Strong normalization of the simply-typed λ-calculus

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We recall the rules of the simply-typed λ-calculus:

$$\frac{}{\Gamma, x : A, \Gamma' \vdash x : A}\ (\text{AX}) \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}\ (\Rightarrow_I) \qquad \frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B}\ (\Rightarrow_E)$$

where, in the first rule, we suppose $x \notin \mathrm{dom}(\Gamma')$. Our goal is to show that every typable term t (in an arbitrary context) is *strongly normalizable*, meaning that there is no infinite reduction from t.

1. Can we show the property by induction on the derivation of the typing of t?

I Weak normalization

We first want to show that every typable term t (in an arbitrary context) is *weakly normalizable*, meaning that a typable term can reduce to a normal form. We write $t \to t'$ for a reduction in the *call-by-value strategy* defined by

$$\frac{t \to t'}{t\,u \to t'\,u} \qquad \frac{u \to u'}{(\lambda x.t)u \to (\lambda x.t)u'} \qquad \frac{}{(\lambda x.t)(\lambda y.u) \to t[\lambda y.u/x]}$$

1. Show that the reduction strategy is deterministic, meaning $t \to t_1$ and $t \to t_2$ implies $t_1 = t_2$.
2. For such a strategy is there a difference between weak and strong normalization?

We define the set $\mathcal{R}(A)$ of *reducible* terms of type A by induction by

- for A atomic, $\mathcal{R}(A)$ is the set of normalizing closed terms of type A,
- for A and B types, $\mathcal{R}(A \Rightarrow B)$ is the set of normalizing closed terms t of type $A \Rightarrow B$ such that $tu \in \mathcal{R}(B)$ for every term $u \in \mathcal{R}(A)$.

Here, normalizing is always understood with respect to the normal order strategy.

3. Show that given terms t and t' such that $t \to t'$, show that t is normalizing if and only if t' is normalizing.
4. Show the property (CR1): if $t \in \mathcal{R}(A)$ then t is normalizing.
5. Show the property (CR2): if $t \in \mathcal{R}(A)$ and $t \to t'$ then $t' \in \mathcal{R}(A)$.
6. Show the property (CR3): if t has type A, $t \to t'$ and $t' \in \mathcal{R}(A)$ then $t \in \mathcal{R}(A)$.
7. Suppose that $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$ is derivable. Show that for all $u_1 \in \mathcal{R}(A_1)$, $\ldots$, $u_n \in \mathcal{R}(A_n)$, we have $t[u_1/x_1, \ldots, u_n/x_n] \in \mathcal{R}(A)$.
8. Show that any typable closed term t is normalizable.
9. Show that typable λ-terms are weakly normalizable.
10. Show that there are non-typable λ-terms.
11. Show that $\mathcal{R}(A)$ is, in fact, the set of closed terms of type A.

II Strong normalization

We now turn to strong normalization. In the course of the proof, will need the following *well-founded induction* principle.

1. Suppose given a set X equipped with a binary relation $\to$ which is *well-founded*: there is no infinite sequence of reductions. Suppose given a property P on the elements of X such that, for every $t \in X$, we have
$$\forall t \in X.\ ((\forall t' \in X.\ t \to t' \Rightarrow P(t')) \Rightarrow P(t))$$
Show that $\forall t \in X.\ P(t)$ holds. How can we recover recurrence as a particular case of this?

A term t is *neutral* when no new redex is created when applied to another term u (all the redexes in tu are either in t or in u).

2. Give an explicit description of neutral terms.

We define $\mathcal{R}(A)$, the *reducible* terms of type A, by induction by

- $\mathcal{R}(A)$, for A atomic, is the set of strongly normalizable terms,
- $\mathcal{R}(A \Rightarrow B)$ is the set of terms t such that $tu \in \mathcal{R}(B)$ for every $u \in \mathcal{R}(A)$.

We are going to show that following conditions hold:

(CR1) if $t \in \mathcal{R}(A)$ then t is strongly normalizable,
(CR2) if $t \in \mathcal{R}(A)$ and $t \to t'$ then $t' \in \mathcal{R}(A)$,
(CR3) if t is neutral and for every t' such that $t \to t'$ we have $t' \in \mathcal{R}(A)$ then $t \in \mathcal{R}(A)$.

3. Show that these conditions imply that a variable x belongs to $\mathcal{R}(A)$ for every type A.
4. Show that if t is strongly normalizable and $t \to t'$ then t' is also strongly normalizable. Does the converse hold?
5. Show the conditions (CR1), (CR2) and (CR3) by induction on A.
6. Suppose that $t[u/x] \in \mathcal{R}(B)$ for every $u \in \mathcal{R}(A)$. Show that $\lambda x.t \in \mathcal{R}(A \Rightarrow B)$.
7. Suppose that $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$ is derivable. Show that for all $u_1 \in \mathcal{R}(A_1)$, $\ldots$, $u_n \in \mathcal{R}(A_n)$, we have $t[u_1/x_1, \ldots, u_n/x_n] \in \mathcal{R}(A)$.
8. Show that $\Gamma \vdash t : A$ derivable implies $t \in \mathcal{R}(A)$.
9. Show that all typable terms are strongly normalizable.
10. Use this to show that typable terms are confluent.

References

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