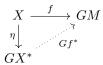
Adjunctions: an alternative formulation

Samuel Mimram samuel.mimram@lix.polytechnique.fr http://lambdacat.mimram.fr

October 11, 2021

1 An alternative formulation of adjunctions

1. Given a set X and a monoid M, show that for every function $f: X \to GM$ there exists a unique morphism of monoids $f^*: X^* \to M$ such that the following diagram commutes:



where $G: \mathbf{Mon} \to \mathbf{Set}$ is the forgetful functor.

Our goal is to show that this situation generalizes to any adjunction. Suppose given a functor $G : \mathcal{D} \to \mathcal{C}$ between categories \mathcal{C} and \mathcal{D} .

2. Show that, given objects A in \mathcal{C} and F_A in \mathcal{D} , every map

$$\eta: A \to GF_A$$

induces a natural transformation

$$\phi: \mathcal{D}(F_A, -) \Rightarrow \mathcal{C}(A, G-): \mathcal{D} \to \mathbf{Set}$$

We say that a pair (F_A, η) with $F_A \in \mathcal{D}$ and $\eta : A \to GF_A$ represents the functor $\mathcal{C}(A, G_-) : \mathcal{D} \to \mathbf{Set}$ when the induced natural transformation $\phi : \mathcal{D}(F_A, -) \Rightarrow \mathcal{C}(A, G_-)$ is a bijection, i.e. when $\phi_B : \mathcal{D}(F_A, B) \Rightarrow \mathcal{C}(A, GB)$ for every $B \in \mathcal{D}$.

3. Show that (F_A, η) represents the functor $\mathcal{C}(A, G_-) : \mathcal{D} \to \mathbf{Set}$ precisely when for every $B \in \mathcal{D}$ and morphism

 $f: A \to GB$

there exists a unique morphism

 $f^*: F_A \to B$

such that the following diagram commutes:

$$\begin{array}{c} A \xrightarrow{f} GB \\ \eta \downarrow \\ GF_A \end{array} \xrightarrow{\gamma} GB \\ GF^* \\ GF_A \end{array}$$

4. Now, suppose that for every object A of C, there exists a pair (F_A, η_A) representing the functor $C(A, G-) : D \to \mathbf{Set}$. Given $f : A \to A'$, construct a morphism $F_f : F_A \to F_{A'}$ in D such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GF_A \\ f & & & \downarrow^{GF_J} \\ A' & \xrightarrow{\eta_{A'}} & GF_{A'} \end{array}$$

5. Construct a functor

$$F: \mathcal{C} \to \mathcal{D}$$
$$A \mapsto F_A$$

6. Construct a natural bijection

$$\phi_{A,B}: \mathcal{D}(FA,B) \to \mathcal{C}(A,GB)$$

What have we just shown?

- 7. Conversely, given an adjunction $F : C \dashv D : G$, show that for every object A of C there is a pair (FA, η_A) which represents the functor $C(A, G-) : D \rightarrow \mathbf{Set}$.
- 8. Apply the result to show that the forgetful functor $U: \mathbf{Mon} \to \mathbf{Set}$ admits a left adjoint.

2 Representable functors

A coproduct in a category \mathcal{C} is a product in \mathcal{C}^{op} .

- 1. Provide an explicit definition of coproducts.
- 2. What is a coproduct in **Set**, **Rel**, **Vect**?
- 3. Given an object A of a category \mathcal{C} , describe the functor $\operatorname{Hom}(A, -) : \mathcal{C} \to \operatorname{\mathbf{Set}}$.

A functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* when it is of the above form, i.e., more precisely, there exists an object A and a natural isomorphism $\phi : \operatorname{Hom}(A, -) \Rightarrow F$. We assume that this notion coincides with the one of previous section.

4. In a category C, fix two objects A and B and consider the functor

$$F: \mathcal{C} \to \mathbf{Set}$$
$$C \mapsto \operatorname{Hom}(A, C) \times \operatorname{Hom}(B, C)$$

Complete the definition of the functor. Show that this functor is representable if and only if A and B admit a coproduct.

- 5. In the category **Vect**, given objects A and B, what is a representation of the functor $\text{Bilin}_{A,B}$: **Vect** \rightarrow **Set** which to a vector space C associates the set of bilinear functions $(A, B) \rightarrow C$?
- 6. Construct the diagonal functor

$$\Delta: \mathcal{C} \to \mathcal{C} \times \mathcal{C}$$
$$A \mapsto (A, A)$$

and show that it has a left adjoint precisely when \mathcal{C} has coproducts.

- 7. When does a category have products?
- 8. Show that the notion of representability given in this part coincides with the one given in previous part. More precisely, given a functor $F : \mathcal{C} \to \mathbf{Set}$ and an object $A \in \mathcal{C}$, show that there is a bijection between
 - natural isomorphisms $\phi : \operatorname{Hom}(A, -) \Rightarrow F$,
 - elements $e \in FA$ such that for every $B \in \mathcal{C}$ and $f' \in FB$, there exists a unique $f : A \to B$ such that (Ff)(e) = f'.

Explain how the second condition coincides with the definition of representability given in previous part.