Realizability

Samuel Mimram

7 November 2022

Second-order logic. We write $\mathcal{T}$ for the set of terms over some fixed signature. The syntax of second-order formulas is

$$A \qquad ::= \qquad X(a_1,\ldots,a_n) \quad | \quad A \Rightarrow B \quad | \quad \forall x.A \quad | \quad \forall X.A$$

where $a_i \in \mathcal{T}$, each second-order variable X having a fixed arity n. We consider typing rules which extend those of simply-typed λ-calculus by

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t : \forall x.A} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash t : \forall X.A} \qquad \frac{\Gamma \vdash t : \forall x.A}{\Gamma \vdash t : A[a/x]} \qquad \frac{\Gamma \vdash t : \forall X.A}{\Gamma \vdash t : A[B/X]}$$

where, in the first two rules, we suppose x and X not free in Γ respectively. Note that we have three kinds of variables in $\Gamma \vdash t : A$: those declared in Γ and occurring in t (standing for λ-terms), those occurring in formulas (standing for terms in $\mathcal{T}$) and second-order variables (standing for formulas).

1. Show that identity can be given the type $\forall X.X \Rightarrow X$.

   Solution.

$$\cfrac{\cfrac{\cfrac{}{x : X \vdash x : X}}{\vdash \lambda x.x \vdash X \Rightarrow X}}{\vdash \lambda x.x \vdash \forall X.X \Rightarrow X}$$

2. Recall the elimination rule for $\vee$. How can we encode this operator into our logic?

   Solution. The elimination rule for disjunction is

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash X \qquad \Gamma, B \vdash X}{\Gamma \vdash X}$$

   This suggests that we define, with X of arity 0,

$$A \vee B = \forall X.(A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X$$

3. Similarly, provide an encoding of the operators $\wedge$, $\bot$, $\neg$, first and second order existential quantifications.

Solution. We define similarly:

$$
\begin{aligned}
A \wedge B &= \forall X.(A \Rightarrow B \Rightarrow X) \Rightarrow X \\
\bot &= \forall X.X \\
\neg A &= A \Rightarrow \bot \\
\exists x.A &= \forall X.(\forall x.(A(x) \Rightarrow X)) \Rightarrow X \\
\exists X.A &= \forall Y.(\forall X.(A(X) \Rightarrow Y)) \Rightarrow Y
\end{aligned}
$$

Realizability. We write Λ for the set of *λ-terms* and Π for the set of *stacks*, which are sequences $t_1 \cdot t_2 \cdots t_n$ of λ-terms. *Processes* are elements (t, π) of $\Lambda \times \Pi$, often written $t \star \pi$. The *reduction* relation $\succ$ between processes is given by the following two rules:

$$
\begin{gathered}
tu \star \pi \succ t \star u \cdot \pi \\
\lambda x.t \star u \cdot \pi \succ t[u/x] \star \pi
\end{gathered}
$$

An element of $\mathcal{P}(\Pi)$ is called a *truth value*. Suppose fixed a set $\perp\!\!\!\perp$ of processes closed under anti-reduction. We define an interpretation $[\![A]\!] \in \mathcal{P}(\Pi)$ by induction on the formula A by

$$
[\![A \Rightarrow B]\!] = \{t \cdot \pi \mid t \in |A|, \pi \in [\![B]\!]\} \qquad [\![\forall x.A]\!] = \bigcup_{a \in \mathcal{T}} [\![A[a/x]]\!] \qquad [\![\forall X.A]\!] = \bigcup_{V \in \mathcal{P}(\Pi)} [\![A[V/X]]\!]
$$

where

$$
|A| = \{t \in \Lambda \mid \forall \pi \in [\![A]\!], t \star \pi \in \perp\!\!\!\perp\}
$$

denotes the set of *realizers* of the formula A. Above, we have supposed fixed an interpretation of the first- and second-order free variables (by abuse of notation, given $V \in \mathcal{P}(\Pi)$, we still write V for a variable whose interpretation is V). We write $t \Vdash A$ when $t \in |A|$ and say that t *realizes* A.

4. What are $[\![\bot]\!]$ and $|\bot|$?

   Solution.

$$
[\![\bot]\!] = [\![\forall X.X]\!] = \bigcup_{V \in \mathcal{P}(\Pi)} V = \Pi \qquad\qquad |\bot| = \{t \in \Lambda \mid \forall \pi \in \Pi, t \star \pi \in \perp\!\!\!\perp\}
$$

5. Show that $|\forall X.A| = \bigcap_{V \in \mathcal{P}(\Pi)} |A[V/X]|$.

   Solution. We have

$$
\begin{aligned}
|\forall X.A| &= \{t \in \Lambda \mid \forall \pi \in [\![\forall X.A]\!], t \star \pi \in \perp\!\!\!\perp\} \\
&= \{t \in \Lambda \mid \forall \pi \in \bigcup_{V \in \mathcal{P}(\Pi)} [\![A[V/X]]\!], t \star \pi \in \perp\!\!\!\perp\} \\
&= \bigcap_{V \in \mathcal{P}(\Pi)} \{t \in \Lambda \mid \forall \pi \in [\![A[V/X]]\!], t \star \pi \in \perp\!\!\!\perp\} \\
&= \bigcap_{V \in \mathcal{P}(\Pi)} |A[V/X]|
\end{aligned}
$$

Identity-like terms. Our goal is now to characterize the behavior of terms of type $\forall X.X \Rightarrow X$.

6. Give examples of terms which are of type $\forall X.X \Rightarrow X$.

   Solution. We have the identity $\lambda x.x$, but also terms which reduce to the identity such as $(\lambda x.x)(\lambda x.x)$.

7. Show that $(\lambda x.x) \star u \cdot \pi \succ u \star \pi$.

   Solution. We have $(\lambda x.x) \star u \cdot \pi \succ x[u/x] \star \pi = u \star \pi$.

A term $t \in \Lambda$ is *identity-like* when $t \star u \cdot \pi \succ u \star \pi$ for every $u \in \Lambda$ and $\pi \in \Pi$.

8. Show that if t is identity-like then $t \Vdash \forall X.X \Rightarrow X$.

   Solution. Given an element $u \cdot \pi$ of

   $$[\![\forall X.X \Rightarrow X]\!] = \bigcup_{V \in \mathcal{P}(\Pi)} |V| \cdot V = \{u \cdot \pi \mid u \in |V|, \pi \in V, V \in \mathcal{P}(\Pi)\}$$

   we have $t \star u \cdot \pi \succ u \star \pi \in \perp\!\!\!\perp$.

We temporarily admit the *adequation lemma*: if $\vdash t : A$ is derivable then $t \Vdash A$.

9. Show the converse to previous question, i.e. $\vdash t : \forall X.X \Rightarrow X$ implies that t is identity-like (hint: use a suitably chosen $\perp\!\!\!\perp$).

   Solution. We take $\perp\!\!\!\perp$ to be the closure by anti-reduction of $\{u \star \pi\}$. By the adequation lemma, we have $t \Vdash \forall X.X \Rightarrow X$. Thus $t \vdash X \Rightarrow X$ for every X. Take $[\![X]\!] = \{\pi\}$. We have $u \Vdash X$, thus $u \cdot \pi \in [\![X \Rightarrow X]\!]$, thus $t \Vdash u \cdot \pi$, i.e. $t \star u \cdot \pi \in \perp\!\!\!\perp$, i.e. $t \star u \cdot \pi \succ u \star \pi$.

10. Give an example of an identity-like term which is not the identity, and even non-typable.

    Solution. $(\lambda xy.x)\Omega$.

Booleans.

11. Suppose that our signatures contains constants 0 and 1. Define a predicate $\mathrm{Bool}(x)$, which encodes the fact that x is a boolean.

    Solution. We define

    $$\mathrm{Bool}(x) = \forall X.X(0) \Rightarrow X(1) \Rightarrow X(x)$$

12. Show that $\vdash t : \mathrm{Bool}(0)$ implies $t \star u \cdot v \cdot \pi \succ t \star \pi$ (and similarly for $\vdash t : \mathrm{Bool}(1)$).

    Solution. We define $\perp\!\!\!\perp$ to be the closure under anti-reduction of $\{u \star \pi\}$ and consider the interpretation of X such that $[\![X(0)]\!] = \{\pi\}$ and $[\![X(x)]\!] = \emptyset$ for $x \neq 0$. We have $u \Vdash X(0)$ and $v \Vdash X(1)$. By the adequation lemma, we have $t \Vdash X(0) \Rightarrow X(1) \Rightarrow X(0)$, thus $t \star u \cdot v \cdot \pi \in \perp\!\!\!\perp$ which gives the result.

Other consequences of the adequation lemma.

13. Show that there is no term such that $\vdash t : \bot$.

    Solution. If this was the case, we would have $t \Vdash \bot$, i.e. $t \star \pi \in \perp\!\!\!\perp$ for every $\pi \in \Pi = [\![\perp\!\!\!\perp]\!]$. This is absurd if we take $\perp\!\!\!\perp = \emptyset$.

14. Show that typable λ-terms are normalizing with respect to the call-by-name strategy.

    Solution. The proposed reduction of the machine corresponds to the call-by-name evaluation of a λ-term. If we take $\perp\!\!\!\perp = \{t \star \pi \mid t \star \pi \text{ normalizes}\}$ we can conclude.

The adequation lemma.

15. Show that $t \Vdash A \Rightarrow B$ and $u \Vdash A$ implies $tu \Vdash B$.

    Solution. Suppose $t \Vdash A \Rightarrow B$ and $u \Vdash A$. Given $\pi \in [\![B]\!]$, $tu \star \pi \succ t \star u \cdot \pi$. Since $u \in |A|$ and $\pi \in [\![B]\!]$, we have $u \cdot \pi \in [\![A \Rightarrow B]\!]$ and thus $t \star u \cdot \pi \in \perp\!\!\!\perp$ since $t \in |A \Rightarrow B|$. This $tu \star \pi$ because $\perp\!\!\!\perp$ is closed under antireduction.

16. Show that if for every $u \in \Lambda$, $u \Vdash A$ implies $t[u/x] \Vdash B$, then $\lambda x.t \Vdash A \Rightarrow B$.

    Solution. Suppose given an element of $[\![A \Rightarrow B]\!]$. It is of the form $u \cdot \pi$ with $u \in |A|$ and $\pi \in [\![B]\!]$. We have $\lambda x.t \star u \cdot \pi \succ t[u/x] \star \pi$ which belongs to $\perp\!\!\!\perp$.

17. Prove the adequation lemma: $\vdash t : A$ derivable implies $t \Vdash A$.

    Solution. We proceed by induction on the proof of $\Gamma \vdash t : A$.

    - Axiom:

      $$\frac{}{\Gamma \vdash x_i : A_i}$$

      We have to show that if $t_i \Vdash A_i$ then $t_i \Vdash A_i$, which is obvious.
    - Application, this can be deduced from question 15.
    - For abstraction to go through, we need to show a more general statement:

      if $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$ is derivable and $\forall i, t_i \Vdash A_i$ then $t[t_1/x_1, \ldots, t_n/x_n] \Vdash A$.

      This can be checked to still be valid for the two previous cases. The abstraction rule is

      $$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}$$

      By induction hypothesis, we have, for every $u \in A$, $t[t_i/x_i][u/x] \Vdash B$, we deduce that $(\lambda x.t)[t_i/x_i] \Vdash A \Rightarrow B$ by question 16.
    - For first-order rules to go through, we actually need to show a stronger induction hypothesis:

If $x_1 : A_1, \dots, x_n : A_n \vdash t : A$ is derivable, with $A_1, \dots, A_n$ having free variables among $y_1, \dots, y_m$ then for every $(t_i)_{1 \le i \le n} \in \Lambda^n$ such that $t_i \Vdash A_i$, for every $(u_i)_{1 \le i \le m} \in \mathcal{T}^m$, we have $t[t_i/x_i] \Vdash A[u_i/y_i]$.

It is straightforward to show that the above cases can be adapted to this generalization.

– First-order introduction:

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t : \forall x.A}$$

By induction hypothesis, we have $t[t_i/x_i] \in |A[u_i/x_i]|$. Therefore,

$$t[t_i/x_i] \in \bigcap_{t \in \mathcal{T}} |A[u/x][u_i/x_i]| = |(\forall x.A)[u_i/x_i]|$$

– First-order elimination:

$$\frac{\Gamma \vdash t : \forall x.A}{\Gamma \vdash t : A[a/x]}$$

By induction hypothesis, we have

$$t[t_i/x_i] \in |\forall x.A[u_i/x_i]| = \bigcap_{u \in \mathcal{T}} |A[u_i/x_i, u/x]| \supseteq |A[u_i/x_i, a/x]| = |A[a/x][u_i/x_i]|$$

- To handle second-order quantification, we also need a similar generalization of the induction hypothesis for second order variables (we can replace any second-order variable X_i of arity k_i by any function $P_i : (\mathcal{T}^*)^{k_i} \to \mathcal{P}(\Pi)$).

Classical logic. We extend λ-terms by adding constants $\texttt{cc}$ and $\texttt{k}_\pi$ for every $\pi \in \Pi$ with reduction rules

$$\texttt{cc} \star t \cdot \pi \succ t \star \texttt{k}_\pi \cdot \pi \qquad\qquad \texttt{k}_\pi \star t \cdot \rho \succ t \star \pi$$

18. Show that the formula $\forall X.(X \vee \neg X)$ is realized.

Solution. We have that $\forall X.(X \vee \neg X)$ is defined as

$$\forall X.\forall Y.(X \Rightarrow Y) \Rightarrow ((X \Rightarrow \bot) \Rightarrow Y) \Rightarrow Y$$

We consider

$$t = \lambda l.r.\texttt{cc}(\lambda k.r(\lambda a.k(la)))$$

Fix X and Y. Given $l \in |X \Rightarrow Y|$, $r \in |(X \Rightarrow \bot) \Rightarrow Y|$ and $\pi \in [\![Y]\!]$, we have

$$t \star l \cdot r \cdot \pi \succ r \star \lambda a.\texttt{k}_\pi(la) \cdot \pi$$

and we can thus conclude if we manage to show

$$\lambda a.\texttt{k}_\pi(la) \Vdash X \Rightarrow \bot$$

We conclude by anti-reduction since

$$\lambda a.\texttt{k}_\pi(la) \star a \cdot \rho \succ la \star \pi \succ l \star a \cdot \pi$$

which belongs to $\perp\!\!\!\perp$ since $a \cdot \pi$ belongs to $[\![X \Rightarrow Y]\!]$.

References

[1] Jean-Louis Krivine. Realizability in classical logic. *Panoramas et synthèses*, 27:197–229, 2009.