Coproducts, pullbacks, monoids

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1 Coproducts

Notions in category theory can always be "dualized" in the following way.

1. Given a category $\mathcal{C}$ define the category $\mathcal{C}^{\mathrm{op}}$ obtained by reversing the morphisms.

   Solution. We define the category $\mathcal{C}^{\mathrm{op}}$ by

   - the objects of $\mathcal{C}^{\mathrm{op}}$ are the same as those of $\mathcal{C}$,
   - $\mathcal{C}^{\mathrm{op}}(A, B) = \mathcal{C}(B, A)$

   For clarity, we write $f^{\mathrm{op}} : A \to B$ for a morphism in $\mathcal{C}^{\mathrm{op}}$ corresponding to a morphism $f : B \to A$ in $\mathcal{C}$. Composition of morphisms $f^{\mathrm{op}} : A \to B$ and $g^{\mathrm{op}} : B \to C$ in $\mathcal{C}^{\mathrm{op}}$ (i.e. morphisms $f : B \to A$ and $g : B \to C$ in $\mathcal{C}$) is defined by

   $$g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$$

   identities in $\mathcal{C}^{\mathrm{op}}$ are the same as in $\mathcal{C}$.

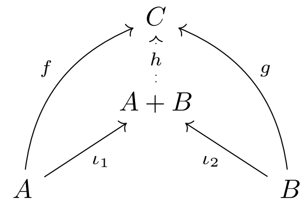
A *co*something in a category $\mathcal{C}$ is a something in $\mathcal{C}^{\mathrm{op}}$.

2. Show that **Set** is a cocartesian category, i.e. has coproducts and an initial object (an initial object is a coterminal object).

   Solution. Let us spell out explicitly the definition of a coproduct. Given two objects A and B a coproduct is an object $A + B$ together with two morphisms

   $$\iota_1 : A \to A + B \qquad\qquad \iota_2 : B \to A + B$$

   such that for every pair of morphisms $f : A \to C$ and $g : B \to C$ with the same target, there exists a unique morphism $h : A + B \to C$ such that $h \circ \iota_1 = f$ and $h \circ \iota_2 = g$:



   In **Set**, the coproduct is given by the disjoint union

   $$A + B = A \sqcup B = \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}$$

and ι_1 and ι_2 are the canonical injections: for $a \in A$ and $b \in B$,

$$\iota_1(a) = (0, a) \qquad\qquad \iota_2(b) = (1, b)$$

Given h as above, we necessarily have, for $a \in A$ and $b \in B$,

$$h \circ \iota_1(a) = h((0, a)) = f(a) \qquad\qquad h \circ \iota_2(b) = b((1, b)) = g(b)$$

Conversely, the function

$$\begin{aligned} h : A \sqcup B &\to C \\ (0, a) &\mapsto f(a) \\ (1, B) &\mapsto g(b) \end{aligned}$$

is suitable for similar reasons as above.

An object I in a category $\mathcal{C}$ is *initial* when, for every object A, there exists a unique morphism $I \to A$. In **Set**, the initial object is the empty set $\emptyset$.

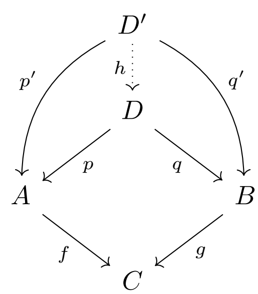
3. Show that the usual categories are cocartesian : **Set**, **Top**, **Rel**, **Vect**, **Cat**.

   Solution.

   - In **Top**, coproducts are given by disjoint union with the usual topology.
   - In **Rel**, we have $\mathbf{Rel}^{\mathrm{op}} \simeq \mathbf{Rel}$ and therefore the coproduct is the same as the product and is given by disjoint union.
   - In **Vect**, the coproduct is the direct sum.

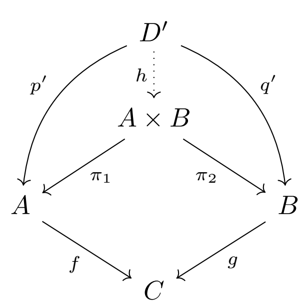
2 Pullbacks

Given two morphisms $f : A \to C$ and $g : B \to C$ with the same target, a *pullback* is given by an object D (sometimes abusively noted $A \times_C B$) together with two morphisms $p : D \to A$ and $q : D \to B$ such that $f \circ p = g \circ q$, and for every pair of morphisms $p' : D' \to A$ and $q' : D' \to B$ (with the same source) such that $f \circ p' = g \circ q'$, there exists a unique morphism $h : D' \to D$ such that $p \circ h = p'$ and $q \circ h = q'$.



1. What is a pullback in the case where C is the terminal object?

   Solution. When C is the terminal object, the morphisms $f : A \to C$ and $g : B \to C$ are the terminal morphisms and their pullback is the same as the product of A and B:



2. What is a pullback in **Set**?

   Solution. Given two functions $f : A \to C$ and $g : B \to C$, their pullback $D = A \times_C B$ is the following subset of $A \times B$:

   $$A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

   and p and q are the restrictions of the projections: for $(a, b) \in A \times_C B$,

   $$p(a, b) = a \qquad\qquad q(a, b) = b$$

   Given two functions $p' : D' \to A$ and $q' : D' \to B$ such that $f \circ p' = g \circ q'$, and a function $h : D' \to A \times_C B$ making the two triangles commute, we necessarily have, for $d \in D'$,

   $$h(d) = (\pi_1(h(d)), \pi_2(h(d))) = (p'(d), q'(d))$$

   and conversely, the function defined in this way suits.

A *pushout* in a category $\mathcal{C}$ is a pullback in $\mathcal{C}^{\mathrm{op}}$.

3. What is a pushout in **Set**? In **Top**?

   Solution. The pushout of two morphisms $f : C \to A$ and $g : C \to B$ is the set $A +_C B$ defined as

   $$A +_C B = (A \sqcup B)/\sim$$

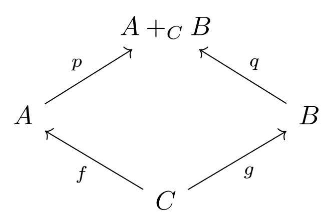
   where $\sim$ is the smallest equivalence relation such that

   $$\iota_1(f(c)) \sim \iota_2(g(c))$$

   for every element $c \in C$.

4. Show that the pushout of an isomorphism is an isomorphism.

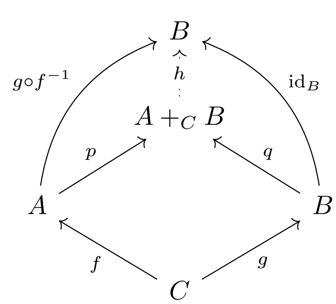
   Solution. Suppose given a pushout



   where f is an isomorphism. The above diagram commutes, i.e.

   $$p \circ f = q \circ g$$

   We want to show that q is also an isomorphism. Since $g \circ f^{-1} \circ f = g = \mathrm{id}_B \circ g$, by universal property of the pushout we have the existence of a morphism $h : A +_C B \to B$



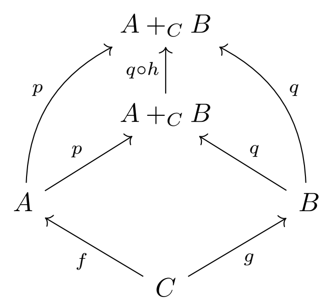
such that

$$h \circ p = g \circ f^{-1} \qquad\qquad h \circ q = \mathrm{id}_B$$

In order to conclude that h is the inverse of q, we are left with proving that $q \circ h = \mathrm{id}_{A+_C B}$. We have

$$q \circ h \circ q = q \circ \mathrm{id}_B = q \qquad\qquad q \circ h \circ p = q \circ g \circ f^{-1} = p \circ f \circ f^{-1} = p$$

The following diagram thus commutes:



Since the same diagram, where the vertical arrow $q \circ h$ has been replaced by $\mathrm{id}_{A+_C B}$ also commutes, we deduced by universal property of the pushout that $q \circ h = \mathrm{id}_{A+_C B}$.

3 Monomorphisms

A *monomorphism* is a morphism $f : A \to B$ such that for every morphisms $g_1, g_2 : A' \to A$, we have that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$:

$$A' \underset{g_2}{\overset{g_1}{\rightrightarrows}} A \xrightarrow{f} B$$

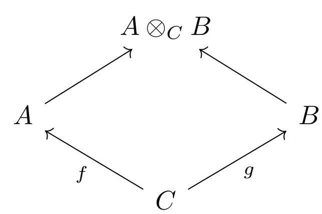
1. What is a monomorphism in **Set**?

   Solution. An injective function.

2. Show that the pullback of a monomorphism along any morphism is a monomorphism.

3. Show that, in **Set**, the pushout of a monomorphism along any morphism is a monomorphism. Does this seem to be true in any category?

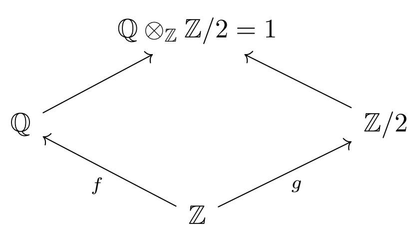
   Solution. It is true in most categories we first think of first, because it is true in ever adhesive category (which includes the case of all toposes). For a counter example, consider the category of commutative rings, where the pushout is given by tensor product:

   when $f : C \to A$ is mono (= injective), we have that the function

   $$\begin{aligned} B &\to A \otimes_C B \\ b &\mapsto 1 \otimes b \end{aligned}$$

   is also a mono when $g : C \to B$ is flat, which is not the case for all ring morphisms. For a concrete counter example take

   The terminal map $\mathbb{Z}/2 \to 1$ is not a monomorphism.

4. Define the dual notion of *epimorphism*. What is an epimorphism in **Set**?

   Solution. A surjective function.

5. In the category of posets, construct a morphism which is both a monomorphism and an epimorphism, but not an isomorphism.

   Solution. Consider the posets $P = (\{x, y\}, \leq)$ with x and y independent, and $Q = (\{x, y\}, \leq)$ with $x \leq y$. The canonical inclusion $P \hookrightarrow Q$ is an example.

4 (Co)monoids in cartesian categories

1. Given a cartesian category $\mathcal{C}$, show that the cartesian product induces a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
2. Generalize the definition of *monoid* to any cartesian category (a monoid in **Set** should be a monoid in the usual sense). When is a monoid commutative?
3. Generalize the notion of morphism of monoid.
4. A *comonoid* in $\mathcal{C}$ is a monoid in $\mathcal{C}^{\mathrm{op}}$. Make explicit the notion of comonoid.
5. What part of the cartesian structure on $\mathcal{C}$ did we really need in order to define the notion of monoid?
6. Show that in a cartesian category every object is a comonoid (with respect to product).
7. Given a category $\mathcal{C}$, show that the category of commutative comonoids and morphisms of comonoids in $\mathcal{C}$ is cartesian.