

Homotopy levels

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École polytechnique

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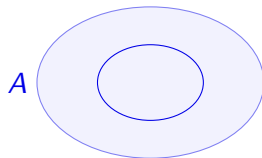
which come equipped with a canonical inclusion

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Note: all disks are contractile and thus homotopy equivalent to a point, i.e. $D^n = 1$.

Homotopy levels

Given a space A , an n -sphere in A is a map $\sigma : S^n \rightarrow A$.

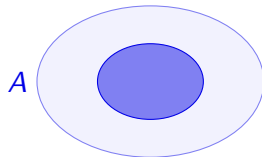


Homotopy levels

Given a space A , an n -sphere in A is a map $\sigma : S^n \rightarrow A$.

An n -sphere is **contractible** when this map extends to D^n :

$$\begin{array}{ccc} S^n & \xrightarrow{\sigma} & A \\ \iota^n \downarrow & \nearrow & \\ D^n & & \end{array}$$

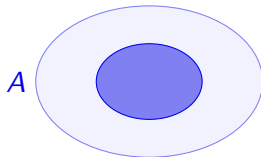


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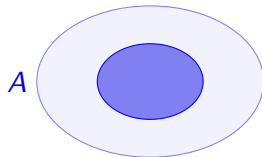


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A type is an n -type (or n -truncated) when all its k -spheres are contractible for $k > n$.

Homotopy levels

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A 0-type is thus a space in which between any two points there is at most one path (up to homotopy).

This kind of observation can be used in order to define n -types in a nice way.

Homotopy levels

In low dimensions, n -types have names

- 2 contractible types

- 1 propositions

- 0 sets

- 1 groupoids

We will begin by studying those before the general case.

Part I

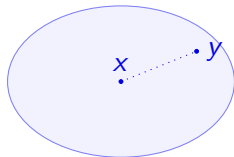
Contractible types

Contractible types

A type A is **contractible** when it satisfies

$$\text{isContr } A \quad \hat{=} \quad \Sigma(x : A). \Pi(y : A). (x = y)$$

The point x is the *center of contraction*.



Contractible types

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$$(\star, f)$$

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Contractible types

The canonical example of a contractible type is $\mathbf{1}$.

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This is fact, essentially, the only contractible type.

Bool is not contractible

Lemma

Bool *is not contractible*.

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Proof.

Suppose that Bool is contractible. There is a point $b_0 : \text{Bool}$ and a family of paths

$$p : (b : \text{Bool}) \rightarrow b_0 = b$$

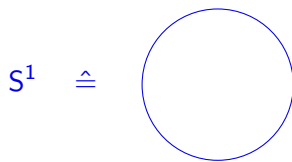
We thus have

$$\text{true} \xrightarrow{\quad (p \text{ true})^{-} \quad} b_0 \xrightarrow{\quad p \text{ false} \quad} \text{false}$$

but we know that this is not the case. □

S^1 is not contractible

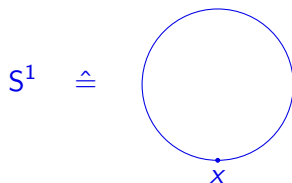
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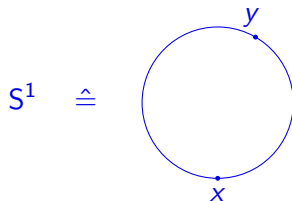
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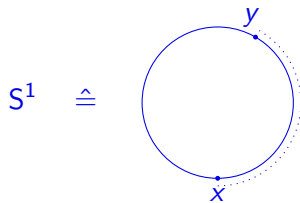
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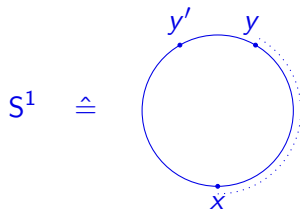
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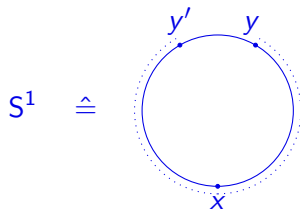
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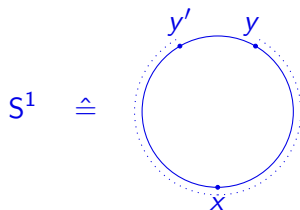
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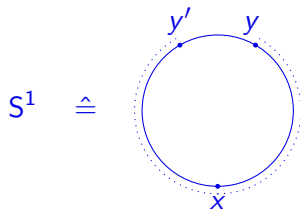
Namely, we have to construct a function

$$(y : S^1) \rightarrow (x = y)$$

which, as any function, has to be continuous.

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which, as any function, has to be continuous.

In topology, this is the difference between a **contractible** and **path connected** space.

Path spaces of contractible types

You might have the false impression that everything can be solved with J / path induction:

Proposition?

Given a contractible type A and paths $p\ q : x = y$, we have $p = q$.

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Proposition ([Uni13, Lemma 3.11.10])

Given a contractible type A and $x\ y : A$, we have that $x = y$ is contractible.

Proof.

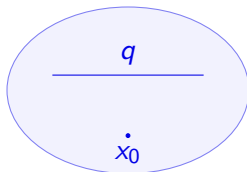
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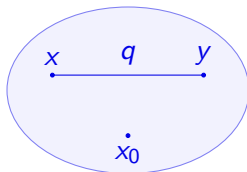
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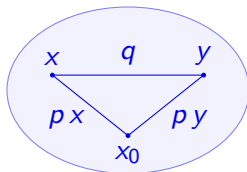
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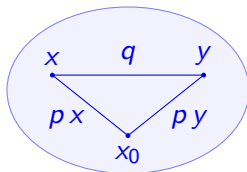
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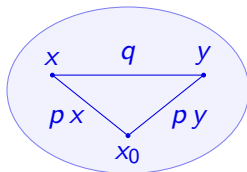
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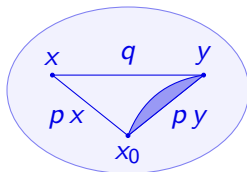
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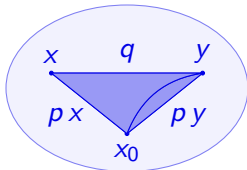
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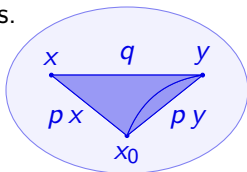
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and we conclude by groupoid laws. □

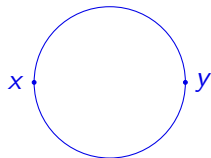


Path spaces of contractible types

Proposition ([Uni13, Lemma 3.11.10])

Given a contractible space A and $x, y : A$, we have that $x = y$ is contractible.

This also shows that S^1 is not contractible:



Clearly $x = y$ is not contractible!

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TODO: circle..... (not the connected component) helix

Singleton

Proposition ([Uni13, Lemma 3.11.8])

Given $x : A$, the type $\text{singl } x \hat{=} \Sigma(y : A).(x = y)$ is contractible.

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By transport in path types [Uni13, Lemma 2.11.2], we have

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and we conclude by unitality. □

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The $\mathbf{1}$ contractible type

Are there contractible types other than $\mathbf{1}$?

For now, we do not have a way to show that $\text{isContr } A \rightarrow (A = \mathbf{1})$ or even $\mathbf{1} = \mathbf{1}$.

But we will once we assume **univalence**, so that you can safely suppose this.

Part II

Propositions

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Every contractible type is a proposition.

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Given a contractible type A and $x\ y : A$, we know that $x = y$ is contractible and thus inhabited.

More explicitly, writing (x_0, p) for the contraction of A , with $p : (x : A) \rightarrow (x_0 = x)$, we can take

$$x \xrightarrow[p]{(p\ x)^{-}} x_0 \xrightarrow[p]{p\ y} y$$

□

Propositions as contractible types

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is not expected to hold.

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is not expected to hold.

Namely, take A to be “the n -th Turing machine halts”: this is a proposition!

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However one can show that

Lemma

For $A : \mathcal{U}$, we have that $\text{isProp } A$ is logically equivalent to $A \rightarrow \text{isContr } A$.

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Conversely suppose $f : A \rightarrow \text{isContr } A$. Given $x y : A$, we have to show $x = y$.

By $f \ x$, we have that A is contractible, therefore $x = y$ is also contractible, and thus inhabited. □

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Proposition thus deserve their name: they are formulas which can be proved in a unique way (when they can).

Lemma ([Uni13, Lemma 3.11.10])

A proposition A has contractible path spaces: $\text{isContr}(x = y)$ for $x, y : A$.

Proof.

Path spaces of propositions

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Proposition

General fact

Whenever you introduce something which looks like a predicate, you should check that this is a family of propositions: what matters is that $\text{bla}(x)$ holds and not which proof of $\text{bla}(x)$ we gave.

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In the following, we will assume this:

Temporary axiom

We assume **function extensionality**: given functions $f\ g : A \rightarrow B$ if

$$(x : A) \rightarrow f\ x = g\ x$$

then $f = g$.

It is “temporary”, because it will follow from univalence.

Being a contractible is a proposition

Proposition ([Uni13, Lemma 3.3.5])

Being contractible is a proposition: $\text{isProp}(\text{isContr } A)$.

Proof.

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By funext this amounts to show, for $z : A$

$$\text{transport}(\lambda x. (y : A) \rightarrow x = y)(p\ y)\ p\ z = q\ z$$

in $y = z$.

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in $y = z$. But A is contractile, therefore $x = z$ also, which is thus a proposition. $\square$

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in $x = y$. but A is a proposition, therefore $x = y$ is contractible, and thus a proposition. □

Subtypes

Given a type A and a family $P : A \rightarrow \mathcal{U}$ of propositions, we can form the **subtype** $\Sigma(x : A).P\ x$ of A .

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which follows from the fact that P is a proposition. □

The type of **propositions** is

$$\mathsf{HProp} \quad \hat{=} \quad \Sigma(A : \mathcal{U}). \mathsf{isProp} \ A$$

This is a subtype of $\mathcal{U}$.

Closure properties: product

We have seen that 0 / 1 are propositions, understood as $\perp$ / $\top$.

What about connectives?

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Propositions are closed under $\times$. We thus have an induced operation

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Proof.

Given A and B which are propositions and (x, y) and (x', y') in $A \times B$, we have

- $p : x = x'$ since A is a proposition
- $q : y = y'$ since B is a proposition

and thus $\text{pair}^= p\ q : (x, y) = (x', y')$.



Closure properties: implication

Proposition

The type $A \rightarrow B$ is a proposition when B is. We thus have an induced operation

$$\Rightarrow : \text{HProp} \rightarrow \text{HProp} \rightarrow \text{HProp}$$

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In particular negation is always a proposition:

$$\neg A \quad \hat{=} \quad A \rightarrow \perp$$

Closure properties: universal quantification

Previous theorem generalizes to dependent types:

Proposition

Given a type A and a family $B : A \rightarrow \mathcal{U}$ of propositions, the type

$$\prod (x : A). B x$$

is a proposition. We thus have an induced operation

$$\forall : (A : \mathcal{U}) \rightarrow (B : A \rightarrow \mathbf{HProp}) \rightarrow \mathbf{HProp}$$

Propositional extensionality

We define equivalence as

$$A \Leftrightarrow B \quad \hat{=} \quad (A \Rightarrow B) \wedge (B \Rightarrow A)$$

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The **proposition extensionality** principle [WR27, Chu40] states that

$$(A \Leftrightarrow B) \rightarrow (A = B)$$

Propositional extensionality

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The **proposition extensionality** principle [WR27, Chu40] states that

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This principle is compatible with the current theory, and in fact will be implied by **univalence** which is a generalization of it.

Closure properties: coproducts

Are propositions closed under coproducts?

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Given propositions A and B such that $\neg(A \times B)$, we have $A \sqcup B$ a proposition

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Given elements $x, x' : A \sqcup B$, we want to show that $x = x'$:

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a	a'	we have $a = a'$ by A proposition

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Closure properties: dependent sums

Propositions are not closed under Σ -types: given a family $B : A \rightarrow \mathcal{U}$ of proposition,

$$\Sigma(x : A). B x$$

is not a proposition in general. For instance,

$$\mathbb{N} = \Sigma(x : \mathbb{N}). 1$$

is not a proposition.

Propositional truncation

In order to correct this, we need an operation which turns a type A into a proposition $\|A\|_{-1}$:

- when $A = 0$, we expect $\|A\|_{-1} = 0$,
- otherwise, we expect $\|A\|_{-1}$ to be contractible.

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Excepting that we cannot reason like this in intuitionistic logic.

Instead, we will introduce $\| - \|_{-1}$ as a new type constructor!

Constructor:

$$\| - \|_{-1} : \mathcal{U} \rightarrow \mathcal{U}$$

Propositional logic

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Introduction rules:

Propositional logic

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$$| - |_{-1} : A \rightarrow \|A\|_{-1}$$

$$\text{pt} : \text{isProp}(\|A\|_{-1})$$

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Introduction rules:

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The recursor is (the eliminator is not more useful)

$$\text{rec} : (B : \mathcal{U}) \rightarrow (A \rightarrow B) \rightarrow \text{isProp } B \rightarrow \|A\|_{-1} \rightarrow B$$

Informally, if you have an $a : \|A\|_{-1}$ and you are trying to prove a proposition B , you can safely assume that you actually have $a : A$.

Otherwise said $a : \|A\|_{-1}$ is in a box that you are only allowed to open when proving a proposition.

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$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ | - |_{-1} \downarrow & \nearrow & \\ \|A\|_{-1} & & \end{array}$$

Merely

We say that A **merely** holds when we have $\|A\|_{-1}$.

The missing connectives

We can thus define, for A and B propositions,

$$A \vee B \quad \hat{=} \quad \|A \sqcup B\|_{-1}$$

and for $A : \mathcal{U}$ and $B : A \rightarrow \mathcal{U}$ a family of propositions,

$$\exists(x : A). B\ x \quad \hat{=} \quad \|\Sigma(x : A). B\ x\|_{-1}$$

The Curry-Howard correspondence

Logic	Type theory	
Formula	Type	
$\top$	1	
$\perp$	0	
$A \wedge B$	$A \times B$	
$A \Rightarrow B$	$A \rightarrow B$	
$A \vee B$	$A \sqcup B$	
$\forall(x : A).B(x)$	$(x : A) \rightarrow B(x)$	
$\exists(x : A).B(x)$	$\Sigma(x : A).B(x)$	

The Curry-Howard correspondence

Logic	Type theory	Propositions
Formula	Type	Proposition
$\top$	1	1
$\perp$	0	0
$A \wedge B$	$A \times B$	$A \times B$
$A \Rightarrow B$	$A \rightarrow B$	$A \rightarrow B$
$A \vee B$	$A \sqcup B$	$\ A \sqcup B\ _{-1}$
$\forall(x : A).B(x)$	$(x : A) \rightarrow B(x)$	$(x : A) \rightarrow B(x)$
$\exists(x : A).B(x)$	$\Sigma(x : A).B(x)$	$\ \Sigma(x : A).B(x)\ _{-1}$

Path connected types

connected

connected component

the connected component is connected

S^1 is path connected $(y : A) \rightarrow \|x = y\|_{-1}$

connected component (subtypes are embeddings) [Uni13, Lemma 3.5.1]

the image

.....

Double negation

Propositional truncation is close to double negation: there is a canonical map

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The converse does not hold unless A is “classical” in the sense that $\neg\neg A \rightarrow A$.

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Propositional truncation is close to double negation: there is a canonical map

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The converse does not hold unless A is “classical” in the sense that $\neg\neg A \rightarrow A$.

Propositional truncation is thus an “intuitionistic” variant of double negation.

The law of excluded middle.....

General excluded middle is inconsistent.....

The axiom of choice

The propositional truncation forgets about proofs only remember provability.
However, we can nevertheless sometimes extract information from it.

Extracting from truncation

Proposition ([Uni13, Exercise 3.19])

Suppose that $P : \mathbb{N} \rightarrow \mathcal{U}$ is a decidable proposition: for $n : \mathbb{N}$, we have $P\ n \sqcup \neg(P\ n)$ and $\text{isProp}(P\ n)$. Then

$$\exists (n : \mathbb{N}). P\ n \rightarrow \Sigma (n : \mathbb{N}). P\ n$$

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Typically: $P\ n \hat{=} (f\ n = 0)$ for some function $f : \mathbb{N} \rightarrow \mathbb{N}$.

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Proof.

Consider the predicate $\text{isFirst}\ n \triangleq (P\ n) \times ((m : \mathbb{N}) \rightarrow m < n \rightarrow \neg(P\ m))$.

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$$\text{findFirst} : (m : \mathbb{N}) \rightarrow P\ m \rightarrow \Sigma(n : \mathbb{N}). \text{isFirst}\ n$$

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Now, suppose given $\exists(n : \mathbb{N}). P\ n$ i.e. $\|\Sigma(n : \mathbb{N}). P\ n\|_{-1}$ and want $\Sigma(n : \mathbb{N}). \text{isFirst}\ n$. Since this is a proposition, we can suppose given an element of $\Sigma(n : \mathbb{N}). P\ n$.

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$$\text{findFirst} : (m : \mathbb{N}) \rightarrow P\ m \rightarrow \Sigma(n : \mathbb{N}). \text{isFirst}\ n$$

Now, suppose given $\exists(n : \mathbb{N}). P\ n$ i.e. $\|\Sigma(n : \mathbb{N}). P\ n\|_{-1}$ and want $\Sigma(n : \mathbb{N}). \text{isFirst}\ n$. Since this is a proposition, we can suppose given an element of $\Sigma(n : \mathbb{N}). P\ n$. By `findFirst`, we can construct an element of $\Sigma(n : \mathbb{N}). \text{isFirst}\ n$

Extracting from truncation

Proposition ([Uni13, Exercise 3.19])

Suppose that $P : \mathbb{N} \rightarrow \mathcal{U}$ is a decidable proposition: for $n : \mathbb{N}$, we have $P n \sqcup \neg(P n)$ and $\text{isProp}(P n)$. Then

$$\exists(n : \mathbb{N}). P n \rightarrow \Sigma(n : \mathbb{N}). P n$$

Proof.

Consider the predicate $\text{isFirst } n \triangleq (P n) \times ((m : \mathbb{N}) \rightarrow m < n \rightarrow \neg(P m))$. This is a proposition and thus $\Sigma(n : \mathbb{N}). \text{isFirst } n$ is also a proposition. We can then construct a function which, given a solution, finds the first one:

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Since this is a proposition, we can suppose given an element of $\Sigma(n : \mathbb{N}). P n$. By `findFirst`, we can construct an element of $\Sigma(n : \mathbb{N}). \text{isFirst } n$ and thus $\Sigma(n : \mathbb{N}). P n$. $\square$

In Agda, $\|A\|_{-1}$ is noted

$\| A \|_1$

(apparently “-” is difficult to type).

Propositional truncation: implementation

One way of implementing propositional truncation is axiomatically which we can do in Agda using rewriting rules:

`postulate`

`||_|1 : Type → Type`

`|_|1 : {A : Type} → A → || A ||1`

`isPropPropTrunc : {A : Type} → isProp || A ||1`

`propTrunc-rec :`

`{A : Type} {B : Type} → isProp B → (A → B) → (|| A ||1 → B)`

`propTrunc-beta :`

`{A : Type} {B : Type}`

`(P : isProp B) (f : A → B) (x : A) → propTrunc-rec P f | x |1 ≡ f x`

Propositional truncation: implementation

Another (better) way is to use higher inductive types (once we have them):

```
data ||_|_1 (A : Type) : Type where
  |_|_1 : A → || A ||_1
  squash_1 : isProp A
```


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Propositional truncation: impredicative encoding

A last way to encode propositional truncation is the **impredicative encoding** which consists in defining truncation by its recursion principle:

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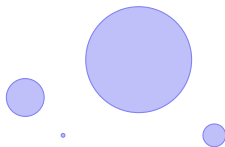
However, we can only eliminate at a specified level ℓ !

Part III

Sets

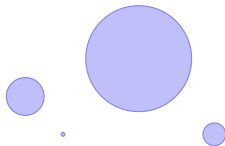
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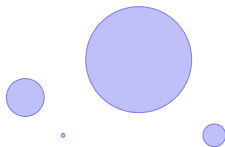


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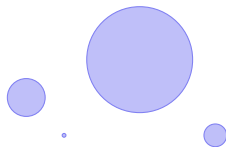
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For instance:

- are sets: 0 , 1 , Bool , $\mathbb{N}$, HProp , ...
- are not sets: S^n with $n > 0$, ...

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Booleans are sets

Proposition(?)

The type `Bool` is a set.

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Proof.

We have to show $(x\ y : \text{Bool}) \rightarrow (p\ q : x \rightarrow y) \rightarrow p = q$.

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By induction on q , it is enough to show $(x : \text{Bool}) \rightarrow (p : x \rightarrow x) \rightarrow p = \text{refl}$.

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With univalence, we will be able to show that $(x : \text{Bool}) \rightarrow \text{isContr}(x = x)$, but we do not have the tools to show this directly for now. Another proof follows. □

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Proposition ([Uni13, Lemma 3.3.4])

Any proposition A is a set: $\text{isProp } A \rightarrow \text{isSet } A$.

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Being a set is a proposition

Proposition ([Uni13, Lemma 3.3.5])

Being a set for a given type is a proposition: `isProp (isSet A)`.

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Being a set for a given type is a proposition: $\text{isProp}(\text{isSet } A)$.

Proof.

We want to show $\text{isProp}((x\ y : \text{Bool}) \rightarrow \text{isProp}(x = y))$.

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Which does hold.



We write

$$\mathsf{HSet} \quad \hat{=} \quad \Sigma(A : \mathcal{U}). \mathsf{isSet} \ A$$

for the type of sets.

Closure properties

We have that

- $A \times B$ is a set when A and B are sets
- $A \sqcup B$ is a set when A and B are sets
- $A \rightarrow B$ is a set when B is a set
- $(x : A) \rightarrow B x$ is a set when the $B x$ are sets
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These operations thus induce operations on $\mathbf{HSet}$.

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These operations thus induce operations on $\mathbf{HSet}$.

In particular, given a set A and a predicate P , we have

$$\Sigma(x : A).P\ x$$

which plays the role of a subset.

Hedberg's theorem

A type A is **decidable** when $A \sqcup \neg A$ holds.

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The following is known as **Hedberg's theorem**:

Theorem ([Hed98],[Uni13, Theorem 7.2.5])
Every discrete type is a set.

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Note: see [Esc04] if you are curious about topological terminology.

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Lemma

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Proof.

Suppose A is decidable, i.e. $A \sqcup \neg A$. Supposing $\neg\neg A$, we have to show A .

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Corollary

Any discrete type is separated.

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Suppose given two types $p\ q : x = y$ in A . We want to show that they are equal.

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Since we have $p : x = y$, we have $\bar{p} : \neg\neg(x = y)$ and thus $\tilde{p} : x = y$ by separation.

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The first equality is shown by induction on p , which amounts to $\text{refl} = \tilde{\text{refl}} \cdot \tilde{\text{refl}}^-$. $\square$

Examples of sets

Corollary

The types `Bool`, $\mathbb{N}$, `Fin n` are sets.

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Proof.

For instance, for `Bool`, it is enough to show that for $x\ y : \text{Bool}$, we have

$(x = y) \sqcup \neg(x = y)$, which can be done by case analysis:

x	y	$(x = y) \sqcup \neg(x = y)$
false	false	$x = y$ by refl
true	true	$x = y$ by refl
false	true	$\neg(x = y)$
true	false	$\neg(x = y)$



.....

We have

$$\pi_0(X) = \|X\|_0$$

The fundamental group

with set truncation

$$\pi_1(A) = \|\Omega A\|_0$$

Part IV

n-types

Groupoids

A **groupoid** is a type which has sets of paths:

$$\text{isGroupoid } A \quad \hat{=} \quad (x\ y : A) \rightarrow \text{isSet}(x = y)$$

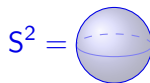
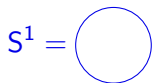
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For instance,

- sets are groupoids (`Bool`, $\mathbb{N}$, etc.)
- S^1 is a groupoid
- S^2 is not a groupoid
- `HSet` is a groupoid



n -types [Uni13, Definition 7.1.1]

We define n -types as

- a (-2) -type is a contractible type,
- an $(n + 1)$ -type is a type A in which $x = y$ is an n -type for every $x y : A$.

Formally,

$$\text{isType } n A \quad \hat{=} \quad \begin{cases} \text{isContr } A & \text{if } n \hat{=} -2, \\ (x y : A) \rightarrow \text{isType } (n - 1) A & \text{otherwise.} \end{cases}$$

Lemma ([Uni13, Lemma 3.11.10])

A (-1) -type is the same as a proposition.

Proposition

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A (-1) -type is the same as a proposition.

Proof.

A proposition satisfies

$$(x\ y : A) \rightarrow (x = y)$$

a (-1) -type satisfies

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Clearly, a (-1) -type is a proposition (project to the contraction center).

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Conversely, we have seen that a proposition has contractible path spaces.



Being an n -type is a proposition

Theorem ([Uni13, Theorem 7.1.10])

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Proof.

By induction on n .

For the base case, we already know that being contractible is a proposition.

For the inductive case, we have to show

$$\text{isProp}((x\ y : A) \rightarrow \text{isType}(n - 1)(x = y))$$

for which it is enough to show $\text{isProp}(\text{isType}(n - 1)(x = y))$ which is the induction hypothesis. □

Theorem ([Uni13, Theorem 7.1.7])
Every n -type is an $(n + 1)$ -type.

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Proof.

For the base case, we have to show that a contractible type has contractible path types, which we already did.

For the inductive case, we apply the induction hypothesis on $x = y$. □

Closure of n -types

Theorem ([Uni13, Theorems 7.1.8 and 7.1.9])

We have that

- $A \times B$ is an n -type when A and B are
- $A \sqcup B$ is an n -type when A and B are
- $A \rightarrow B$ is an n -type when B is
- $\Sigma A.B$ is an n -type when A and the $B\ x$ are
- $\Pi A.B$ is an n -type when the $B\ x$ are

Proof.

Not enough tools for now



The type of n -types

Theorem ([Uni13, Theorem 7.1.11])

The type of n -types is an $(n + 1)$ -type.

Proof.

Not enough tools for now



Higher truncations

We can define the n -truncation $\|A\|_n$ of a type, which comes equipped with

$$|-|_n : A \rightarrow \|A\|_n$$

Given an n -type B , a map $f : A \rightarrow B$ extends uniquely as a map $\|A\|_n \rightarrow B$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow |-|_n & \nearrow & \uparrow \\ \|A\|_n & & \end{array}$$

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