Graphs and the Yoneda lemma

Samuel Mimram
samuel.mimram@lix.polytechnique.fr
http://lambdacat.mimram.fr

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I Graphs as presheaf categories

By a *graph*, we always mean here a directed multigraph (i.e. edges have a direction, there can be multiple edges with the same source and the same target, there can be edges from a vertex to the same vertex, and the sets of vertices and edges can be infinite).

1. Give a proper definition of the notion of graph considered here.
2. Define a category $\mathbf{G}$ such that functors $\mathbf{G}^{\mathrm{op}} \to \mathbf{Set}$ are in bijection with graphs.
3. Show that natural transformations between functors $\mathbf{G}^{\mathrm{op}} \to \mathbf{Set}$ correspond to morphisms of graphs.

Given a category $\mathcal{C}$, the category $\hat{\mathcal{C}}$ of *presheaves* over $\mathcal{C}$ is the category whose objects are functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and morphisms are natural transformations.

II The Yoneda lemma

1. Define a graph Y_0 such that given a graph G, the vertices of G are in bijection with graph morphisms from Y_0 to G. Similarly, define a graph Y_1 such that we have a bijection between edges of G and graph morphisms from Y_1 to G.
2. Given a category $\mathcal{C}$, we define the *Yoneda functor* $Y : \mathcal{C} \to \hat{\mathcal{C}}$ by $YAB = \mathcal{C}(B, A)$ for objects $A, B \in \mathcal{C}$. Complete the definition of Y.
3. In the case of $\mathcal{C} = \mathbf{G}$, what are the graphs obtained as the image of the two objects?

A presheaf of the form YA for some object A is called a *representable* presheaf.

4. *Yoneda lemma:* show that for any category $\mathcal{C}$, presheaf $P \in \hat{\mathcal{C}}$, and object $A \in \mathcal{C}$, we have $PA \cong \hat{\mathcal{C}}(YA, P)$ [hint: first define a function $\hat{\mathcal{C}}(YA, P) \to PA$].
5. Show that the Yoneda functor $Y : \mathcal{C} \to \hat{\mathcal{C}}$ is full and faithful.
6. Show that the category of graphs has finite products.
7. Show that any presheaf category is cartesian closed. Describe this structure in the case of the category of graphs.

III 2-graphs

1. Explain that a category is a graph with additional structure.
2. Define a notion of 2-graph such that a 2-category is a 2-graph with additional structure. Express this notion as a presheaf category $\hat{\mathbf{G}}'$.
3. Show that any functor $F : \mathcal{C} \to \mathcal{D}$ canonically induces a functor $\hat{F} : \hat{\mathcal{D}} \to \hat{\mathcal{C}}$.
4. Describe the "inclusion" functor $\hat{I} : \hat{\mathbf{G}}' \to \hat{\mathbf{G}}$.
5. Show that this function $\hat{I}$ admits both a left and a right adjoint.

IV The simplicial category

The *presimplicial category* Δ_+ is the category whose objects are the sets

$$[n] = \{0, 1, \ldots, n\}$$

for $n \in \mathbb{N}$ and morphisms are increasing injective functions.

1. What is the full subcategory of Δ_+ on the objects [0] and [1] and what are presheaves over it?
2. What is the full subcategory of Δ_+ on the objects [0], [1] and [2] and what are presheaves over it?
3. For each $n \in \mathbb{N}$, define a "small" family of functions $s_i^n : [n] \to [n+1]$ (indexed by i) such that every morphism of Δ_+ can be obtained as a composite of such morphisms.
4. Give relations satisfied by the above generating functions.
5. Describe the category of presheaves over Δ_+.

The *simplicial category* Δ is the category with the sets $[n]$ as objects for $n \in \mathbb{N}$ and morphisms are weakly increasing functions.

6. Describe the presheaves over Δ.