

Identity types

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École polytechnique

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of proofs that t is the same as u (equalities/identities/paths).

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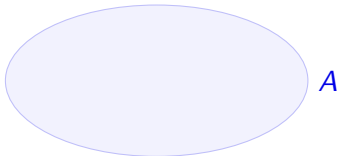
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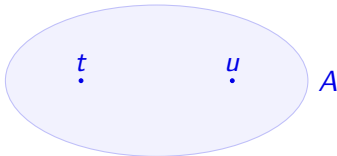
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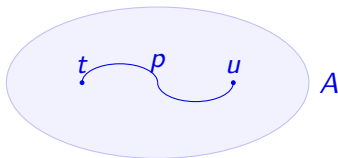
For $t, u : A$, we have a type

$$t = u$$

of proofs that t is the same as u (equalities/identities/paths).

On a semantic point of view,

- A corresponds to a *space*,
- t and u correspond to *points* in the space A ,
- $p : t = u$ corresponds to a *path* from t to u in A :



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Type former:

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$$- = - : (A : \mathcal{U}) \rightarrow A \rightarrow A \rightarrow \mathcal{U}$$

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Eliminator:

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For instance, we can prove that equality is symmetric, i.e. the property

$$\begin{array}{ccc} P : (y : A) \rightarrow x = y \rightarrow \mathcal{U} \\ y & p & \mapsto y = x \end{array}$$

for given $A : \mathcal{U}$ and $x : A$ by

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This corresponds to a pattern matching:

$\text{sym} : \{A : \text{Type}\} \{x\ y : A\} \rightarrow x \equiv y \rightarrow y \equiv x$

$\text{sym}\ p = ?$

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```
sym : {A : Type} {x y : A} → x ≡ y → y ≡ x
sym refl = refl
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Similarly, we can prove that `sym` is involutive, i.e. the property

$$\begin{array}{c} P : (y : A) \rightarrow x = y \rightarrow \mathcal{U} \\ y \qquad \qquad p \qquad \mapsto \text{sym}(\text{sym } p) = p \end{array}$$

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for given $A : \mathcal{U}$ and $x : A$ by

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namely, by the computation rule, we have

$$\text{sym}(\text{sym } \text{refl}) \hat{=} \text{refl}$$

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- uniqueness of identity proofs

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$$K : (P : (x = x) \rightarrow \mathcal{U}) \rightarrow P\ \text{refl} \rightarrow (p : x = x) \rightarrow P\ p$$

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All are equivalent (see lab).

Can they be proved?

At first, it seems that the question was settled by the uniqueness rule.

Uniqueness rule

The uniqueness rule for booleans states that a function

$$f : (b : \text{Bool}) \rightarrow A(b)$$

is entirely determined by the two values

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Similarly, we expect that a function

$$f : (y : A) \rightarrow (p : x = y) \rightarrow B(y, p)$$

is entirely determined by

$$f \ x \ \text{refl} : B(x, \text{refl})$$

Uniqueness rule

We consider the following **uniqueness rule**: given $x : A$ and

$$f : (y : A) \rightarrow (p : x = y) \rightarrow B(y, p)$$

we have

$$f \quad \hat{=} \quad J A x (f \times \text{refl})$$

This rule was actually present in Martin-Löf's original type system [MLS84].

It implies the equality reflection rule, which is problematic in some ways.

Equality reflection rule

The **equality reflection rule** states that from $a = b$ we can deduce $a \hat{=} b$.

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Proof.

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Thus,

$$x \hat{=} (\lambda y. \lambda p. x) y p \hat{=} (\lambda y. \lambda p. y) y p \hat{=} y$$



Equality reflection rule

Lemma

The equality reflection rule

$$\frac{\Gamma \vdash p : x = y}{\Gamma \vdash x \hat{=} y}$$

implies UIP.

Proof.

Consider the type

$$(x\ y : A) \rightarrow (p : x = y) \rightarrow (p = \text{refl})$$

Note that this is well-typed because of the equality reflection rule!

By **J**, in order to prove this, it is enough to prove it for $x \hat{=} y$ and $p \hat{=} \text{refl}$.

This can be done by **refl**.



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Proposition ([Hof95, Theorem 3.2.1])

The equality reflection rule

$$\frac{\Gamma \vdash p : x = y}{\Gamma \vdash x \hat{=} y}$$

makes typechecking undecidable.

Because of this, the uniqueness rule for identities is almost never considered.

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Don't think *we cannot prove what we want*, but rather *we have more models*!

The groupoid model

In the set semantics of logic we interpret:

- a type A as a set $\llbracket A \rrbracket$
- a term $t : A$ as an element of $\llbracket t \rrbracket \in \llbracket A \rrbracket$

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More generally, we interpret

- a dependent type $x : A \vdash B(x)$ as a function $\llbracket B \rrbracket : \llbracket A \rrbracket \rightarrow \mathbf{Set}$,
- a dependent term $x : A \vdash t : B(x)$ as a function $\llbracket t \rrbracket : (x : \llbracket A \rrbracket) \rightarrow \llbracket B \rrbracket(x)$

For instance, we interpret

$$n : \mathbb{N} \vdash \mathbf{Vec\ Bool}\ n$$

as the expected function $\mathbb{N} \rightarrow \mathbf{Set}$ and similarly for

$$n : \mathbb{N} \vdash t : \mathbf{Vec\ Bool}\ n$$

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For instance, we interpret

$$x : A, y : A \vdash x = y$$

as the function

$$\begin{aligned} &\llbracket A \rrbracket \times \llbracket A \rrbracket \rightarrow \mathbf{Set} \\ &(\tilde{x}, \tilde{y}) \mapsto \begin{cases} \{\star\} & \text{if } \tilde{x} = \tilde{y} \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

The groupoid model

In the model of Hoffman and Streicher,
we interpret types as **groupoids** instead of sets.

The groupoid model

A category $\mathcal{C}$ consists of

- a set $\mathcal{C}_0$ of objects,
- a set $\mathcal{C}(A, B)$ of morphisms for every objects A and B ,
- composition operations $\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$
- identities $\text{id}_A \in \mathcal{C}(A, A)$

such that composition is associative: for $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

and identities are neutral elements: for $f : A \rightarrow B$,

$$\text{id}_B \circ f = f = f \circ \text{id}_A$$

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For instance, we have

- the category **Set** of sets and functions,
- the category **Vect** of vector spaces and linear maps,
- the category **Top** of topological spaces and continuous maps,
- the category **Cat** of categories and functors,
- ...

The groupoid model

In a category $\mathcal{C}$, a morphism $f : A \rightarrow B$ is **invertible** when there exists a morphism $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

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In a category $\mathcal{C}$, a morphism $f : A \rightarrow B$ is **invertible** when there exists a morphism $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

A category is a **groupoid** when every morphism is invertible.

For instance,

- the category $\mathbf{Bij}$ of sets and bijections,
- the category $\mathbb{Z}$ with one object, $\mathbb{Z}$ as morphisms and composition given by addition,
- ...

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In particular, we interpret $x : A, y : A \vdash x = y$ as the functor

$$\llbracket A \rrbracket \times \llbracket A \rrbracket \rightarrow \mathbf{Gpd}$$

which to objects $x, y : \llbracket A \rrbracket$ associates groupoid whose objects are morphisms in $\llbracket A \rrbracket(x, y)$ and only trivial morphisms.

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which to objects $x, y : \llbracket A \rrbracket$ associates groupoid whose objects are morphisms in $\llbracket A \rrbracket(x, y)$ and only trivial morphisms.

By taking $\llbracket A \rrbracket$ a groupoid such as $\mathbf{Bij}$, we see that our model does not validate UIP!

(note: we are leaving out lots of details, see [HS98])

The space model

We have a more general semantics here where we interpret

- a type A as a **space** $\llbracket A \rrbracket$,
- a term $t : A$ as a point $\llbracket t \rrbracket : A$.

More generally, we interpret

- a type $\Gamma \vdash A$ as a **fibration** $\llbracket \Gamma \vdash A \rrbracket : \llbracket A \rrbracket \rightarrow \llbracket \Gamma \rrbracket$,
- a term $\Gamma \vdash t : A$ as a section $\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$ of the fibration $\llbracket \Gamma \vdash A \rrbracket$.

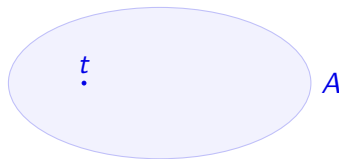
Setting this up precisely is quite subtle [Rie24]:

- MLTT with identity types can be modeled in **model categories** [GG08, AW09],
- Voevodsky constructed a model of HoTT in **simplicial sets** [KL21],
- Shulman extended this to ∞ -**toposes** [Shu19].

The space model

In the model, we interpret

- a type A as a **space** (think: topological space),
- a term $t : A$ as a **point** of A .

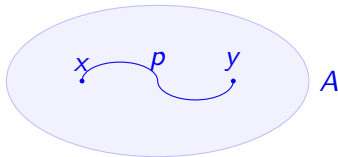


The space model: paths

A path p in a space A is a continuous map

$$p : I \rightarrow A$$

with $I = [0, 1]$. The points $p(0)$ and $p(1)$ are its *source* and *target*.



The space model: paths

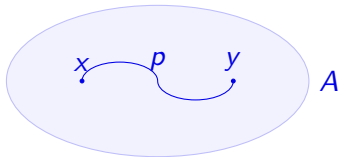
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Given $x, y : A$, we interpret the type $x = y : A$ as the space of paths from x to y in A .

We thus interpret a term $p : x = y$ as a path from x to y .



The space model: paths

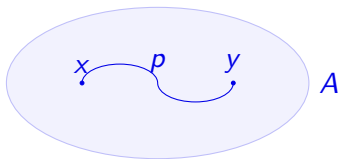
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In particular, we interpret $\text{refl}_x : x = x$ as the constant path $p : I \rightarrow A$ with $p(t) \hat{=} x$.

The space model: paths between paths

Given points $x, y : A$ and paths $p, q : x = y$, we can consider the type $p = q$.

Its points are **homotopies** between p and q , i.e. continuous deformations from p to q , i.e. maps

$$\alpha : I \rightarrow I \rightarrow A$$

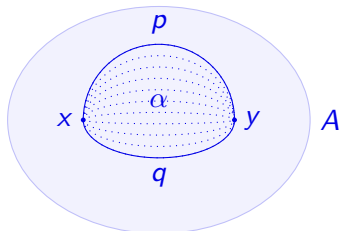
such that

$$\alpha 0 \hat{=} p$$

$$\alpha 1 \hat{=} q$$

$$\alpha t 0 \hat{=} x$$

$$\alpha t 1 \hat{=} y$$



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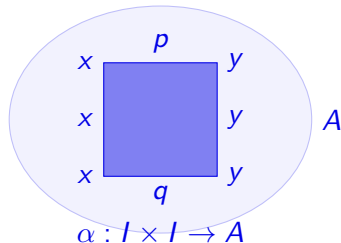
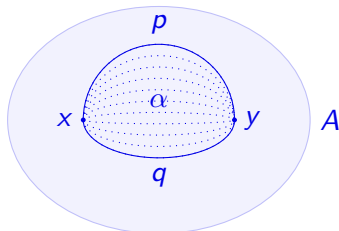
such that

$$\alpha 0 \hat{=} p$$

$$\alpha 1 \hat{=} q$$

$$\alpha t 0 \hat{=} x$$

$$\alpha t 1 \hat{=} y$$



The space model: paths between paths between paths

Given $x, y : A$, $p, q : x = y$, $\alpha, \beta : p = q$, a path $\Psi : \alpha = \beta$ is a homotopy between homotopies, i.e. a map

$$I \rightarrow I \rightarrow I \rightarrow A$$

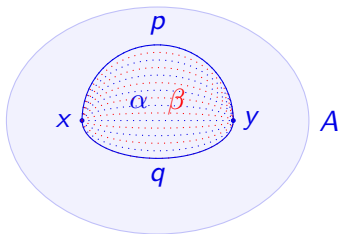
such that

$$\Psi 0 \triangleq \alpha$$

$$\Psi 1 \triangleq \beta$$

$$\Psi t 0 \triangleq p$$

$$\Psi t 1 \triangleq q$$



and so on...

The space model: path types

Let's have a look at some path types.

- In 1, we have

$$(\star = \star) = ?$$

The space model: path types

Let's have a look at some path types.

- In 1, we have

$$(\star = \star) = 1$$

The space model: path types

Let's have a look at some path types.

- In $\mathbb{1}$, we have

$$(\star = \star) = 1$$

- In $\mathbb{N}$, we have

$$(m = n) = ?$$

The space model: path types

Let's have a look at some path types.

- In $\mathbf{1}$, we have

$$(\star = \star) = 1$$

- In $\mathbf{N}$, we have

$$(m = n) = \begin{cases} 1 & \text{if } m \hat{=} n \\ 0 & \text{otherwise} \end{cases}$$

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we have

$$(x = y) \simeq 1$$

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- In $\mathbf{I}$



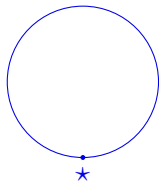
we have

$$(x = y) = \text{blue circle} \simeq 1$$

The space model: path types

Let's have a look at some path types.

- In S^1 , we have

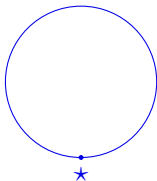


$$(\star = \star) = ?$$

The space model: path types

Let's have a look at some path types.

- In S^1 , we have

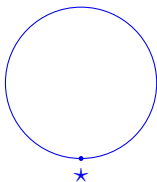


$$(\star = \star) \simeq \mathbb{Z}$$

The space model: path types

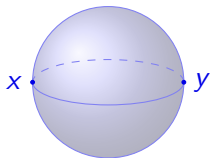
Let's have a look at some path types.

- In S^1 , we have



$$(\star = \star) \simeq \mathbb{Z}$$

- In S^2 , we have

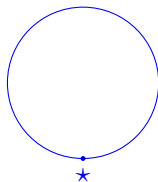


$$(x = y) = ?$$

The space model: path types

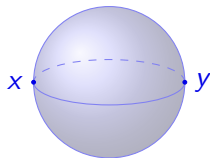
Let's have a look at some path types.

- In S^1 , we have



$$(\star = \star) \simeq \mathbb{Z}$$

- In S^2 , we have



$$(x = y) = ? \neq S^1$$

The space model: $\mathcal{J}$

The axiom

$$\begin{aligned} \mathcal{J} : (A : \mathcal{U}) \rightarrow (x : A) \rightarrow (P : (y : A) \rightarrow x = y \rightarrow \mathcal{U}) \rightarrow \\ P\ x\ (\text{refl}\ x) \rightarrow \\ (y : A) \rightarrow (p : x = y) \rightarrow P\ y\ p \end{aligned}$$

states that in order to prove a property P on a path p , we can always suppose that p is refl *provided that y is a generic point.*

The space model: $\mathcal{J}$

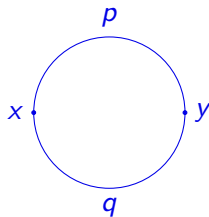
The axiom

$$\begin{aligned} \mathcal{J} : (A : \mathcal{U}) \rightarrow (x : A) \rightarrow (P : (y : A) \rightarrow x = y \rightarrow \mathcal{U}) \rightarrow \\ P x (\text{refl } x) \rightarrow \\ (y : A) \rightarrow (p : x = y) \rightarrow P y p \end{aligned}$$

states that in order to prove a property P on a path p , we can always suppose that p is *refl* provided that y is a generic point.

For instance, suppose that we want to prove UIP:

$$(x : A) \rightarrow (y : A) \rightarrow (p : x = y) \rightarrow (q : x = y) \rightarrow p = q$$



The space model: $\mathcal{J}$

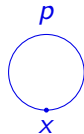
The axiom

$$\begin{aligned} \mathcal{J} : (A : \mathcal{U}) \rightarrow (x : A) \rightarrow (P : (y : A) \rightarrow x = y \rightarrow \mathcal{U}) \rightarrow \\ P\ x\ (\text{refl}\ x) \rightarrow \\ (y : A) \rightarrow (p : x = y) \rightarrow P\ y\ p \end{aligned}$$

states that in order to prove a property P on a path p , we can always suppose that p is *refl* provided that y is a generic point.

For instance, suppose that we want to prove UIP:

$$(x : A) \rightarrow (p : x = x) \rightarrow p = \text{refl}$$



which is URP. We cannot use $\mathcal{J}$ anymore!

The space model: J

In Agda, we can do this by using pattern matching. With

```
UIP : {A : Type} {x y : A} (p q : x ≡ y) → p ≡ q
```

```
UIP p q = ?
```

we have to prove

```
Goal: p ≡ q
```

```
q : x ≡ y
```

```
p : x ≡ y
```

```
y : A    (not in scope)
```

```
x : A    (not in scope)
```

```
A : Type  (not in scope)
```

The space model: J

In Agda, we can do this by using pattern matching. With

```
UIP : {A : Type} {x y : A} (p q : x ≡ y) → p ≡ q
```

```
UIP p refl = ?
```

we have to prove

```
Goal: p ≡ refl
```

```
p : x ≡ x
```

```
x : A    (not in scope)
```

```
A : Type  (not in scope)
```

The space model: J

In Agda, we can do this by using pattern matching. With

```
UIP : {A : Type} {x y : A} (p q : x ≡ y) → p ≡ q
```

```
UIP refl refl = ?
```

we have to prove

```
Goal: refl ≡ refl
```

```
x : A    (not in scope)
```

```
A : Type (not in scope)
```

The space model: J

In Agda, we can do this by using pattern matching. With

```
UIP : {A : Type} {x y : A} (p q : x ≡ y) → p ≡ q
```

```
UIP refl refl = refl
```

we have to prove

The space model: J

In Agda, we can do this by using pattern matching. With

```
{-# OPTIONS --without-K #-}  
UIP : {A : Type} {x y : A} (p q : x ≡ y) → p ≡ q  
UIP p refl = ?
```

we have to prove

```
error: [SplitError.UnificationStuck]
```

I'm not sure if there should be a case for the constructor refl, because I get stuck when trying to solve the following unification problems (inferred index [?] = expected index):

$$x_1 \overset{?}{=} x_1$$

Possible reason why unification failed:

Cannot eliminate reflexive equation $x_1 = x_1$ of type A_1 because K has been disabled.

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