Graphs and the Yoneda lemma

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I Graphs as presheaf categories

By a *graph*, we always mean here a directed multigraph (i.e. edges have a direction, there can be multiple edges with the same source and the same target, there can be edges from a vertex to the same vertex, and the sets of vertices and edges can be infinite).

1. Give a proper definition of the notion of graph considered here.

   Solution. A graph is a diagram

$$\begin{tikzcd} V & E \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \end{tikzcd}$$

   in **Set**.

2. Define a category $\mathbf{G}$ such that functors $\mathbf{G}^{\text{op}} \to \mathbf{Set}$ are in bijection with graphs.

   Solution. We define $\mathbf{G}$ as the category with two objects 0, 1 and four morphisms

$$\mathrm{id}_0 : 0 \to 0 \qquad s, t : 0 \to 1 \qquad \mathrm{id}_1 : 1 \to 1$$

   A morphism $G : \mathbf{G}^{\text{op}} \to \mathbf{Set}$ is determined by

   - a set $G0$ of vertices,
   - a set $G1$ of edges,
   - two functions $Gs : G1 \to G0$ and $Gt : G1 \to G0$ associating to each edge its source and target.

3. Show that natural transformations between functors $\mathbf{G}^{\text{op}} \to \mathbf{Set}$ correspond to morphisms of graphs.

   Solution. Given two graphs $G, H : \mathbf{G}^{\text{op}} \to \mathbf{Set}$, a natural transformation ϕ consists of two functions

   - $\phi_0 : G0 \to H0$ between vertices and
   - $\phi_1 : G1 \to H1$ between edges

   such that the two diagrams

$$\begin{tikzcd} G1 \arrow[r, "\phi_1"] \arrow[d, "Gs"'] & H1 \arrow[d, "Hs"] \\ G0 \arrow[r, "\phi_0"'] & H0 \end{tikzcd} \qquad \begin{tikzcd} G1 \arrow[r, "\phi_1"] \arrow[d, "Gt"'] & H1 \arrow[d, "Ht"] \\ G0 \arrow[r, "\phi_0"'] & H0 \end{tikzcd}$$

   commute, i.e. the image of the source of an edge is the source of its image and similarly for targets.

Given a category $\mathcal{C}$, the category $\hat{\mathcal{C}}$ of *presheaves* over $\mathcal{C}$ is the category whose objects are functors $\mathcal{C}^{\text{op}} \to \mathbf{Set}$ and morphisms are natural transformations.

II The Yoneda lemma

1. Define a graph Y_0 such that given a graph G, the vertices of G are in bijection with graph morphisms from Y_0 to G. Similarly, define a graph Y_1 such that we have a bijection between edges of G and graph morphisms from Y_1 to G.

   Solution. We define the graphs

   $$Y_0 = \quad \cdot \qquad\qquad\qquad\qquad Y_1 = \quad \cdot \longrightarrow \cdot$$

2. Given a category $\mathcal{C}$, we define the *Yoneda functor* $Y : \mathcal{C} \to \hat{\mathcal{C}}$ by $YAB = \mathcal{C}(B, A)$ for objects $A, B \in \mathcal{C}$. Complete the definition of Y.

   Solution. Suppose fixed A in $\mathcal{C}$. The functor $YA : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is defined

   - on an object B by $YAB = \mathcal{C}(B, A)$,
   - on a morphism $f : B \to B'$ of $\mathcal{C}^{\mathrm{op}}$ (i.e. $f : B' \to B$ in $\mathcal{C}$) by

   $$\begin{aligned} YAf : YAB &\to YAB' \\ \mathcal{C}(B, A) &\to \mathcal{C}(B', A) \\ g &\mapsto g \circ f \end{aligned}$$

   which is sometimes written

   $$\mathcal{C}(f, A) : \mathcal{C}(B, A) \to \mathcal{C}(B', A)$$

   Suppose given $h : A \to A'$ in $\mathcal{C}$. We define the natural transformation $Yh : YA \to YA'$ whose components are

   $$\begin{aligned} (Yh)_B : YAB &\to YA'B \\ \mathcal{C}(B, A) &\to \mathcal{C}(B, A') \\ g &\mapsto h \circ g \end{aligned}$$

   which is sometimes written

   $$\mathcal{C}(A, h) : \mathcal{C}(B, A) \to \mathcal{C}(B, A')$$

   The naturality condition is that for every morphism $f : B \to B'$ in $\mathcal{C}^{\mathrm{op}}$, we have the commutation of

   $$\begin{tikzcd} YAB \arrow[r, "(Yh)_B"] \arrow[d, "YAf"'] & YA'B \arrow[d, "YA'f"] \\ YAB' \arrow[r, "(Yh)_{B'}"'] & YA'B' \end{tikzcd}$$

   i.e. of

   $$\begin{tikzcd} \mathcal{C}(B, A) \arrow[r, "\mathcal{C}(B,h)"] \arrow[d, "\mathcal{C}(f,A)"'] & \mathcal{C}(B, A') \arrow[d, "\mathcal{C}(f,A')"] \\ \mathcal{C}(B', A) \arrow[r, "\mathcal{C}(B',h)"'] & \mathcal{C}(B', A') \end{tikzcd}$$

   i.e. for every $g \in YAB = \mathcal{C}(B, A)$,

   $$h \circ (g \circ f) = (h \circ g) \circ f$$

   which follows from the associativity of composition in $\mathcal{C}$.

3. In the case of $\mathcal{C} = \mathbf{G}$, what are the graphs obtained as the image of the two objects?

   Solution. For 0, we have

$$Y00 = \mathbf{G}(0,0) = \{\mathrm{id}_0\} \qquad\qquad Y01 = \mathbf{G}(1,0) = \emptyset$$

   and the maps $Y0s : Y01 \to Y00$ and $Y0t : Y01 \to Y00$ are both the initial map since $Y01 = \emptyset$. The graph $Y0$ is thus

$$Y0 = \ \mathrm{id}_0$$

   For 1, we have

$$Y10 = \mathbf{G}(0,1) = \{s,t\} \qquad\qquad Y11 = \mathbf{G}(1,1) = \{\mathrm{id}_1\}$$

   and we have the maps

$$\begin{aligned} Y1s : Y11 &\to Y10 \\ \mathrm{id}_1 &\mapsto \mathrm{id}_1 \circ s = s \end{aligned} \qquad\qquad \begin{aligned} Y1t : Y11 &\to Y10 \\ \mathrm{id}_1 &\mapsto \mathrm{id}_1 \circ t = t \end{aligned}$$

   The graph $Y1$ is thus

$$s \xrightarrow{\mathrm{id}_1} t$$

A presheaf of the form YA for some object A is called a *representable* presheaf.

4. *Yoneda lemma:* show that for any category $\mathcal{C}$, presheaf $P \in \hat{\mathcal{C}}$, and object $A \in \mathcal{C}$, we have $PA \cong \hat{\mathcal{C}}(YA, P)$ [hint: first define a function $\hat{\mathcal{C}}(YA, P) \to PA$].

   Solution. We define a function by

$$\begin{aligned} \Phi : \hat{\mathcal{C}}(YA, P) &\to PA \\ \phi &\mapsto \phi_A(\mathrm{id}_A) \end{aligned}$$

   Namely, given $\phi \in \hat{\mathcal{C}}(YA, P)$, i.e. natural transformation $YA \Rightarrow P$, the component at A, $\phi_A : YAA \to PA$ is a function $\mathcal{C}(A, A) \to PA$. The naturality condition ensures that ϕ is entirely determined by $\phi_A(\mathrm{id}_A) \in PA$ since, for any $f : A \to B$ in $\mathcal{C}^{\mathrm{op}}$, we have

$$\begin{tikzcd} YAA \arrow[r, "\phi_A"] \arrow[d, "YAf"'] & PA \arrow[d, "Pf"] \\ YAB \arrow[r, "\phi_B"'] & PB \end{tikzcd}$$

   i.e.

$$\begin{tikzcd} \mathcal{C}(A,A) \arrow[r, "\phi_A"] \arrow[d, "YAf"'] & PA \arrow[d, "Pf"] \\ \mathcal{C}(B,A) \arrow[r, "\phi_B"'] & PB \end{tikzcd}$$

   i.e. for every $g : A \to A$,

$$\phi_B \circ (g \circ f) = Pf(\phi_A(g))$$

   and in particular, for $g = \mathrm{id}_A$,

$$\phi_B(f) = Pf(\phi_A(\mathrm{id}_A))$$

   This shows that Φ is injective. Conversely, every element $u \in PA$ defines a natural transformation by

$$\phi_B(f) = Pf(u)$$

for $B \in \mathcal{C}^{\mathrm{op}}$ and $f : A \to B$ in $\mathcal{C}^{\mathrm{op}}$. It is indeed natural since for $f : B \to B'$ in $\mathcal{C}^{\mathrm{op}}$, we have the commutation of

$$\begin{tikzcd} YAB \arrow[r, "\phi_B"] \arrow[d, "YAf"'] & PB \arrow[d, "Pf"] \\ YAB' \arrow[r, "\phi_{B'}"'] & PB \end{tikzcd}$$

i.e.

$$\begin{tikzcd} \mathcal{C}(B,A) \arrow[r, "\phi_B"] \arrow[d, "\mathcal{C}(f,A)"'] & PB \arrow[d, "Pf"] \\ \mathcal{C}(B',A) \arrow[r, "\phi_{B'}"'] & PB \end{tikzcd}$$

since for $g \in \mathcal{C}(B, A)$, we have

$$\phi_{B'}(g \circ f) = P(g \circ f)u = Pf(Pg(u)) = Pf(\phi_B(g))$$

by functoriality of P. This shows that ϕ is surjective.

5. Show that the Yoneda functor $Y : \mathcal{C} \to \hat{\mathcal{C}}$ is full and faithful.

   Solution. Given A, B in $\mathcal{C}$, we have by the Yoneda lemma

   $$\hat{\mathcal{C}}(YA, YB) \simeq YBA = \mathcal{C}(A, B)$$

   which shows that Y is full and faithful.

6. Show that the category of graphs has finite products.

   Solution. Given graphs G and H, their product $G \times H$ is the graph whose

   - vertices are pairs (x, y) of a vertex x of G and y of H and
   - edges are pairs $(e, f) : (x, y) \to (x', y')$ consisting of an edge $e : x \to x'$ of G and an edge $f : y \to y'$ of H.

   The terminal graph has one vertex and one edge.

7. Show that any presheaf category is cartesian closed. Describe this structure in the case of the category of graphs.

   Solution. Suppose given a category $\mathcal{C}$. It is not very difficult to see that the category $\hat{\mathcal{C}}$ is cartesian with, for $P, Q \in \hat{\mathcal{C}}$ and $A \in \mathcal{C}^{\mathrm{op}}$,

   $$(P \times Q)A = PA \times QA$$

   (and similarly on morphisms) and the terminal presheaf is the constant presheaf

   $$\begin{aligned} P : \mathcal{C}^{\mathrm{op}} &\to \mathbf{Set} \\ A &\mapsto \{\star\} \end{aligned}$$

   For the closure things are much less obvious: the first guess is to define $(P \Rightarrow Q)A$ as the set of functions $PA \to QA$ but there is no reasonable way to extend this definition of P on morphisms, i.e. define Pf.

   Suppose that $\hat{\mathcal{C}}$ is closed and write $P \Rightarrow Q$ for the exponential object. By the Yoneda lemma, we should have

   $$(P \Rightarrow Q)A \simeq \hat{\mathcal{C}}(YA, P \Rightarrow Q) \simeq \hat{\mathcal{C}}(YA \times P, Q)$$

   Conversely, we can define

   $$P \Rightarrow Q = \hat{\mathcal{C}}(Y - \times P, Q)$$

   and check that this induces a closure.

   For graphs, given graphs G and H, $G \Rightarrow H$ is the graph whose

- vertices are graph morphisms $Y0 \times P \to Q$: $Y0 \times P$ is the graph with the vertices of P and no edge and thus vertices are functions $f : G0 \to H0$ from the vertices of G to those of H,
- edges are graph morphisms $Y1 \times P \to Q$: $Y1 \times P$ is the graph whose vertices are of the form x^- or x^+ where x is a vertex of P and edges are $e : x^- \to y^+$ where $e : x \to y$ is an edge of P, an edge in $G \Rightarrow H$ is thus a function from the edges of G to the edges of H such that two edges with the same are sent to edges with the same source and two edges with the same target are sent to edges with the same target.

III 2-graphs

1. Explain that a category is a graph with additional structure.

   Solution. A category is a graph (its vertices are the objects and the edges are morphisms) together with distinguished identity morphisms and a composition function, satisfying axioms.

2. Define a notion of 2-graph such that a 2-category is a 2-graph with additional structure. Express this notion as a presheaf category $\hat{\mathbf{G}}'$.

   Solution. We take presheaves on the category

$$\begin{tikzcd} G_0 & G_1 \arrow[l, shift right, "s_0"'] \arrow[l, shift left, "t_0"] & G_2 \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"] \end{tikzcd}$$

   such that

$$s_0 \circ s_1 = s_0 \circ t_1 \qquad\qquad t_0 \circ s_1 = t_0 \circ t_1$$

3. Show that any functor $F : \mathcal{C} \to \mathcal{D}$ canonically induces a functor $\hat{F} : \hat{\mathcal{D}} \to \hat{\mathcal{C}}$.

   Solution. The functor $\hat{F}$ sends $P \in \hat{\mathcal{D}}$, i.e. $P : \mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$, to $P \circ F^{\mathrm{op}}$. Similarly, a natural transformation $\phi : P \Rightarrow Q$, is sent to $\phi_{F^{\mathrm{op}}}$:

$$\begin{tikzcd} & \mathbf{Set} \\ \mathcal{C}^{\mathrm{op}} \arrow[r, "F^{\mathrm{op}}"'] & \mathcal{D}^{\mathrm{op}} \arrow[u, bend left, "P"] \arrow[u, bend right, "Q"'] \arrow[u, phantom, "\overset{\phi}{\Rightarrow}"] \end{tikzcd}$$

4. Describe the "inclusion" functor $\hat{I} : \hat{\mathbf{G}}' \to \hat{\mathbf{G}}$.

5. Show that this function $\hat{I}$ admits both a left and a right adjoint.

   Solution. The left adjoint sends a graph to the corresponding 2-graph with no 2-cell. The right adjoint constructs a 2-graph from a graph by adding one 2-cell between every pair of parallel 1-cells.

IV The simplicial category

The *presimplicial category* Δ_+ is the category whose objects are the sets

$$[n] = \{0, 1, \dots, n\}$$

for $n \in \mathbb{N}$ and morphisms are increasing injective functions.

1. What is the full subcategory of Δ_+ on the objects [0] and [1] and what are presheaves over it?
2. What is the full subcategory of Δ_+ on the objects [0], [1] and [2] and what are presheaves over it?
3. For each $n \in \mathbb{N}$, define a "small" family of functions $s_i^n : [n] \to [n+1]$ (indexed by i) such that every morphism of Δ_+ can be obtained as a composite of such morphisms.
4. Give relations satisfied by the above generating functions.
5. Describe the category of presheaves over Δ_+.

The *simplicial category* Δ is the category with the sets $[n]$ as objects for $n \in \mathbb{N}$ and morphisms are weakly increasing functions.

6. Describe the presheaves over Δ.