Travaux Dirigés
Equalizers, epi-mono factorization, first-order logic

λ-calculs et catégories

1 Equalizers and coequalizers

In this exercise, we study the notion of equalizer (called "égalisateur" in French) and its dual notion of coequalizer. Suppose given a pair of coinitial and cofinal arrows

$$f, g : X \rightrightarrows Y$$

in a category $\mathscr{C}$. An equalizer of f and g is an arrow $m : E \longrightarrow X$ such that

$$f \circ m = g \circ m$$

and such that, for every arrow $n : F \longrightarrow X$ such that

$$f \circ n = g \circ n$$

there exists a unique arrow $h : F \longrightarrow E$ such that

$$n = m \circ h.$$

§1. Show that every pair of functions $f, g : X \longrightarrow Y$ has an equalizer $m : E \longrightarrow X$ in the category Sets and describe this equalizer.

§2. Show that when it exists in a category $\mathscr{C}$, the equalizer $m : E \longrightarrow X$ of a pair of arrows $f, g : X \longrightarrow Y$ is a mono.

§3. Formulate the dual notion of coequalizer $e : Y \longrightarrow Q$ of two arrows

$$f, g : X \rightrightarrows Y$$

in a category $\mathscr{C}$.

§4. Show that when it exists in a category $\mathscr{C}$, the coequalizer $e : Y \longrightarrow Q$ of two arrows $f, g : X \longrightarrow Y$ is an epi.

§5. Show that every pair of functions $f, g : X \longrightarrow Y$ has a coequalizer $e : Y \longrightarrow Q$ in the category Sets and describe this coequalizer.

§6. An epi $e : Y \longrightarrow Q$ is called *regular* when there exists a pair of arrows $f, g : X \longrightarrow Y$ such that e is a coequalizer of f and g as in the diagram below:

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{e} Q$$

Show that every surjective function $e : A \longrightarrow B$ is a regular epi in the category Sets.

2 Epi-mono factorization

An arrow $f : A \to B$ is orthogonal to an arrow $g : X \to Y$ in a category $\mathscr{C}$ when for every pair of arrows $u : A \to X$ and $v : B \to Y$ making the diagram below commute

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] & Y \end{tikzcd}$$

there exists a unique arrow $h : B \longrightarrow X$ making the diagram below commute

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

in the sense that

$$u = h \circ f \qquad \text{and} \qquad v = g \circ h.$$

We write in that case

$$f \perp g.$$

A factorisation system $(\mathcal{E}, \mathcal{M})$ is a pair of collections $\mathcal{E}$ and $\mathcal{M}$ of arrows of the category $\mathscr{C}$ satisfying the three properties below:

A. every arrow

$$X \xrightarrow{f} Y$$

of the category $\mathscr{C}$ factors as

$$X \xrightarrow{e} U \xrightarrow{m} Y$$

where $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

B. every arrow $e \in \mathcal{E}$ is orthogonal to every arrow $m \in \mathcal{M}$, what we write

$$\mathcal{E} \perp \mathcal{M}.$$

C. both collections $\mathcal{E}$ and $\mathcal{M}$ are closed under composition and contain the isos.

The purpose of the exercise is to show that the category **Sets** is equipped with a factorisation system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ and $\mathcal{M}$ are respectively the collections of surjective and of injective functions.

§1. Show that every function

$$X \xrightarrow{f} Y$$

factors as

$$X \xrightarrow{e} U \xrightarrow{m} Y$$

where $e : X \longrightarrow U$ is a surjective function and $m : U \longrightarrow Y$ is an injective function.

§2. Show that every surjective function $e : A \to B$ is orthogonal to every injective function $m : X \to Y$ in the category **Sets**.

§3. Deduce from §1 and §2 that $(\mathcal{E}, \mathcal{M})$ defines a factorization system in **Sets**, where $\mathcal{E}$ and $\mathcal{M}$ are respectively the collections of surjective and injective functions in **Sets**.

§4. Suppose given a category $\mathscr{C}$ equipped with a factorization system $(\mathcal{E}, \mathcal{M})$ and a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "e_1"] \arrow[d, "u"'] & U_1 \arrow[r, "m_1"] & Y_1 \arrow[d, "v"] \\ X_2 \arrow[r, "e_2"'] & U_2 \arrow[r, "m_2"'] & Y_2 \end{tikzcd}$$

where $e_1, e_2 \in \mathcal{E}$ and $m_1, m_2 \in \mathcal{M}$. Show that there exists a unique arrow $h : U_1 \to U_2$ making the diagram below commute:

$$\begin{tikzcd} X_1 \arrow[r, "e_1"] \arrow[d, "u"'] & U_1 \arrow[r, "m_1"] \arrow[d, "h"] & Y_1 \arrow[d, "v"] \\ X_2 \arrow[r, "e_2"'] & U_2 \arrow[r, "m_2"'] & Y_2 \end{tikzcd}$$

in the category $\mathscr{C}$.

§5. Suppose given a category $\mathscr{C}$ whose collections $\mathcal{E}$ of epis and $\mathcal{M}$ of monos define a factorization system $(\mathcal{E}, \mathcal{M})$. Show that every arrow $f : X \to Y$ induces a subobject $(U, m) \in \mathbf{Sub}(Y)$ defined as the unique subobject of Y such that the arrow $f : X \to Y$ factors as

$$X \xrightarrow{e} U \xrightarrow{m} Y$$

for a given epi $e : X \to U$. Show that in the case of the category **Sets**, the construction associates to every function $f : X \to Y$ its image in the set Y. For that reason, one often calls the subobject $m : U \to Y$ the image of the arrow $f : X \to Y$.

§6. Suppose that we are still in the situation of §5. Show that every arrow $f : A \to B$ of the category $\mathscr{C}$ induces a monotone function

$$f_* \quad : \quad \mathbf{Sub}(A) \quad \longrightarrow \quad \mathbf{Sub}(B)$$

which transports every subobject (U, m) to the image $f_*(U, m)$ of the composite arrow

$$U \xrightarrow{m} A \xrightarrow{f} B$$

using the notion of "image" of an arrow $f \circ m : U \to B$ formulated in §5.

§7. Show that in the particular case $\mathscr{C} = \mathbf{Sets}$, one associates in this way to every function $f : A \to B$ the monotone function

$$f_* \quad : \quad \mathscr{P}(A) \quad \longrightarrow \quad \mathscr{P}(B)$$

which transports every subset $U \subseteq A$ to its image $f(U) \subseteq B$.

§8. Suppose that we are in the situation of §5 and that the category $\mathscr{C}$ has moreover pullbacks. We have seen in the previous TD that every arrow

$$f \,:\, A \longrightarrow B$$

induces in that case a monotone function

$$f^* \quad : \quad \mathbf{Sub}(B) \quad \longrightarrow \quad \mathbf{Sub}(A)$$

defined by "pulling back" subobjects $(V, n) \in \mathbf{Sub}(B)$ into subobjects $(U, m) \in \mathbf{Sub}(A)$. Show that the monotone function f_* is left adjoint to f^* in the sense that

$$f_*(U, m) \leq (V, n) \quad \iff \quad (U, m) \leq f^*(V, n)$$

for every pair of subobjects $(U, m) \in \mathbf{Sub}(A)$ and $(V, n) \in \mathbf{Sub}(B)$.

3 Application to first-order logic

Consider a family of sets $X_1, \ldots, X_n$ and their cartesian product $\Gamma = X_1 \times \ldots \times X_n$. Every first-order formula φ with free variables $x_1, \ldots, x_n$ induces a subset

$$[\,\varphi\,] \quad \subseteq \quad \Gamma$$

consisting of all the elements $(x_1, \ldots, x_n) \in \Gamma$ satisfying the formula φ. Note that the interpretation $[\,\varphi\,]$ of the formula φ can be also seen as an element of the powerset:

$$[\,\varphi\,] \quad \in \quad \mathscr{P}(\Gamma).$$

§1. Every set X induces a function

$$\pi : \Gamma \times X \longrightarrow \Gamma$$

defined by the first projection. Given a first-order formula φ with free variables $x_1, \ldots, x_n$, show that the subset

$$\pi^*[\,\varphi\,] \quad = \quad \{(x_1, \ldots, x_n, x) \in \Gamma \times X \,|\, \varphi(x_1, \ldots, x_n)\}$$

coincides with the interpretation of the same formula φ seen as a formula with free variables $x_1, \ldots, x_n, x$.

§2. Given a first-order formula ψ with free variables $x_1, \ldots, x_n, x$ and with interpretation

$$[\,\psi\,] \in \mathscr{P}(\Gamma \times X)$$

show that

$$\pi_*[\,\psi\,] \quad = \quad \{\ (x_1, \ldots, x_n) \in \Gamma \mid \exists x \in X, \psi(x_1, \ldots, x_n, x)\ \}.$$

coincides with the interpretation $[\,\exists_{x\in X}\,\psi\,]$ of the formula $\exists_{x\in X}\,\psi$.

§3. From this, deduce that

$$[\,\exists_{x\in X}\,\psi\,] \leq_\Gamma [\,\varphi\,] \quad \iff \quad [\,\psi\,] \leq_{\Gamma\times X} [\,\varphi\,]$$

where we write $U \leq_\Gamma V$ for the inclusion $U \subseteq V$ between subsets $U, V \in \mathscr{P}(\Gamma)$. Justify this equivalence from the point of view of first-order logic.