# Operations on identity types

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2025

École polytechnique

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**Warning**
The naming of stuff is slightly different in the HoTT book and in the cubical Agda library.

I will use HoTT book notation in the slides and cubical Agda in the labs.

## Symmetry

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More informally, in order to define $\mathsf{sym}$ on an arbitrary path $p : x = y$, by induction on $p$ it is enough to define it for $y \hat{=} x$ and $p \hat{=} \mathsf{refl}$, which we do using $\mathsf{refl}$.

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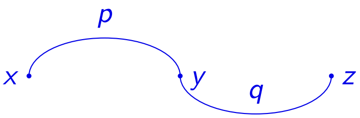
This proves that equality is symmetric.

# Transitivity / concatenation

We can also show that equality is **transitive**, which amounts to concatenation of paths:



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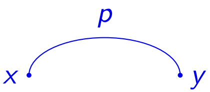


In order to define the concatenation $p \cdot q : x = z$ of paths $p : x = y$ and $q : y = z$, by induction on $q$ it is enough to define when $z \triangleq y$ and $q \triangleq \mathsf{refl} : y = y$, in which case we define it as $p$.

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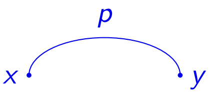


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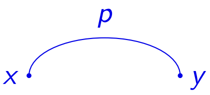


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$$(p \cdot q)(t) = \begin{cases} p(2t) & \text{if } 0 \leq t \leq 1/2 \\ q(2t-1) & \text{if } 1/2 \leq t \leq 2 \end{cases}$$

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They are not definitionally equal, but they can be proved to be propositionally equal.

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Other are proved similarly by induction on paths (see the lab).

# The fundamental $\infty$-groupoid of a space

It seems that this states that to any type/space we can associate a groupoid such that

- the objects are the points $x : A$,
- the morphisms $x \to y$ are the paths $p : x = y$,
- identities are given by $\mathsf{refl}$ and composition by concatenation,
- we have seen that the axioms are satisfied:

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However this is not exactly the case because "axioms" are homotopies!

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Consider the space

$$A \quad \hat{=} \quad x \bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\bullet\, y$$

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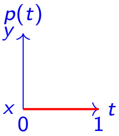


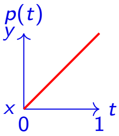


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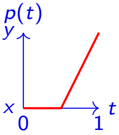

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A space induces an $\infty$-**groupoid** [Lum10, VDBG11]: we have

- 0-cells: the points of $x : A$
- 1-cells: the paths $p : x = y$ in $A$
- 2-cells: the homotopies $\alpha : p = q : x = y$ between paths in $A$
- ...

such that

- composition of $n$-cells is unital and associative *up to* $(n+1)$*-cells*
- the unitality and associativity satisfy coherence laws up to higher cells, etc.

$$\begin{array}{ccccc}
(((p \cdot q) \cdot r) \cdot s) & = & (p \cdot (q \cdot r)) \cdot s & & \\
 & & & \diagdown & \\
\| & & & & p \cdot ((q \cdot r) \cdot s) \\
 & & & & \| \\
(p \cdot q) \cdot (r \cdot s) & & = & & p \cdot (q \cdot (r \cdot s))
\end{array}$$

# Grothendieck's homotopy hypothesis

In fact, the $\infty$-groupoid is expected to contain all the relevant information of the space:

**Hypothesis**
The **Grothendieck homotopy hypothesis** [Gro83] states that spaces should be equivalent to $\infty$-groupoids.

Note: we would have to detail what we mean by "space", by "$\infty$-groupoid" and by "equivalent", which is out of the scope of this course. There are various answers for that, and the hypothesis has been proved for some of them.

# Congruence

An important property of equality is that it is a **congruence**:

**Proposition**
*Given a function* $f : A \to B$ *and an equality* $p : x = y$ *in* $A$*, we have an equality* $f(x) = f(y)$*.*

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We therefore have a function

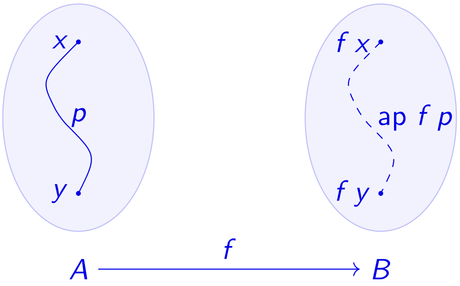
$$\mathsf{ap} : (A \to B) \to \{x\,y : A\} \to (x = y) \to (f(x) = f(y))$$

also named `cong` in Agda, which can be read as

- we can **apply** a function to a path,
- all functions induce **functors** between the corresponding $\infty$-groupoids.

# Congruence: continuity

Geometrically, ap also means that every function is **continuous**:



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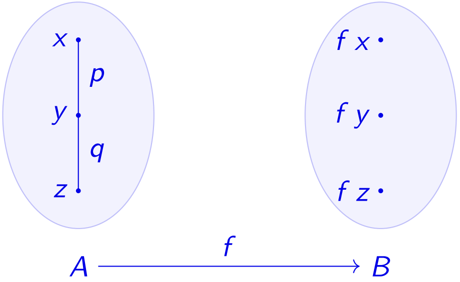

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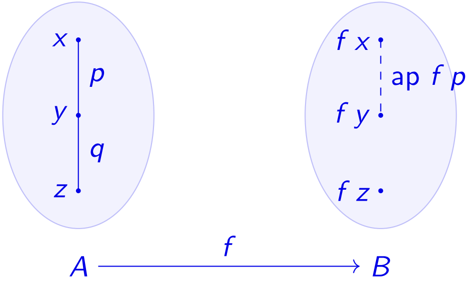

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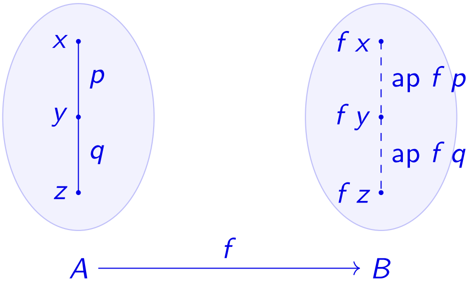

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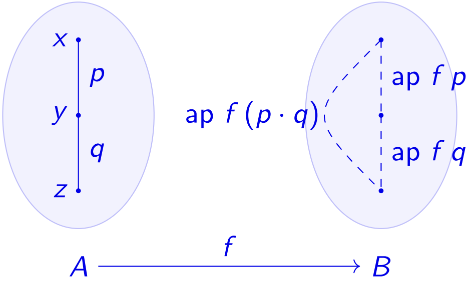

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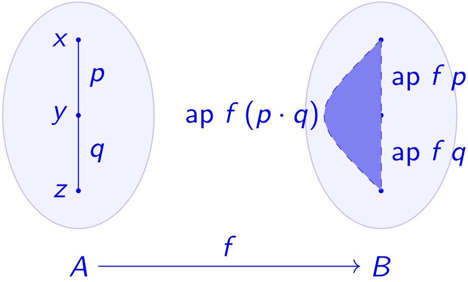

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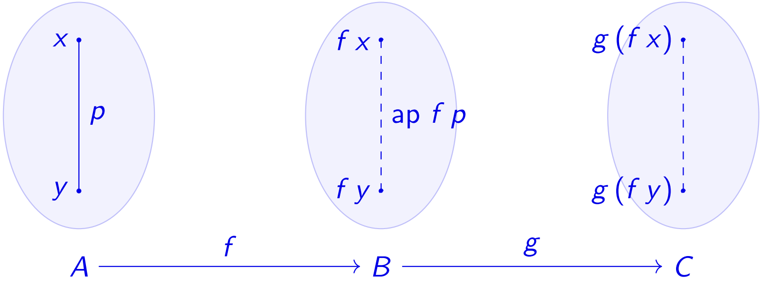
$$\mathsf{ap}\,(g \circ f)\,p = \mathsf{ap}\ g\,(\mathsf{ap}\ f\ p) \qquad \mathsf{ap}\ \mathsf{id}\ p = p$$



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**Proposition**
*We have a function called* ***transport*** *or* `subst` *in Agda:*

$$\text{transport} : \{A : \mathcal{U}\} \to (P : A \to \mathcal{U}) \to \{x\, y : A\} \to (x = y) \to P\, x \to P\, y$$

**Proof.**
By induction, it is enough to provide a function $P\, x \to P\, x$ and we take $\text{id}$. □

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We thus have $\bot$ by applying it to $\star : 1$. □

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This property of **indiscernability of identicals** can be taken as the definition of **Leibniz equality** [Lei86]: on $A$, we define

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**Lemma**
*This is a symmetric relation.*

**Proof.**
Suppose $x \overset{\mathsf{L}}{=} y$. Given $P : A \to \mathcal{U}$ such that $P\,y$, we have to show $P\,x$.

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This property of **indiscernability of identicals** can be taken as the definition of **Leibniz equality** [Lei86]: on $A$, we define

$$(x \stackrel{\mathsf{L}}{=} y) \quad \hat{=} \quad ((P : A \to \mathcal{U}) \to P\,x \to P\,y)$$

**Lemma**
*This is a symmetric relation.*

**Proof.**
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Consider the property $Q(y) \hat{=} P\,y \to P\,x$. We have $Q(x) \hat{=} P\,x \to P\,y$ by $\mathsf{id}$, therefore $Q(y) \hat{=} P\,y \to P\,x$ because $x \overset{\mathsf{L}}{=} y$,

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In fact, Leibniz equality is logically equivalent to identity [ACD+20]:

$$(x \overset{\mathsf{L}}{=} y) \leftrightarrow (x = y)$$

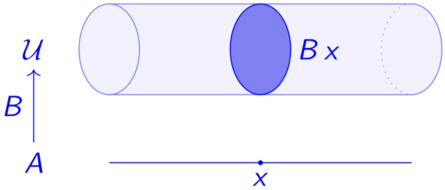
Let's provide a geometric interpretation for transport.

# Type families

A **type family** is a function

$$B : A \to \mathcal{U}$$

which can be thought of as a family of spaces $B\,x$ continuously indexed by $x : A$



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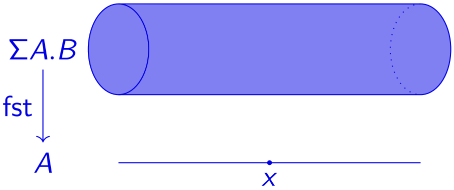


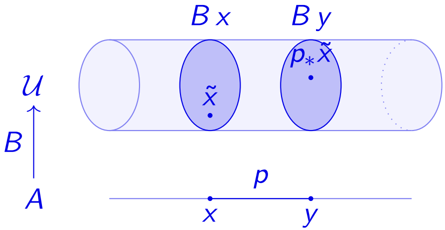

and $\Sigma(x : A).B\,x$ is the **total space**.

# Transport

Given a type family $B : A \to \mathcal{U}$ the **transport** (or `subst`) operation associates to a path $p : x = y$ in $A$ a function

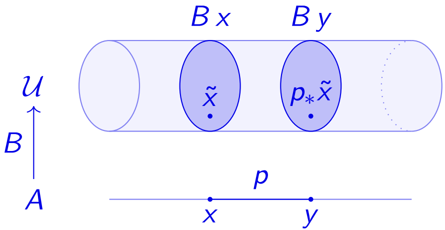
$$p_* : B\,x \to B\,y$$

which can be pictured as



# Transport

By ap all the fibers have to be equal when $A$ is connected, but the transport can still be non-trivial!



# Transport

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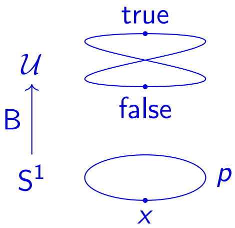
For instance, consider the non-trivial fibration $B : \mathsf{S}^1 \to \mathcal{U}$ with $B\,x \triangleq \mathsf{Bool}$.




We have

$$p_*\,\mathsf{false} = \mathsf{true}$$

# Transport: properties

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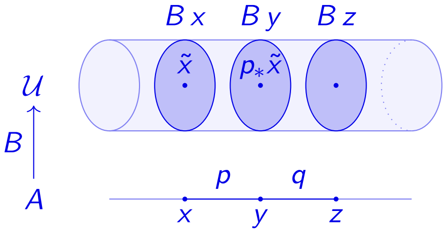
**Proposition ([Uni13, Lemma 2.3.9])**
*For $p : x = y$ and $q : y = z$ in $A$, and $\tilde{x} : B\,x$, we have*

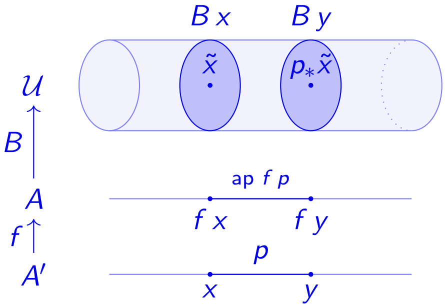
$$(p \cdot q)_* \, \tilde{x} = q_*(p_* \, \tilde{x})$$



# Transport: properties

**Proposition ([Uni13, Lemma 2.3.10])**
*Given* $f : A' \to A$, $B : A \to \mathcal{U}$, $p : x = y$ *in* $A'$ *and* $\tilde{x} : B\,x$,

$$\text{transport}\,(B \circ f)\,p\,\tilde{x} = \text{transport}\ B\,(\text{ap}\ f\ p)\,\tilde{x}$$



# Transport: a variant

We have the following variant of transport

$$\mathsf{transport} : (B : A \to \mathcal{U}) \to \{x\, y : A\} \to (x = y) \to B\, x \to B\, y$$

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$$\text{coe } p\, x = \text{transport}\, (\lambda X.X)\, p\, x$$

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# Type families are fibrations

**Proposition ([Uni13, Lemma 2.3.2])**
*Consider a type family* $B : A \to \mathcal{U}$*, the map*

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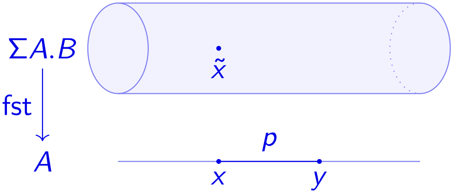

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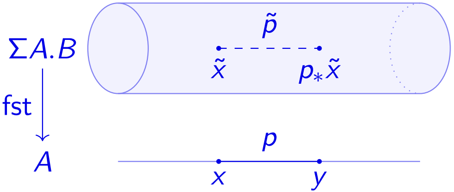

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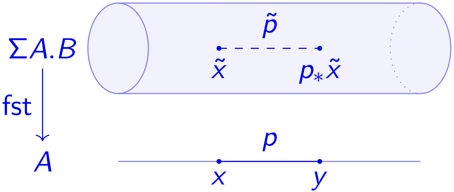


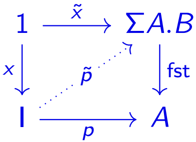

# Dependent application

We have the congruence/application function

$$\mathsf{ap} : \{A\, B : \mathcal{U}\} \to (f : A \to B) \to \{x\, y : A\} \to (p : x = y) \to f\, x = f\, y$$

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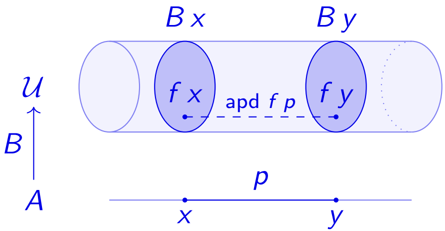
We want to have a path between elements of $B\, x$ and $B\, y$ which is not allowed, but intuitively fine because we have a path $p : x = y$.



# Dependent application

One way out is to define from

$$f : (x : A) \to B\,x$$

the application to the total space

$$\begin{aligned} F &: A \to \Sigma A.B \\ x &\mapsto (x, f\,x) \end{aligned}$$

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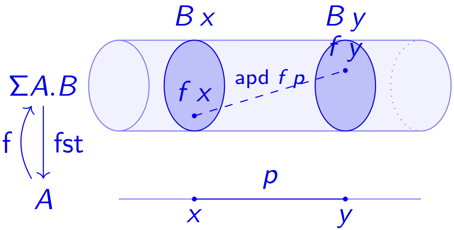


But we loose the fact that we are over $p$!

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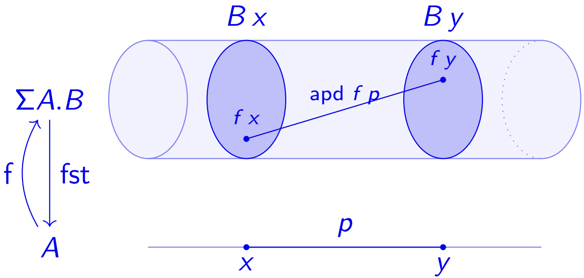
A better idea is to encode the path $\mathsf{apd}\ f\ p$ as a path in $B\ y$.

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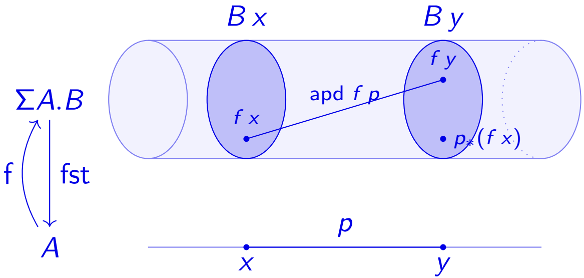


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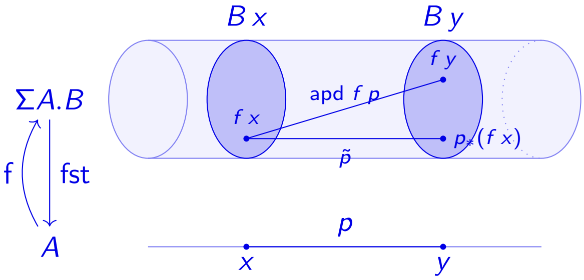


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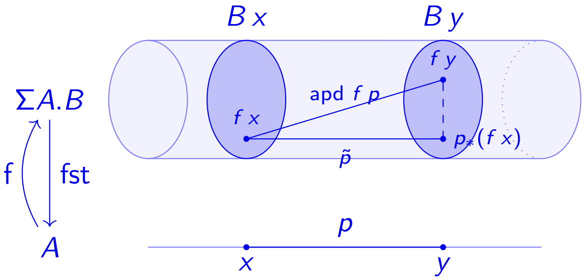


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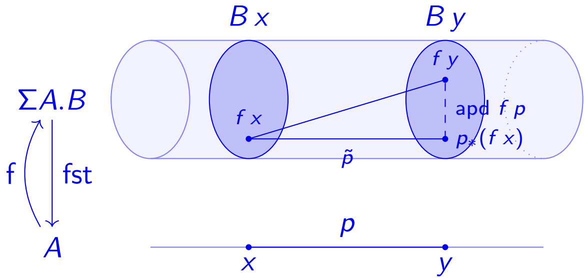


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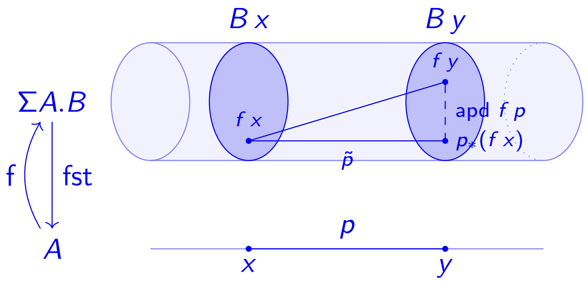


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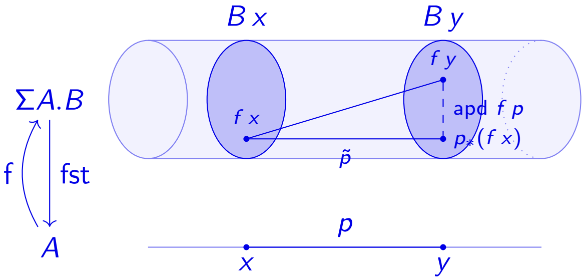


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by induction by

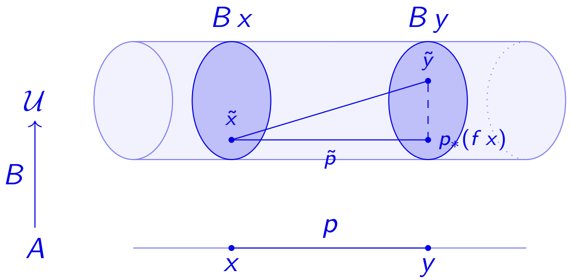
$$\mathsf{apd}\ f\ \mathsf{refl} \triangleq \mathsf{refl}_{f\ x}$$

# Paths over

More generally, given $x = y$ in $A$, $B : A \to \mathcal{U}$, $\tilde{x} : B\,x$ and $\tilde{y} : B\,y$, the type of **paths** in $B$ **over** $p$ between $\tilde{x}$ and $\tilde{y}$ is

$$\tilde{x} =^{B}_{p} \tilde{y} \quad \hat{=} \quad p_{*}\,\tilde{x} = \tilde{y}$$

which can be pictured as



# Paths in product types [Uni13, Section 2.6]

Suppose given $x\, x' : A$ and $y\, y' : B$.

A path $p : (x, y) = (x', y')$ induces paths

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Conversely, we have a function

$$\mathsf{pair}^{=} : (x = x') \to (y = y') \to (x, y) = (x', y')$$

which is defined by path induction and is useful to construct paths in products.

# Path in dependent sum types [Uni13, Section 2.7]

Suppose given $x\, x' : A$, $y : B\, x$ and $y' : B\, x'$.

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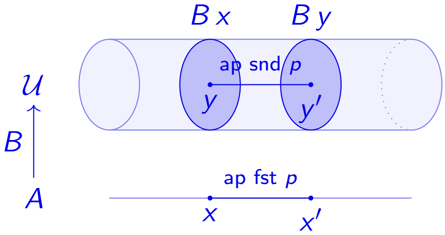

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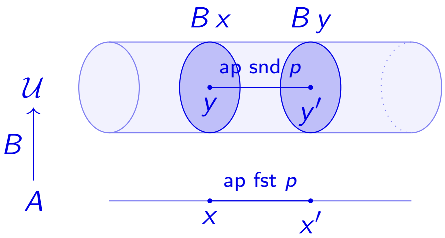

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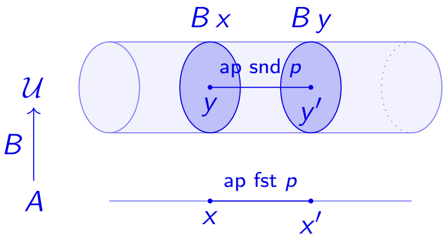

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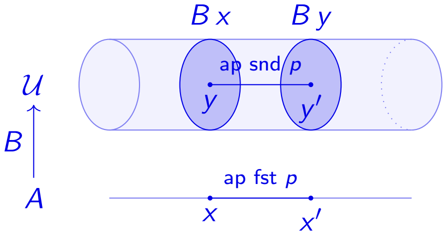

# Transporting paths

**Proposition ([Uni13, Lemma 2.11.2])**
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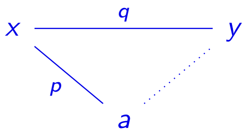

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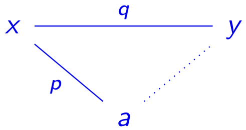


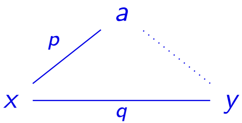

**Proposition ([Uni13, Lemma 2.11.2])**
*Given paths $p : a = x$ and $q : x = y$, we have*

$$\mathsf{transport}\,(\lambda x.a = x)\; q\; p = p \cdot q$$

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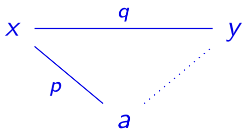


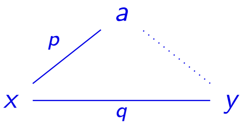


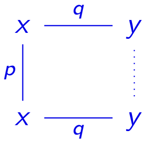


**Proof.**
By path induction on $p$. □

# Transporting paths

More generally,

**Proposition ([Uni13, Lemma 2.11.3])**
*Given* $f\, g : A \to B$, $p : f\, x = g\, x$ *in* $B$ *and* $q : x = y$ *in* $A$,

$$\mathsf{transport}\,(\lambda x. f\, x = g\, x)\, q\, p \quad = \quad (\mathsf{ap}\ f\ q)^{-} \cdot p \cdot \mathsf{ap}\ g\, q$$

*in* $f\, y = g\, y$.

# Transporting functions

Writing $\mathbb{N}$ for the unary natural number and $\mathbb{B}$ for the binary ones, we have

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# Transporting functions

**Proposition**
*Given a function* $f : A \to B$ *and paths* $p : A = A'$ *and* $q : B = B'$*, we have*

$$\text{transport}\,(\lambda X.A \to X)\,q\,f = \text{coe}\ q \circ f$$

$$\text{transport}\,(\lambda X.X \to B)\,p\,f = f \circ \text{coe}\ p^{-}$$

# Transporting functions

**Proposition ([Uni13, (2.9.4)])**
*Given type families $A : X \to \mathcal{U}$ and $B : X \to \mathcal{U}$, a path $p : x = y$ in $X$ and a function $f : A\,x \to B\,x$, we have*

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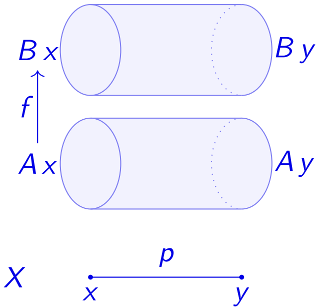

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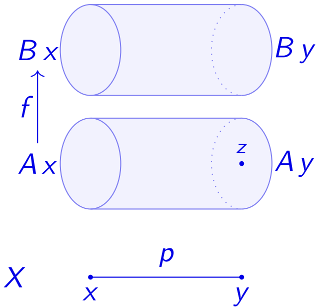

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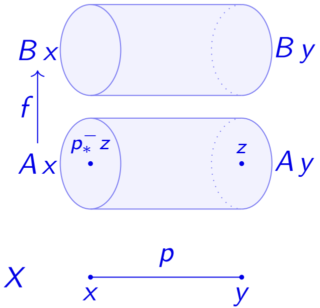

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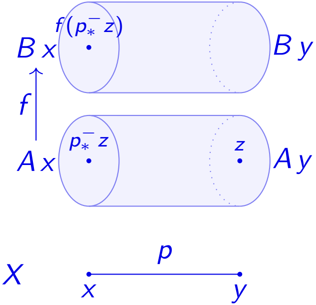

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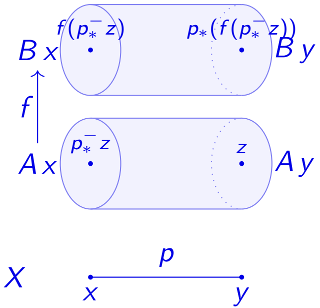

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