Cartesian categories

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September 26, 2022

0 Categories

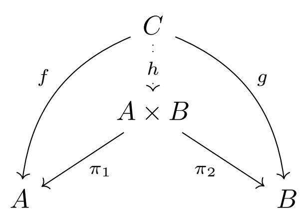
0. Recall the definition of *category* and provide some examples (e.g. **Set**, **Top**, **Vect**, **Grp**, **Rel**, **Cat**, etc.).

1 Cartesian categories

Suppose fixed a category $\mathcal{C}$. A *cartesian product* of two objects A and B is given by an object $A \times B$ together with two morphisms

$$\pi_1 : A \times B \to A \qquad \text{and} \qquad \pi_2 : A \times B \to B$$

such that for every object C and morphisms $f : C \to A$ and $g : C \to B$, there exists a unique morphism $h : C \to A \times B$ making the diagram



commute. We also recall that a *terminal object* in a category is an object 1 such that for every object A there exists a unique morphism $f_A : A \to 1$. A category is *cartesian* when it has finite products, i.e. has a terminal object and every pair of objects admits a product.

1. Suppose that $(E, \leq)$ is a poset. We associate to it category whose objects are elements of E and such that there exists a unique morphism between object a and b iff $a \leq b$. What is a terminal object and a product in this category?

   Solution. A terminal object is an element $b \in E$ such that, for every other element $a \in E$, there is a unique morphism $a \to b$, i.e. $a \leq b$. A terminal object is thus precisely a maximal element of the set.

   Given two elements a and b of E, a product is an element $a \times b$ equipped with two morphisms $\pi_1 : a \times b \to a$ and $\pi_2 : a \times b \to b$ such that for every element c equipped with two morphisms $f : c \to a$ and $f : c \to b$ there exists a unique morphism $h : c \to a \times b$ making some diagrams commute. This is thus precisely an element $a \times b$ such that $a \times b \leq a$ and $a \times b \leq b$ such that for every element c with $c \leq a$ and $c \leq b$, we have $c \leq a \times b$. Otherwise said, $a \times b$ is an infimum of a and b.

2. Show that the category **Set** of sets and functions is cartesian.

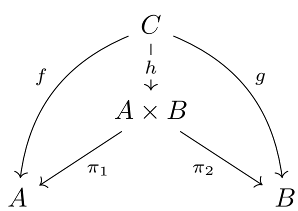
   Solution. Given two sets A and B, we define

   $$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

   and

   $$\begin{aligned} \pi_1 : A \times B &\to A \\ (a,b) &\mapsto a \end{aligned} \qquad\qquad \begin{aligned} \pi_2 : A \times B &\to B \\ (a,b) &\mapsto b \end{aligned}$$

   Given a set C and functions $f : C \to A$ and $g : C \to B$, suppose that there exists a function $h : C \to A \times B$ making the following diagram commute:



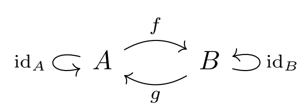
   Given $c \in C$, we have $h(c) \in A \times B$, i.e. $h(c)$ is of the form $h(c) = (a,b)$. The commutation of the left triangle imposes

   $$a = \pi_1(a,b) = \pi_1 \circ h(c) = f(c)$$

   and the one of the right that $b = g(c)$. Therefore, necessarily, we have $h(c) = (f(c), g(c))$ (if h exists it is unique). Conversely, the function h thus defined makes the two triangle commutes (h actually exists).

3. Show that two terminal objects in a category are necessarily isomorphic.

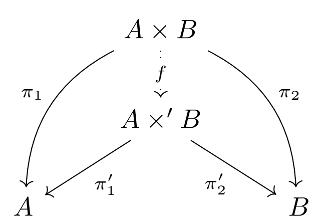
   Solution. Suppose given two terminal objects A and B. Since B is terminal, we have a unique morphism $f : A \to B$ and, since A is terminal, we have a unique morphism $g : B \to A$.



   The composite $g \circ f : A \to A$ and $\mathrm{id}_A : A \to A$ are both morphisms with the same source and A as target: since A is terminal, we therefore have $g \circ f = \mathrm{id}_A$. Similarly, we have $f \circ g = \mathrm{id}_B$ and we deduce that A and B are isomorphic.

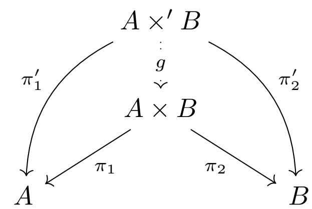
4. Similarly, show that the cartesian product of two objects is defined up to isomorphism.

   Solution. Fix two objects A and B and suppose that they admit two products $(A \times B, \pi_1, \pi_2)$ and $(A \times' B, \pi'_1, \pi'_2)$. From the following diagram,

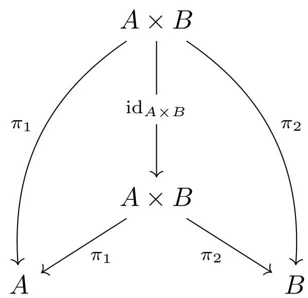


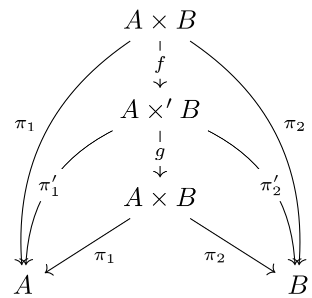
   by definition of the product $A \times' B$, we deduce the existence of a unique morphism $f : A \times B \to A \times' B$ such that $\pi'_1 \circ f = \pi_1$ and $\pi'_2 \circ f = \pi_2$. Similarly, there exists a unique morphism

$g : A \times' B \to A \times B$ such that $\pi_1 \circ g = \pi'_1$ and $\pi_2 \circ g = \pi'_2$:



Now, we have two morphisms from $A \times B$ to $A \times B$, namely $\mathrm{id}_{A\times B}$ and $g \circ f$:





and those make the two triangles commute: we have

$$\pi_1 \circ \mathrm{id}_{A\times B} = \pi_1 \qquad\qquad \pi_2 \circ \mathrm{id}_{A\times B} = \pi_2$$

and

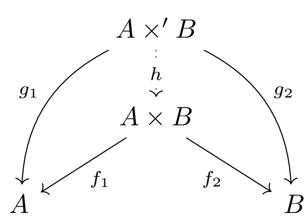
$$\pi_1 \circ g \circ f = \pi'_1 \circ f = \pi_1 \qquad\qquad \pi_2 \circ g \circ f = \pi'_2 \circ f = \pi_2$$

Therefore by universal property of the product, we have $g \circ f = \mathrm{id}_{A\times B}$. Similarly, we have $f \circ g = \mathrm{id}_{A\times' B}$ and thus $A \times B$ and $A \times' B$ are isomorphic.

5. How could you show previous question using question 3.?

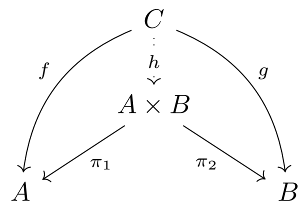
Solution. Suppose fixed two objects A and B of our ambient category $\mathcal{C}$. The idea is to construct another category $\mathcal{D}$ in which a terminal object is precisely a product of A and B. We therefore define the category

- whose objects are triple (C, f_1, f_2) where C is an object of $\mathcal{C}$ and $f_1 : C \to A$ and $f_2 : C \to B$ are morphisms of $\mathcal{C}$,
- a morphism in $\mathcal{D}((C, f_1, f_2), (D, g_1, g_2))$ is a morphism $h : C \to D$ of $\mathcal{C}$ such that $g_1 \circ h = f_1$ and $g_2 \circ h = f_2$:



- identities are identities of $\mathcal{C}$,
- composition is the same as in $\mathcal{C}$.

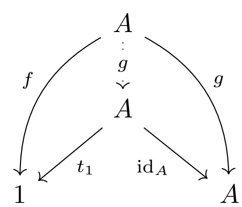
(We leave the reader check the composite is well-defined, i.e. that the composite of two morphisms is still a morphism, and that identities are actually morphisms). A terminal object in $\mathcal{D}$ is an object $(A \times B, \pi_1, \pi_2)$ such that for every other object (C, f, g) there is a unique morphism $h : (C, f, g) \to (A \times B, \pi_1, \pi_2)$ in $\mathcal{D}$, i.e. there exists a unique morphism $h : C \to A \times B$ in $\mathcal{C}$ making the following diagram commute:



i.e. $A \times B$ is a product of A and B. By question 3, two such objects are isomorphic in $\mathcal{D}$, and they will thus be isomorphic in $\mathcal{C}$ since composition and identities in $\mathcal{D}$ are induced by those of $\mathcal{C}$.

6. Show that for every object A of a cartesian category, the objects $1 \times A$, A and $A \times 1$ are isomorphic.

   Solution. Of course the same reasoning "by hand" as above can be performed here. Another way to proceed in order to show that $1 \times A$ and A are isomorphic is to show that (A, t_A, id_A) is a product of 1 and A (where $t_A : 1 \to A$ is the terminal map) and conclude by question 4. Namely, given two morphisms $f : C \to 1$ and $g : C \to A$, we have the morphism g which make the following diagram commute:



   The left triangle commutes because 1 is terminal and therefore $\pi_1 \circ g = f$, and the right triangle commutes by definition of identities: $\mathrm{id}_A \circ g = g$. Conversely, g is the only such morphism by commutation of the right triangle.

7. Show that for every objects A and B, $A \times B$ and $B \times A$ are isomorphic.

8. Show that for every objects A, B and C, $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic.

2 Examples of cartesian categories

1. Show that the category **Rel** of sets and relations is cartesian.

   Solution. Let us first properly define the category **Rel**. An object is a set, a morphism in $\mathbf{Rel}(A, B)$ is a relation between A and B, i.e. a subset of $A \times B$:

$$\mathbf{Rel}(A, B) = \mathcal{P}(A, B)$$

   Composition of $R : A \to B$ and $S : B \to C$, i.e. $R \subseteq A \times B$ and $S \subseteq B \times C$, is the following subset of $A \times C$:

$$S \circ R = \{(a, c) \mid \exists b \in B.(a, b) \in R \land (b, c) \in S\}$$

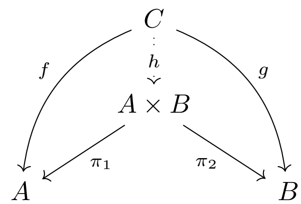
   the identity on A is the diagonal subset of $A \times A$:

$$\mathrm{id}_A = \{(a, a) \mid a \in A\}$$

   It turns out that that product of A and B is the disjoint union $A \sqcup B$ of the sets A and B (we write $\iota_1(a)$, resp. $\iota_2(b)$, for the canonical injections of an element $a \in A$, resp. $b \in B$, in $A \sqcup B$). The projections are

$$\pi_1 = \{(\iota_1(a), a) \mid a \in A\} \subseteq (A \sqcup B) \times A \qquad \pi_2 = \{(\iota_2(b), b) \mid b \in B\} \subseteq (A \sqcup B) \times B$$

Given a pair of morphisms $f : C \to A$ and $g : C \to B$,



one can check that the unique mediating morphism $h : C \to A \times B$ is

$$h = \{(c, \iota_1(a)) \mid (c, a) \in f\} \cup \{(c, \iota_1(b)) \mid (c, b) \in g\} \subseteq C \times (A \sqcup B)$$

2. We write **Vect** for the category of $\mathbb{k}$-vector spaces (where $\mathbb{k}$ is a fixed field) and linear functions. Show that this category is cartesian. Given a basis for A and B, describe a basis for $A \times B$.

3. Show that the category **Cat** is cartesian.

3 Cartesian product as a functor

1. Recall the definition of a *functor* and provide some examples.

2. Define the category **Cat** of categories and functors.

3. Given a cartesian category $\mathcal{C}$, show that the cartesian product induces a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.