# Coproducts, pullbacks, monoids

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## 1 Coproducts

Notions in category theory can always be "dualized" in the following way.

1. Given a category $\mathcal{C}$ define the category $\mathcal{C}^{\mathrm{op}}$ obtained by reversing the morphisms.

A *<u>co</u>something* in a category $\mathcal{C}$ is a something in $\mathcal{C}^{\mathrm{op}}$.

2. Show that **Set** is a cocartesian category, i.e. has coproducts and an initial object (an initial object is a coterminal object).

3. Show that the usual categories are cocartesian : **Set**, **Top**, **Rel**, **Vect**, **Cat**.

## 2 Pullbacks

Given two morphisms $f : A \to C$ and $g : B \to C$ with the same target, a *pullback* is given by an object $D$ (sometimes abusively noted $A \times_C B$) together with two morphisms $p : D \to A$ and $q : D \to B$ such that $f \circ p = g \circ q$, and for every pair of morphisms $p' : D' \to A$ and $q' : D' \to B$ (with the same source) such that $f \circ p' = g \circ q'$, there exists a unique morphism $h : D' \to D$ such that $p \circ h = p'$ and $q \circ h = q'$.

$$
\begin{tikzcd}
 & D' \arrow[d, dotted, "h"] \arrow[ddl, bend right, "p'"'] \arrow[ddr, bend left, "q'"] & \\
 & D \arrow[dl, "p"'] \arrow[dr, "q"] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\
 & C &
\end{tikzcd}
$$

1. What is a pullback in the case where $C$ is the terminal object?

2. What is a pullback in **Set**?

A *pushout* in a category $\mathcal{C}$ is a pullback in $\mathcal{C}^{\mathrm{op}}$.

3. What is a pushout in **Set**? In **Top**?

4. Show that the pushout of an isomorphism is an isomorphism.

## 3 Monomorphisms

A *monomorphism* is a morphism $f : A \to B$ such that for every morphisms $g_1, g_2 : A' \to A$, we have that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$:

$$A' \underset{g_2}{\overset{g_1}{\rightrightarrows}} A \xrightarrow{f} B$$

1. What is a monomorphism in **Set**?
2. Show that the pullback of a monomorphism along any morphism is a monomorphism.
3. Show that, in **Set**, the pushout of a monomorphism along any morphism is a monomorphism. Does this seem to be true in any category?
4. Define the dual notion of *epimorphism*. What is an epimorphism in **Set**?
5. In the category of posets, construct a morphism which is both a monomorphism and an epimorphism, but not an isomorphism.

## 4 (Co)monoids in cartesian categories

1. Given a cartesian category $\mathcal{C}$, show that the cartesian product induces a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
2. Generalize the definition of *monoid* to any cartesian category (a monoid in **Set** should be a monoid in the usual sense). When is a monoid commutative?
3. Generalize the notion of morphism of monoid.
4. A *comonoid* in $\mathcal{C}$ is a monoid in $\mathcal{C}^{\mathrm{op}}$. Make explicit the notion of comonoid.
5. What part of the cartesian structure on $\mathcal{C}$ did we really need in order to define the notion of monoid?
6. Show that in a cartesian category every object is a comonoid (with respect to product).
7. Given a category $\mathcal{C}$, show that the category of commutative comonoids and morphisms of comonoids in $\mathcal{C}$ is cartesian.