.

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But we cannot do this!

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is a proposition.

A first plan is to show that $\Sigma(g : B \to A).(g \circ f \sim \mathrm{id})$ is a proposition (and similarly for the other component) and conclude because propositions are closed under product.

But we cannot do this! For instance

$$f : \begin{tikzcd}[row sep=small] 0 \arrow[r, mapsto] & 0 \\ 1 \arrow[r, mapsto] & 1 \\ & 2 \end{tikzcd}$$

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For the first one, we have

$$g\,y = g'(f(g\,y)) = g'\,y$$

so that $g = g'$ (if we assume funext!).
The second one can be done, but is less clear for now... (TBC) $\square$

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