Adjunctions

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We recall that a functor $U : \mathcal{D} \to \mathcal{C}$ has a *left adjoint* $F : \mathcal{C} \to \mathcal{D}$

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D} \qquad (F \dashv U,\ \perp)$$

when, for every objects $A \in \mathcal{C}$ and $B \in \mathcal{D}$ there is a bijection

$$\phi_{A,B} : \mathcal{D}(FA, B) \to \mathcal{C}(A, UB)$$

which is natural in A and B, in the sense that for every $f : A \to A'$ in $\mathcal{C}$ and $g : B \to B'$ in $\mathcal{D}$ we have the commutation of the diagram

$$\begin{array}{ccc}
\mathcal{D}(FA', B) & \xrightarrow{\phi_{A',B}} & \mathcal{C}(A', UB) \\
{\scriptstyle \mathcal{D}(Ff,g)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(f,Gg)} \\
\mathcal{D}(FA, B') & \xrightarrow[\phi_{A,B'}]{} & \mathcal{C}(A, UB')
\end{array}$$

In this case, U is the *right adjoint* to F.

1 First examples of adjunctions

1. Show that the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ admits a left adjoint $F : \mathbf{Set} \to \mathbf{Top}$.

   Solution. We can guess that the functor F associate to a set A the topological space FA which is the set A together with an appropriate topology. Since we should have

   $$\mathbf{Top}(FA, B) \simeq \mathbf{Set}(A, UB)$$

   we need to put a topology on A such that every possible function $f : A \to B$ is continuous, i.e. we always have that $f^{-1}(U)$ is an open set of A for an open set U of B. There is an easy way to have that: take the topology on A consisting of all subsets of A (this is called the *discrete topology* on A).

2. Show that the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ admits a right adjoint $F : \mathbf{Set} \to \mathbf{Top}$.

   Solution. We should now have

   $$\mathbf{Top}(A, FB) \simeq \mathbf{Set}(UA, B)$$

   which means that FB should be the set B equipped with the less possible number of open sets: simply take the *trivial topology* consisting of $\emptyset$ and B as open sets.

3. Show that the functor $\mathbf{Graph} \to \mathbf{Set}$ sending a graph to the underling set of vertices has both a left and a right adjoint.

2 The adjoint functor theorem between posets

In this exercise, we will study adjunctions between posets (seen as categories) and prove, in this restricted case, the adjoint functor theorem which provides conditions for the existence of adjoints to functors.

1. What is a functor between posets?

   Solution. An increasing function.

2. Consider the inclusion $F : \mathbb{Z} \to \mathbb{R}$ between posets. What is a left/right adjoint?

   Solution. Writing $G : \mathbb{R} \to \mathbb{Z}$ for the right adjoint we should have, for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, $F(n) \leq x$ in $\mathbb{R}$ if and only if $n \leq G(x)$ in $\mathbb{Z}$. Therefore

   $$G(x) = \lfloor x \rfloor = \sup\{m \in \mathbb{Z} \mid m \leq x\}$$

   Namely,

   $$\begin{aligned} n \leq G(x) &\Leftrightarrow n \leq \sup\{m \in \mathbb{Z} \mid m \leq x\} \\ &\Leftrightarrow n \in \{m \in \mathbb{Z} \mid m \leq x\} \\ &\Leftrightarrow n \leq x \end{aligned}$$

   A left adjoint is dually given by $G(x) = \lceil x \rceil$.

3. Consider the function $F : \mathbb{Z} \to \mathbb{Z}$ between posets such that $F(n) = 2n$. What is a left/right adjoint?

   Solution. The right adjoint is $G(n) = \lfloor n/2 \rfloor$. Namely,

   $$\begin{aligned} F(m) \leq n &\Leftrightarrow 2m \leq n \\ &\Leftrightarrow m \leq n/2 \\ &\Leftrightarrow m \leq \lfloor n/2 \rfloor \end{aligned}$$

   The right adjoint is $G(n) = \lceil n/2 \rceil$.

We suppose fixed a functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are posets.

4. Show that if F is a left adjoint then it preserves arbitrary joins.

   Solution. We write $G : \mathcal{D} \to \mathcal{C}$ for the right adjoint. Fix a family $(A_i)_{i \in I}$ of objects. For every index i, we have $A_i \leq \bigvee_i A_i$ thus, since F is increasing (this is a functor), $F(A_i) \leq F(\bigvee_i A_i)$, and therefore

   $$\bigvee_i F(A_i) \leq F(\bigvee_i A_i)$$

   In order to show the converse inequality, observe that for any $i \in I$, we have $F(A_i) \leq \bigvee_i F(A_i)$, therefore (we have an adjunction) $A_i \leq G(\bigvee_i F(A_i))$, thus $\bigvee_i A_i \leq G(\bigvee_i F(A_i))$, and finally

   $$F(\bigvee_i A_i) \leq \bigvee_i F(A_i)$$

5. Suppose that $\mathcal{C}$ has all joins. Show that F is a left adjoint if and only if it preserves arbitrary joins.

   Solution. The left-to-right implication is the previous question. Conversely, suppose that F preserves arbitrary joins. We define $G : \mathcal{D} \to \mathcal{C}$ by

   $$G(B) = \bigvee\{A \in \mathcal{C} \mid F(A) \leq B\}$$

This is clearly a functor in the sense that G is monotone. Finally, let us show that $FA \leq B$ if and only if $A \leq GB$. If $FA \leq B$ then $A \in \{A \in \mathcal{C} \mid F(A) \leq B\}$ and thus $A \leq G(B)$. Conversely, if $A \leq GB$, then

$$FA \leq FGB = F(\bigvee\{A \in \mathcal{C} \mid F(A) \leq B\}) = \bigvee\{FA \mid FA \leq B\} \leq B$$

6. A natural generalization of previous question is: a functor F is a left adjoint if and only if it preserves arbitrary colimits. Do you foresee any problem with proving this?

3 Free monoids and categories

We write **Mon** for the category of monoids and morphisms of monoids.

1. Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ admits a left adjoint $F : \mathbf{Set} \to \mathbf{Mon}$.

Solution. The forgetful functor U associates, to a monoid $(M, \times, 1)$, the underlying set M. Given a set A, the free monoid $(A^*, \cdot, [])$ is the monoid whose elements are words $[a_1 \ldots a_n]$ whose letters are in A, multiplication is given by composition and neutral element is the empty word $[]$. The left adjoint is the functor which to a set A associates the free monoid $FA = A^*$ and to a function $f : A \to B$ associates the morphism of monoids

$$\begin{aligned} f^* : A^* &\to B^* \\ [a_1 \ldots a_n] &\mapsto [f(a_1) \ldots f(a_n)] \end{aligned}$$

Given a set A and a monoid $(M, \times, 1)$, let us construct bijections:

$$\mathbf{Mon}(FA, (M, \times, 1)) \underset{\psi_{A,M}}{\overset{\phi_{A,M}}{\rightleftarrows}} \mathbf{Set}(A, U(M, \times, 1))$$

i.e.

$$\mathbf{Mon}(A^*, (M, \times, 1)) \underset{\psi_{A,M}}{\overset{\phi_{A,M}}{\rightleftarrows}} \mathbf{Set}(A, M)$$

Given a morphism of monoids $f : A^* \to (M, \times, 1)$, we define the function

$$\begin{aligned} \phi_{A,M}(f) : A &\to M \\ a &\mapsto f([a]) \end{aligned}$$

i.e. the restriction of f to one-letter functions, and given a function $g : A \to M$, we define

$$\begin{aligned} \psi_{A,M}(g) : A^* &\to (M, \times, 1) \\ [a_1 \ldots a_n] &\mapsto f(a_1) \times \ldots \times f(a_n) \end{aligned}$$

We now have to check that those functions are mutually inverse. Given a morphism of monoids $f : A^* \to (M, \times, 1)$, we have

$$\psi_{A,M} \circ \phi_{A,M}(f)([a_1 \ldots a_n]) = f([a_1]) \times \ldots \times f([a_n]) = f([a_1 \ldots a_n])$$

because f is a morphism of monoids. Conversely, given a function $g : A \to M$, we have

$$\phi_{A,M} \circ \psi_{A,M}(g)(a) = \psi_{A,M}(g)([a]) = g(a)$$

Finally, we have to check that these bijections are natural, this proof being left to the reader.

2. Show that the forgetful functor $U : \mathbf{Cat} \to \mathbf{Graph}$ admits a left adjoint $F : \mathbf{Graph} \to \mathbf{Cat}$.

   Solution. This situation is a generalization of the previous one: monoids are bijection with categories with one object, and sets are in bijection with graphs with one vertex. Given a graph G, we define FG to be the free category on G: its objects are the vertices of G, a morphism from x to y is a (directed) path from x to y in G, composition is given by concatenation of paths, identities are given by empty paths. The rest of the proof is pretty similar to previous question.

4 Terminal objects and products by adjunctions

1. Given a category $\mathcal{C}$, show that the terminal functor $T : \mathcal{C} \to \mathbf{1}$ has a right (resp. left) adjoint iff the category $\mathcal{C}$ admits a terminal (resp. initial) object.

2. Given a category $\mathcal{C}$, describe the *diagonal functor* $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ and show that the category $\mathcal{C}$ admins cartesian products (resp. coproducts) iff the diagonal functor admits a right (resp. left) adjoint.

5 Cartesian closed categories

A category is *cartesian closed* when for every object B, the functor $- \times B$ admits a right adjoint $B \Rightarrow -$.

1. Show that **Set** is cartesian closed.

2. Show that the category **POSet** of partially ordered sets and increasing functions is a cartesian closed category.

   Solution. The product of two posets $(M, \leq_M)$ and $(N, \leq_N)$ is $(M \times N, \leq_{M\times N})$ with $(m, n) \leq_{M\times N} (m', n')$ whenever both $m \leq_M m'$ and $n \leq_N n'$. The exponential closure $(N^M, \leq_{N^M})$ is the set of increasing functions $M \to N$ ordered by $f \leq_{N^M} g$ whenever $f(m) \leq_N g(m)$ for every $m \in M$.

3. Show that the category **Mon** is cartesian but not closed (hint: look at the properties satisfied by the terminal object).

   Solution. The cartesian product of monoids M and N is the usual cartesian product $M \times N$ of sets equipped with pointwise multiplication: $(u, u') \times (v, v') = (u \times v, u' \times v')$. The trivial monoid 1 is both initial and terminal. If **Mon** was closed, we would have

   $$\mathbf{Mon}(M, N) \simeq \mathbf{Mon}(1 \times M, N) \simeq \mathbf{Mon}(1, N^M)$$

   and therefore there should be exactly one morphism between any two objects (i.e. the category is equivalent to the terminal one).

4. Show that the category **Cat** is cartesian closed.