Operations on identity types

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Operations on identity types

We now see that we can define most of the expected operations involving identity types.

Warning
The naming of stuff is slightly different in the HoTT book and in the cubical Agda library.

I will use HoTT book notation in the slides and cubical Agda in the labs.

Symmetry

Given a path $p : \mathsf{I} \to A$ there is a "symmetric path" $p^- : \mathsf{I} \to A$ defined by $p^-(t) = p(1 - t)$:

We have seen that we can define this in HoTT using J:

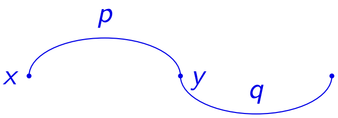
$$\mathsf{sym} : (x : A) \to (y : A) \to (x = y) \to (y = x)$$
$$x \mapsto \mathsf{J}\ A\, x\, (\lambda yp.y = x)\ \mathsf{refl}$$

More informally, in order to define sym on an arbitrary path $p : x = y$, by induction on p it is enough to define it for $y \mathrel{\hat{=}} x$ and $p \mathrel{\hat{=}} \mathsf{refl}$, which we do using refl.

This proves that equality is symmetric.

Transitivity / concatenation

We can also show that equality is **transitive**, which amounts to concatenation of paths:



In order to define the concatenation $p \cdot q : x = z$ of paths $p : x = y$ and $q : y = z$, by induction on q it is enough to define when $z \triangleq y$ and $q \triangleq \mathsf{refl} : y = y$, in which case we define it as p.

Geometrically, this corresponds to the concatenation of paths $p : \mathbb{I} \to A$ and $q : \mathbb{I} \to A$ defined by

$$(p \cdot q)(t) = \begin{cases} p(2t) & \text{if } 0 \le t \le 1/2 \\ q(2t-1) & \text{if } 1/2 \le t \le 2 \end{cases}$$

Concatenation

Note that this is not the only way we could define concatenation:

- $p \cdot \mathsf{refl} \triangleq p$
- $\mathsf{refl} \cdot p \triangleq p$
- $\mathsf{refl} \cdot \mathsf{refl} \triangleq \mathsf{refl}$

They are not definitionally equal, but they can be proved to be propositionally equal.

We have the following operation on paths: constant path (refl), concatenation ($p \cdot q$), symmetry (p^-).

Proposition ([Uni13, Section 2.1])
The above operations satisfy the expected laws: for $p : x = y$, $q : y = z$ and $r : z = w$,

$$\mathsf{refl} \cdot p = p \qquad (p \cdot q) \cdot r = p \cdot (q \cdot r) \qquad p \cdot p^- = \mathsf{refl} \qquad (p \cdot q)^- = q^- \cdot p^-$$

$$p \cdot \mathsf{refl} = p \qquad p^- \cdot p = \mathsf{refl} \qquad (p^-)^- = p$$

For instance, we have $p \cdot \mathsf{refl} = p$ by definition so that we can take refl.

To show $\mathsf{refl} \cdot p = p$, by induction on p it is enough to show $\mathsf{refl} \cdot \mathsf{refl} = \mathsf{refl}$ which holds by previous point.

Other are proved similarly by induction on paths (see the lab).

The fundamental ∞-groupoid of a space

It seems that this states that to any type/space we can associate a groupoid such that

- the objects are the points $x : A$,
- the morphisms $x \to y$ are the paths $p : x = y$,
- identities are given by refl and composition by concatenation,
- we have seen that the **axioms** are satisfied:

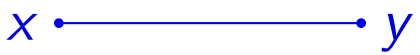
$$\mathsf{refl} \cdot p = p \qquad p \cdot \mathsf{refl} = p \qquad (p \cdot q) \cdot r = p \cdot (q \cdot r)$$

  and we have inverses.

However this is not exactly the case because "axioms" are homotopies!

The fundamental ∞-groupoid of a space

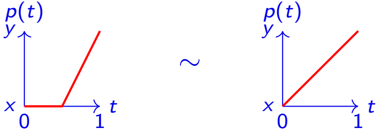
Consider the space

$A \;\hat{=}\;$ 

we have a path $p : x = y$, i.e. a function $p : \mathsf{I} \to A$ which can be pictured as

we also have $\mathsf{refl} : x = x$ pictured on the left.

Their concatenation is $\mathsf{refl} \cdot p \sim p$:



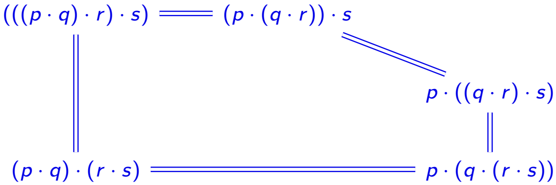
The fundamental ∞-groupoid of a space

A space induces an ∞-**groupoid** [Lum10, VDBG11]: we have

- 0-cells: the points of $x : A$
- 1-cells: the paths $p : x = y$ in A
- 2-cells: the homotopies $\alpha : p = q : x = y$ between paths in A
- ...

such that

- composition of n-cells is unital and associative *up to* $(n+1)$*-cells*
- the unitality and associativity satisfy coherence laws up to higher cells, etc.



Grothendieck's homotopy hypothesis

In fact, the ∞-groupoid is expected to contain all the relevant information of the space:

Hypothesis
The **Grothendieck homotopy hypothesis** [Gro83] states that spaces should be equivalent to ∞-groupoids.

Note: we would have to detail what we mean by "space", by "∞-groupoid" and by "equivalent", which is out of the scope of this course. There are various answers for that, and the hypothesis has been proved for some of them.

Congruence

An important property of equality is that it is a **congruence**:

Proposition
Given a function $f : A \to B$ *and an equality* $p : x = y$ *in* A*, we have an equality* $f(x) = f(y)$*.*

Proof.
By induction on p, it is enough to show that we have $f(x) = f(x)$, done by refl. □

We therefore have a function

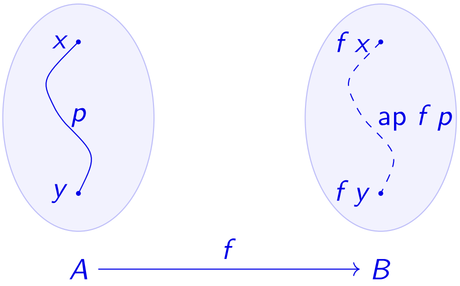
$$\mathrm{ap} : (A \to B) \to \{x\, y : A\} \to (x = y) \to (f(x) = f(y))$$

also named `cong` in Agda, which can be read as

- we can **apply** a function to a path,
- all functions induce **functors** between the corresponding ∞-groupoids.

Congruence: continuity

Geometrically, ap also means that every function is **continuous**:

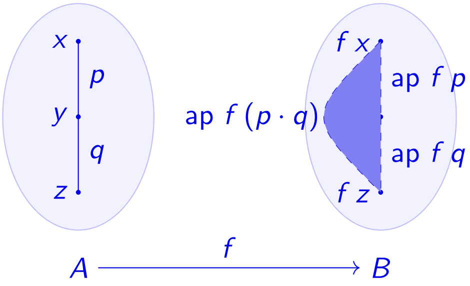


Congruence: properties

Proposition ([Uni13, Lemma 2.2.2])
We have that:

- ap *is compatible with the groupoid structure:*

$$\mathsf{ap}\ f\ (p \cdot q) = (\mathsf{ap}\ f\ p) \cdot (\mathsf{ap}\ f\ q) \qquad \mathsf{ap}\ f\ \mathsf{refl} = \mathsf{refl} \qquad \mathsf{ap}\ f\ p^{-} = (\mathsf{ap}\ f\ p)^{-}$$



- ap *is compatible with composition:*

Substitutivity

An important property of equality is that it is **substitutive**:
given a property $P : A \to \mathcal{U}$, if x satisfies P and $x = y$ then y also satisfies P.

Ex: in $\mathbb{Q}$, we have that $4/2$ is an integer and $4/2 = 6/3$ therefore $6/3$ is also an integer.

Proposition
We have a function called ***transport*** *or* `subst` *in Agda:*

$$\mathsf{transport} : \{A : \mathcal{U}\} \to (P : A \to \mathcal{U}) \to \{x\, y : A\} \to (x = y) \to P\, x \to P\, y$$

Proof.
By induction, it is enough to provide a function $P\, x \to P\, x$ and we take id. □

Difference between booleans

Proposition
In Bool*, we have* $\neg(\mathit{false} = \mathit{true})$*.*

Proof.
Suppose given $p : \mathsf{false} = \mathsf{true}$. Consider the function

$$\begin{aligned} F : \mathsf{Bool} &\to \mathcal{U} \\ \mathsf{false} &\mapsto 1 \\ \mathsf{true} &\mapsto \bot \end{aligned}$$

By transport, we have

$$\mathsf{transport}\ F\ p\ : 1 \to \bot$$

We thus have $\bot$ by applying it to $\star : 1$. □

Leibniz equality

This property of **indiscernability of identicals** can be taken as the definition of **Leibniz equality** [Lei86]: on A, we define

$$(x \stackrel{\mathsf{L}}{=} y) \quad \hat{=} \quad ((P : A \to \mathcal{U}) \to P\,x \to P\,y)$$

Lemma
This is a symmetric relation.

Proof.
Suppose $x \stackrel{\mathsf{L}}{=} y$. Given $P : A \to \mathcal{U}$ such that $P\,y$, we have to show $P\,x$.

Consider the property $Q(y) \mathrel{\hat{=}} P\,y \to P\,x$. We have $Q(x) \mathrel{\hat{=}} P\,x \to P\,y$ by id, therefore $Q(y) \mathrel{\hat{=}} P\,y \to P\,x$ because $x \stackrel{\mathsf{L}}{=} y$, and we deduce $P\,x$ since we have $P\,y$. □

In fact, Leibniz equality is logically equivalent to identity [ACD+20]:

$$(x \stackrel{\mathsf{L}}{=} y) \leftrightarrow (x = y)$$

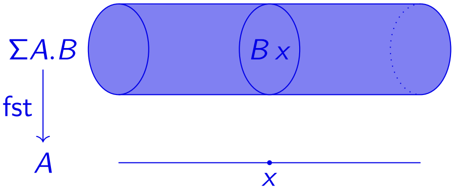
Let's provide a geometric interpretation for transport.

Type families

A **type family** is a function

$$B : A \to \mathcal{U}$$

which can be thought of as a family of spaces $B\,x$ continuously indexed by $x : A$



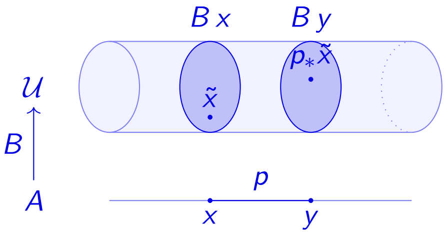
and $\Sigma(x : A).B\,x$ is the **total space**.

Transport

Given a type family $B : A \to \mathcal{U}$ the **transport** (or `subst`) operation associates to a path $p : x = y$ in A a function

$$p_* : B\,x \to B\,y$$

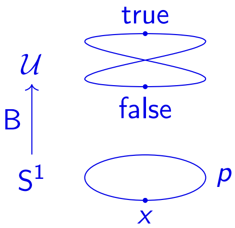
which can be pictured as



Transport

By ap all the fibers have to be equal when A is connected, but the transport can still be non-trivial!

For instance, consider the non-trivial fibration $B : \mathsf{S}^1 \to \mathcal{U}$ with $B\,x \triangleq \mathsf{Bool}$.



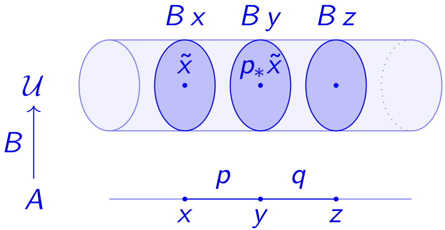
We have

$$p_* \,\mathsf{false} = \mathsf{true}$$

Transport satisfies the expected properties:

Proposition ([Uni13, Lemma 2.3.9])
For $p : x = y$ *and* $q : y = z$ *in* A, *and* $\tilde{x} : B\,x$, *we have*

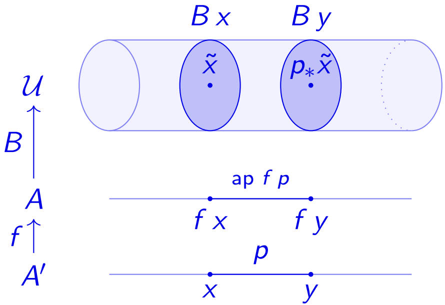
$$(p \cdot q)_* \, \tilde{x} = q_*(p_* \, \tilde{x})$$



Transport: properties

Proposition ([Uni13, Lemma 2.3.10])
Given $f : A' \to A$, $B : A \to \mathcal{U}$, $p : x = y$ *in* A' *and* $\tilde{x} : B\,x$,

$$\text{transport}\,(B \circ f)\,p\,\tilde{x} = \text{transport}\ B\,(\text{ap}\ f\ p)\,\tilde{x}$$



Transport: a variant

We have the following variant of transport

$$\mathsf{transport} : (B : A \to \mathcal{U}) \to \{x\, y : A\} \to (x = y) \to B\, x \to B\, y$$

sometimes called coe for **coercion** and noted `transport` in Agda:

$$\mathsf{coe} : A = B \to A \to B$$

Proposition
The functions $\mathsf{transport}$ *and* coe *are logically equivalent.*

Proof.
We have

$$\mathsf{coe}\ p\, x = \mathsf{transport}\,(\lambda X.X)\, p\, x$$

$$\mathsf{transport}\ b\, p\, \tilde{x} = \mathsf{coe}\,(\mathsf{ap}\ B\ p)\,\tilde{x}$$

□

Type families are fibrations

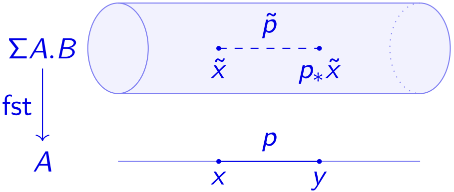
Proposition ([Uni13, Lemma 2.3.2])
Consider a type family $B : A \to \mathcal{U}$*, the map*

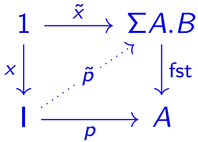
$$\mathsf{fst} : \Sigma(x : A).B\,x \to A$$

is a ***fibration****: given a path* $p : x = y$ *and* $\tilde{x} : B\,x$*, there is a path*

$$\tilde{p} : (x, \tilde{x}) = (y, p_*\,\tilde{x})$$

such that $\mathsf{ap}\ \mathsf{fst}\ \tilde{p} = p$*.*





Dependent application

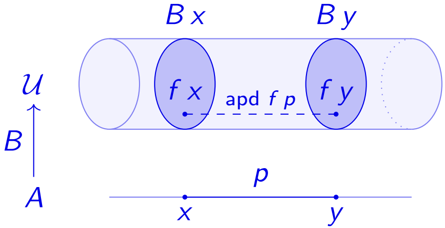
We have the congruence/application function

$$\mathsf{ap} : \{A\, B : \mathcal{U}\} \to (f : A \to B) \to \{x\, y : A\} \to (p : x = y) \to f\, x = f\, y$$

We would now like to generalize it to the dependent case

$$\mathsf{apd} : \{A : \mathcal{U}\}\, \{B : A \to \mathcal{U}\} \to (f : (x : A) \to B\, x) \to \{x\, y : A\} \to (p : x = y) \to f\, x = f\, y$$

We want to have a path between elements of $B\, x$ and $B\, y$ which is not allowed, but intuitively fine because we have a path $p : x = y$.



Dependent application

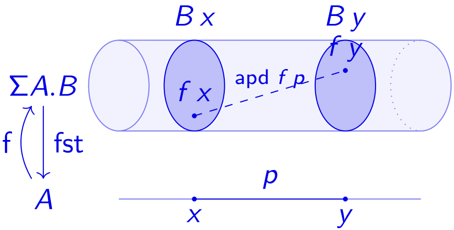
One way out is to define from

$$f : (x : A) \to B\,x$$

the application to the total space

$$F : A \to \Sigma A.B$$
$$x \mapsto (x, f\,x)$$

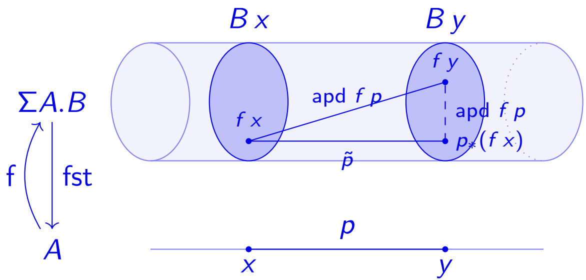
which is a section of $\mathsf{fst} : \Sigma A.B \to A$, i.e. $\mathsf{fst} \circ F(x) \mathrel{\hat=} x$, and use ap on $\tilde{f}$.



But we loose the fact that we are over p!

Dependent application [Uni13, Lemma 2.3.4]

A better idea is to encode the path $\mathsf{apd}\ f\ p$ as a path in $B\ y$.



and we define

$$\mathsf{apd} : (f : (x : A) \to B\,x) \to \{x\,y : A\} \to (p : x = y) \to p_*(f\,x) = f\,y$$

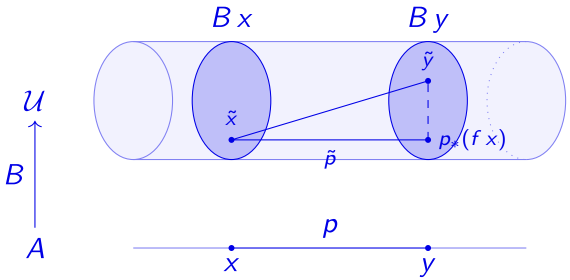
by induction by

$$\mathsf{apd}\ f\ \mathsf{refl} \mathrel{\hat{=}} \mathsf{refl}_{f\,x}$$

More generally, given $x = y$ in A, $B : A \to \mathcal{U}$, $\tilde{x} : B\,x$ and $\tilde{y} : B\,y$, the type of **paths** in B **over** p between $\tilde{x}$ and $\tilde{y}$ is

$$\tilde{x} =^{B}_{p} \tilde{y} \quad \hat{=} \quad p_*\,\tilde{x} = \tilde{y}$$

which can be pictured as



Paths in product types [Uni13, Section 2.6]

Suppose given $x\, x' : A$ and $y\, y' : B$.

A path $p : (x, y) = (x', y')$ induces paths

$$\mathsf{ap}\ \mathsf{fst}\ p : x = x' \qquad\qquad \mathsf{ap}\ \mathsf{snd}\ p : y = y'$$

Conversely, we have a function

$$\mathsf{pair}^{=} : (x = x') \to (y = y') \to (x, y) = (x', y')$$

which is defined by path induction and is useful to construct paths in products.

Path in dependent sum types [Uni13, Section 2.7]

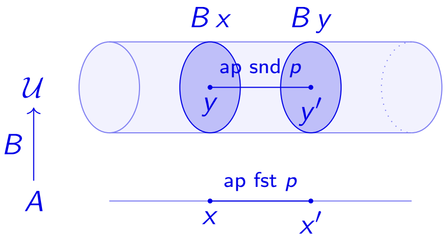
Suppose given $x\,x' : A$, $y : B\,x$ and $y' : B\,x'$.

A path $p : (x,y) = (x',y')$ induces paths

$$\mathsf{ap}\ \mathsf{fst}\ p : x = x' \qquad\qquad \mathsf{ap}\ \mathsf{snd}\ p : y =^B_p y'$$

Conversely, we have a function

$$\mathsf{pair}^= : (p : x = x') \to (y =^B_p y') \to (x,y) = (x',y')$$



Proposition ([Uni13, Lemma 2.11.2])
Given paths $p : a = x$ *and* $q : x = y$, *we have*

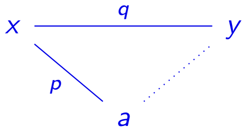
$$\mathsf{transport}\,(\lambda x. a = x)\; q\; p = p \cdot q$$

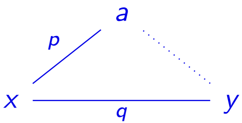
Similarly, given paths $p : x = a$ *and* $q : x = y$, *we have*

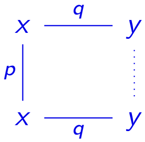
$$\mathsf{transport}\,(\lambda x. a = x)\; q\; p = q^{-} \cdot p$$

Similarly, given paths $p : x = x$ *and* $q : x = y$, *we have*

$$\mathsf{transport}\,(\lambda x. x = x)\; q\; p = q^{-} \cdot p \cdot q$$







Proof.
By path induction on p. □

Transporting paths

More generally,

Proposition ([Uni13, Lemma 2.11.3])
Given $f\, g : A \to B$, $p : f\,x = g\,x$ *in* B *and* $q : x = y$ *in* A,

$$\mathsf{transport}\,(\lambda x. f\,x = g\,x)\,q\,p \quad = \quad (\mathsf{ap}\ f\ q)^{-} \cdot p \cdot \mathsf{ap}\ g\ q$$

in $f\,y = g\,y$.

Transporting functions

Writing $\mathbb{N}$ for the unary natural number and $\mathbb{B}$ for the binary ones, we have

$$p : \mathbb{N} = \mathbb{B}$$

By transport, we obtain

$$\text{coe } p : \mathbb{N} \to \mathbb{B} \qquad\qquad \text{coe } p^{-} : \mathbb{B} \to \mathbb{N}$$

Writing $\text{suc} : \mathbb{N} \to \mathbb{N}$ for the successor function, by transport we have

$$\text{transport}\,(\lambda X.X \to X)\; p \;\text{suc} : \mathbb{B} \to \mathbb{B}$$

This function is

$$\text{transport}\,(\lambda X.X \to X)\; p \;\text{suc} = (\text{coe } p) \circ \text{suc} \circ (\text{coe } p^{-})$$

Transporting functions

Proposition
Given a function $f : A \to B$ *and paths* $p : A = A'$ *and* $q : B = B'$, *we have*

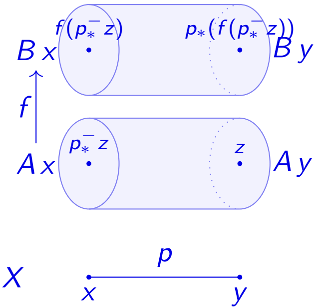
$$\mathrm{transport}\,(\lambda X. A \to X)\, q\, f = \mathrm{coe}\ q \circ f$$

$$\mathrm{transport}\,(\lambda X. X \to B)\, p\, f = f \circ \mathrm{coe}\ p^{-}$$

Transporting functions

Proposition ([Uni13, (2.9.4)])
Given type families $A : X \to \mathcal{U}$ and $B : X \to \mathcal{U}$, a path $p : x = y$ in X and a function $f : A\,x \to B\,x$, we have

$$\mathsf{transport}\,(\lambda x. A\,x \to B\,x)\,p\,f = \mathit{transport}\;B\,p \circ f \circ \mathsf{transport}\;A\,p^{-}$$



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