Strong normalization of the simply-typed λ-calculus

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We recall the rules of the simply-typed λ-calculus:

$$\frac{}{\Gamma, x : A, \Gamma' \vdash x : A}\ (\text{AX}) \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}\ (\Rightarrow_I) \qquad \frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B}\ (\Rightarrow_E)$$

where, in the first rule, we suppose $x \notin \text{dom}(\Gamma')$. Our goal is to show that every typable term t (in an arbitrary context) is *strongly normalizable*, meaning that there is no infinite reduction from t.

1. Can we show the property by induction on the derivation of the typing of t?

   Solution. No, in the third rule we cannot show that if t and u are SN then tu also is, because a reduction in tu is not necessarily a reduction in t or a reduction in u (take $t = u = \lambda x.xx$).

I Weak normalization

We first want to show that every typable term t (in an arbitrary context) is *weakly normalizable*, meaning that a typable term can reduce to a normal form. We write $t \to t'$ for a reduction in the *call-by-value strategy* defined by

$$\frac{t \to t'}{t\,u \to t'\,u} \qquad \frac{u \to u'}{(\lambda x.t)u \to (\lambda x.t)u'} \qquad \frac{}{(\lambda x.t)(\lambda y.u) \to t[\lambda y.u/x]}$$

1. Show that the reduction strategy is deterministic, meaning $t \to t_1$ and $t \to t_2$ implies $t_1 = t_2$.

   Solution. We notice that a term of the form $\lambda x.t$ can never be reduced (because there is no rule to do so). In a term reducible by the first rule t is therefore not an abstraction, and the two other rules cannot apply. Similarly, in a term reducible by the second rule, u is not an abstraction and the third rule cannot apply. Thus at most one rule applies and the strategy is deterministic.

2. For such a strategy is there a difference between weak and strong normalization?

   Solution. No, by determinism, there exists a path to a normal form if and only if every path ends on a normal form.

We define the set $\mathcal{R}(A)$ of *reducible* terms of type A by induction by

- for A atomic, $\mathcal{R}(A)$ is the set of normalizing closed terms of type A,
- for A and B types, $\mathcal{R}(A \Rightarrow B)$ is the set of normalizing closed terms t of type $A \Rightarrow B$ such that $tu \in \mathcal{R}(B)$ for every term $u \in \mathcal{R}(A)$.

Here, normalizing is always understood with respect to the normal order strategy.

3. Show that given terms t and t' such that $t \to t'$, show that t is normalizing if and only if t' is normalizing.

   Solution. Immediate by determinism of the reduction strategy.

4. Show the property (CR1): if $t \in \mathcal{R}(A)$ then t is normalizing.

   Solution. Immediate by induction on the type A.

5. Show the property (CR2): if $t \in \mathcal{R}(A)$ and $t \to t'$ then $t' \in \mathcal{R}(A)$.

   Solution. By induction on A.

   - For atomic types, we conclude using question 3 and subject reduction.

- Suppose $t \in \mathcal{R}(A \Rightarrow B)$ and $t \to t'$. By question 3, t' is normalizing and by subject reduction t' has type A. Given $u \in \mathcal{R}(A)$, we have $tu \in \mathcal{R}(B)$ and $tu \to t'u$ (because the strategy is CBV) thus $t'u \in \mathcal{R}(B)$ by induction hypothesis. Thus $t' \in \mathcal{R}(A \Rightarrow B)$.

6. Show the property (CR3): if t has type A, $t \to t'$ and $t' \in \mathcal{R}(A)$ then $t \in \mathcal{R}(A)$.

   Solution. By induction on A.

   - For atomic types, we conclude using question 3 again.
   - Suppose $t \to t'$ with $t' \in \mathcal{R}(A \Rightarrow B)$. Given $u \in \mathcal{R}(A)$, we have $t'u \in \mathcal{R}(B)$ and $tu \to t'u$ (because the strategy is CBV) thus $tu \in \mathcal{R}(B)$ by induction hypothesis. Thus $t \in \mathcal{R}(A \Rightarrow B)$.

7. Suppose that $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$ is derivable. Show that for all $u_1 \in \mathcal{R}(A_1)$, …, $u_n \in \mathcal{R}(A_n)$, we have $t[u_1/x_1, \ldots, u_n/x_n] \in \mathcal{R}(A)$.

   Solution. By induction on the derivation of $\Gamma \vdash t : A$. We write $\mathcal{R}(\Gamma) = \mathcal{R}(A_1) \times \ldots \times \mathcal{R}(A_n)$. We write $\overline{u}$ instead of $u_1, \ldots, u_n$.

   - If the last rule is
   $$\frac{}{\Gamma, x : A, \Gamma' \vdash x : A}$$
   the result is immediate.
   - If the last rule is
   $$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}$$
   Suppose given $\overline{u} \in \mathcal{R}(\Gamma)$ and $u \in \mathcal{R}(A)$. By (CR1), we have $u \xrightarrow{*} \hat{u}$ for some normal term $\hat{u}$, and, by (CR2), we have $\hat{u} \in \mathcal{R}(A)$. We have $(\lambda x.t)u \xrightarrow{*} (\lambda x.t)\hat{u} \to t[\hat{u}/x]$ (because the strategy is CBV), thus $((\lambda x.t)u)[\overline{u}/\overline{x}] \to t[\overline{u}/\overline{x}, \hat{u}/x]$ and, by induction hypothesis, $t[\overline{u}/\overline{x}, \hat{u}/x] \in \mathcal{R}(B)$. Thus $((\lambda x.t)u)[\overline{u}/\overline{x}] \in \mathcal{R}(B)$ by (CR3). We conclude that $(\lambda x.t)[\overline{u}/\overline{x}] \in \mathcal{R}(A \to B)$.
   - If the last rule is
   $$\frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B}$$
   Given $\overline{u} \in \mathcal{R}(\Gamma)$, we have, by induction hypothesis, $t[\overline{u}/\overline{x}] \in \mathcal{R}(A \Rightarrow B)$ and $u[\overline{u}/\overline{x}] \in \mathcal{R}(A)$ and thus $t[\overline{u}/\overline{x}]u[\overline{u}/\overline{x}] \in \mathcal{R}(B)$, i.e. $(tu)[\overline{u}/\overline{x}] \in \mathcal{R}(B)$.

8. Show that any typable closed term t is normalizable.

   Solution. Given a typable term t, we have that $\vdash t : A$ is derivable, thus $t \in \mathcal{R}(A)$ by previous question.

9. Show that typable λ-terms are weakly normalizable.

   Solution. Given a typable term t, we have $t \in \mathcal{R}(A)$ by previous question, and thus t is normalizable wrt the normal order strategy, i.e. t is weakly normalizable.

10. Show that there are non-typable λ-terms.

    Solution. Ω is not weakly normalizable and thus not typable.

11. Show that $\mathcal{R}(A)$ is, in fact, the set of closed terms of type A.

    Solution. By definition of $\mathcal{R}(A)$, it contains only closed terms of type A. Conversely, given a closed term of type A, we have that $t \in \mathcal{R}(A)$ by question 7.

II Strong normalization

We now turn to strong normalization. In the course of the proof, will need the following *well-founded induction* principle.

1. Suppose given a set X equipped with a binary relation $\to$ which is *well-founded*: there is no infinite sequence of reductions. Suppose given a property P on the elements of X such that, for every $t \in X$, we have

$$\forall t \in X.\ ((\forall t' \in X.\ t \to t' \Rightarrow P(t')) \Rightarrow P(t))$$

Show that $\forall t \in X.\ P(t)$ holds. How can we recover recurrence as a particular case of this?

Solution. By absurd, if there exists t_0 such that $\neg P(t_0)$, then there exists t_1 such that $t_0 \to t_1$ and $\neg P(t_1)$. Going on in this way, we construct an infinite sequence $t_0 \to t_1 \to t_2 \to \ldots$ such that $\neg P(t_i)$ for every index i, which is absurd.

A term t is *neutral* when no new redex is created when applied to another term u (all the redexes in tu are either in t or in u).

2. Give an explicit description of neutral terms.

Solution. A term is neutral when it is not an abstraction.

We define $\mathcal{R}(A)$, the *reducible* terms of type A, by induction by

- $\mathcal{R}(A)$, for A atomic, is the set of strongly normalizable terms,
- $\mathcal{R}(A \Rightarrow B)$ is the set of terms t such that $tu \in \mathcal{R}(B)$ for every $u \in \mathcal{R}(A)$.

We are going to show that following conditions hold:

(CR1) if $t \in \mathcal{R}(A)$ then t is strongly normalizable,

(CR2) if $t \in \mathcal{R}(A)$ and $t \to t'$ then $t' \in \mathcal{R}(A)$,

(CR3) if t is neutral and for every t' such that $t \to t'$ we have $t' \in \mathcal{R}(A)$ then $t \in \mathcal{R}(A)$.

3. Show that these conditions imply that a variable x belongs to $\mathcal{R}(A)$ for every type A.

Solution. By (CR3).

4. Show that if t is strongly normalizable and $t \to t'$ then t' is also strongly normalizable. Does the converse hold?

Solution. If there was an infinite sequence of reductions starting from t', there would also be one starting from t which is supposed to be SN.

5. Show the conditions (CR1), (CR2) and (CR3) by induction on A.

Solution. We simultaneously show the three conditions by induction on A.

- Case of an atomic type A.

(CR1) Obvious.

(CR2) If t is SN and $t \to t'$ then t' is also SN by previous question.

(CR3) Suppose that t is neutral and all its reductions are reducible and thus SN. Then t must be SN.

- Case of $A \Rightarrow B$.

(CR1) Suppose $t \in \mathcal{R}(A \Rightarrow B)$. A variable x is neutral and normal and thus in $\mathcal{R}(A)$. Thus $tx \in \mathcal{R}(B)$. An infinite reduction from t would induce one from tx, which is impossible by (CR1) on B.

(CR2) Suppose $t \in \mathcal{R}(A \Rightarrow B)$ and $t \to t'$. Given $u \in \mathcal{R}(A)$, we have $tu \in \mathcal{R}(B)$ and thus $t'u \in \mathcal{R}(B)$ by (CR2) on B since $tu \to t'u$. Thus $t' \in \mathcal{R}(A \Rightarrow B)$.

(CR3) Suppose t neutral and $t' \in \mathcal{R}(A \Rightarrow B)$ for every t' such that $t \to t'$. Fix $u \in \mathcal{R}(A)$. In order to show that $tu \in \mathcal{R}(B)$, and thus conclude that $t \in \mathcal{R}(A \to B)$, we use (CR3) on B: it is enough to show that if $tu \to v$ then $v \in \mathcal{R}(B)$. Since t is neutral, there are two possible reductions starting from tu.

  * $tu \to t'u$ (with $t \to t'$). Then $t'u \in \mathcal{R}(B)$ because $t' \in \mathcal{R}(B)$ by hypothesis.
  * $tu \to tu'$ (with $u \to u'$). Then $u' \in \mathcal{R}(A)$ by (CR2) and thus $tu' \in \mathcal{R}(B)$ (we reason by well-founded induction on u).

6. Suppose that $t[u/x] \in \mathcal{R}(B)$ for every $u \in \mathcal{R}(A)$. Show that $\lambda x.t \in \mathcal{R}(A \Rightarrow B)$.

Solution. We have to show that, for $u \in \mathcal{R}(A)$, we have $(\lambda x.t)u \in \mathcal{R}(B)$. Since x is $\mathcal{R}(A)$, we have $t = t[x/x]$ in $\mathcal{R}(B)$ and thus both t and u are SN by (CR1). We can reason by induction on the pair (t, u). The term $(\lambda x.t)u$ can reduce to

- $t[u/x]$ which is in $\mathcal{R}(B)$ by general hypothesis,
- $(\lambda x.t')u$ with $t \to t'$, which is in $\mathcal{R}(B)$ by induction hypothesis,
- $(\lambda x.t)u'$ with $u \to u'$, which is in $\mathcal{R}(B)$ by induction hypothesis.

Since $(\lambda x.t)u$ can only reduce to terms in $\mathcal{R}(B)$, it is in $\mathcal{R}(B)$ by (CR3).

7. Suppose that $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$ is derivable. Show that for all $u_1 \in \mathcal{R}(A_1)$, …, $u_n \in \mathcal{R}(A_n)$, we have $t[u_1/x_1, \ldots, u_n/x_n] \in \mathcal{R}(A)$.

   Solution. Same as in previous part.

8. Show that $\Gamma \vdash t : A$ derivable implies $t \in \mathcal{R}(A)$.

   Solution. We have $x_i \in \mathcal{R}(A_i)$ and thus, by previous question $t = t[\overline{x}/\overline{x}] \in \mathcal{R}(A)$.

9. Show that all typable terms are strongly normalizable.

   Solution. Every typable term is reducible and thus SN by (CR1).

10. Use this to show that typable terms are confluent.

    Solution. We can show that the λ-calculus is locally confluent and deduce SN by Newman's lemma.

References

[1] Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and types*, volume 7. Cambridge university press Cambridge, 1989.

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