Travaux Dirigés
An equivalent formulation of adjunctions
Application to cartesian and to cartesian closed categories

λ-calculs et catégories

1 An equivalent formulation of adjunctions

§1. Suppose given a functor

$$R \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{A}$$

between two categories $\mathscr{A}$ and $\mathscr{B}$. Show that every map

$$\eta \quad : \quad A \quad \longrightarrow \quad R(L_A)$$

from an object A of the category $\mathscr{A}$ into the image by R of an object noted L_A of the category $\mathscr{B}$ induces a function

$$\varphi_B \quad : \quad \mathscr{B}(L_A, B) \quad \longrightarrow \quad \mathscr{A}(A, RB)$$

for every object B of the category $\mathscr{B}$.

§2. Show that the family of functions φ_B is natural in B in the sense that it defines a natural transformation

$$\varphi \quad : \quad \mathscr{B}(L_A, -) \ \Rightarrow \ \mathscr{A}(A, R-) \quad : \quad \mathscr{B} \ \longrightarrow \ \mathbf{Set}$$

between the set-valued functors

$$\mathscr{B}(L_A, -) = B \mapsto \mathscr{B}(L_A, B) \qquad\qquad \mathscr{A}(A, R-) = B \mapsto \mathscr{A}(A, RB)$$

from the category $\mathscr{B}$ to the category Set of sets and functions.

§3. One says that a pair (L_A, η) consisting of an object L_A of the category $\mathscr{B}$ and of a map

$$\eta \quad : \quad A \ \longrightarrow \ R(L_A)$$

represents the set-valued functor

$$\mathscr{A}(A, R-) \quad : \quad \mathscr{B} \ \longrightarrow \ \mathbf{Set}$$

when the function φ_B defined in §1 is a bijection

$$\varphi_B \quad : \quad \mathscr{B}(L_A, B) \quad \cong \quad \mathscr{A}(A, RB)$$

for every object B of the category $\mathscr{B}$. Show that the pair (L_A, η) represents the set-valued functor $\mathscr{A}(A, R-)$ precisely when the following property holds: for every object B and for every map

$$f \quad : \quad A \quad \longrightarrow \quad RB$$

there exists a unique map

$$f^\dagger \quad : \quad L_A \quad \longrightarrow \quad B$$

such that the diagram below commutes:

$$\begin{tikzcd} & RB \\ A \arrow[ur, "f"] \arrow[r, "\eta"'] & R(L_A) \arrow[u, "R(f^\dagger)"'] \end{tikzcd}$$

§4. We suppose from now on that every object A of the category $\mathscr{A}$, there exists a pair (L_A, η_A) which represents the set-valued functor $\mathscr{A}(A, R-)$. For every map $f : A_1 \to A_2$ of the category $\mathscr{A}$, construct a map

$$L_f \quad : \quad L_{A_1} \quad \longrightarrow \quad L_{A_2}$$

of the category $\mathscr{B}$ such that the diagram below commutes:

$$\begin{tikzcd} A_2 \arrow[r, "\eta_{A_2}"] & R(L_{A_2}) \\ A_1 \arrow[u, "f"] \arrow[r, "\eta_{A_1}"] & R(L_{A_1}) \arrow[u, "R(L_f)"'] \end{tikzcd}$$

§5. Deduce from the construction in §4 that the function $A \mapsto L_A$ defines a functor

$$L \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

and a family of bijections

$$\phi_{A,B} \quad : \quad \mathscr{B}(LA, B) \quad \cong \quad \mathscr{A}(A, RB)$$

Show moreover that this family of bijections ϕ is natural in A and B.

§6. Conclude that given a functor $R : \mathscr{B} \to \mathscr{A}$, the existence of a pair (LA, η_A) representing the set-valued functor

$$\mathscr{A}(A, R-) \quad : \quad \mathscr{B} \longrightarrow \mathbf{Set}$$

for every object A of the category $\mathscr{A}$ implies the existence of a left adjoint functor $L : \mathscr{A} \to \mathscr{B}$ to the functor $R : \mathscr{B} \to \mathscr{A}$.

§7. Conversely, show that whenever we have a pair of adjoint functors

$$L : \mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} : R$$

every object A of the category $\mathscr{A}$ comes equipped with a pair (LA, η_A) which represents the set-valued functor

$$\mathscr{A}(A, R-) \quad = \quad B \mapsto \mathscr{A}(A, RB) \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathbf{Set}.$$

§8. Apply the results of §6 to establish that the forgetful functor

$$R = U \quad : \quad \mathbf{Mon} \longrightarrow \mathbf{Set}$$

from the category $\mathscr{B} = \mathbf{Mon}$ of monoids and homomorphisms to the category $\mathscr{A} = \mathbf{Set}$ of sets and functions has the free monoid functor

$$L = A \mapsto A^* \quad : \quad \mathbf{Set} \to \mathbf{Mon}$$

as left adjoint.

2 Application to cartesian closed categories

§1. Show that every adjoint pair

$$L : \mathscr{A} \rightleftarrows \mathscr{B} : R$$

where L is left adjoint to R induces an adjoint pair

$$R^{op} : \mathscr{B}^{op} \rightleftarrows \mathscr{A}^{op} : L^{op}$$

where the functor L^{op} is right adjoint to R^{op}.

§2. From this and exercise 1, deduce that a functor $L : \mathscr{A} \to \mathscr{B}$ has a right adjoint precisely when for every object B of the category $\mathscr{C}$ there exists a pair (RB, ε_B) consisting of an object RB of the category $\mathscr{A}$ and of a map

$$\varepsilon_B \quad : \quad L(RB) \quad \longrightarrow \quad B$$

such that the following property holds: for every object A of the category $\mathscr{A}$ and for every map

$$f \quad : \quad LA \quad \longrightarrow \quad B$$

there exists a unique map

$$f^\ddagger \quad : \quad A \quad \longrightarrow \quad RB$$

such that the diagram below commutes:

$$\begin{tikzcd}
L(RB) \arrow[r, "\varepsilon_B"] & B \\
L(A) \arrow[u, "L(f^\ddagger)"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Terminology: one says in that case that the pair (RB, ε_B) *represents* the functor

$$\mathscr{B}(L-, B) \quad = \quad A \mapsto \mathscr{B}(LA, B) \quad : \quad \mathscr{A}^{op} \quad \longrightarrow \quad \mathbf{Set}.$$

§3. Apply this alternative formulation of adjunctions to the functor

$$L \quad = \quad B \mapsto A \times B \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

associated to an object A of a cartesian category $\mathscr{C}$ with

– the object RB noted $A \Rightarrow B$
– the map $\varepsilon_B : L(RB) \to B$ noted $\mathrm{eval}_B : A \times (A \Rightarrow B) \to B$

Show that one recovers in this way the equivalence between the two formulations of cartesian closed category given in the course.

3 Application to cartesian categories

As we explained during the course, the category $\mathbb{1}$ with one object $*$ and one map (=the identity map) is terminal in the category **Cat**. This means that for every category $\mathscr{C}$, there exists a unique functor

$$! \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathbb{1}. \tag{1}$$

At the same time, every object A of the category $\mathscr{C}$ gives rise to a functor, also noted

$$A \quad : \quad \mathbb{1} \quad \longrightarrow \quad \mathscr{C} \tag{2}$$

which transports the unique object $*$ of the category $\mathbb{1}$ to the object A.

§1. Show that an object A is terminal in the category $\mathscr{C}$ if and only if the associated functor (2) is right adjoint to the canonical functor (1).

§2. Show that an object A is initial in the category $\mathscr{C}$ if and only if the associated functor (2) is left adjoint to the canonical functor (1).

§3. Show that the operation $A \mapsto (A, A)$ which transports every object A of the category $\mathscr{C}$ to the object (A, A) of the category $\mathscr{C} \times \mathscr{C}$ defines a functor

$$\Delta \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C} \times \mathscr{C}.$$

This functor Δ is called the diagonal functor of the category $\mathscr{C}$.

§4. Suppose given a pair of objects A, B in a category $\mathscr{C}$. Show that a triple

$$(A \times B, \pi_1, \pi_2)$$

consisting of an object $A \times B$ and of two maps

$$\pi_1 : A \times B \to A \qquad\qquad \pi_2 : A \times B \to B$$

defines a cartesian product of A and B precisely when the pair $(A \times B, \pi)$ consisting of the object $A \times B$ and of the map in the category $\mathscr{C} \times \mathscr{C}$

$$\pi = (\pi_1, \pi_2) \quad : \quad \Delta(A \times B) \quad \longrightarrow \quad (A, B)$$

represents the functor

$$\mathscr{C} \times \mathscr{C}(\Delta -, (A, B)) \quad : \quad \mathscr{C}^{op} \quad \longrightarrow \quad \mathbf{Set}.$$

§5. From this and the exercise 2, deduce that a category $\mathscr{C}$ is cartesian precisely when the two canonical functors

$$! \; : \; \mathscr{C} \; \longrightarrow \; \mathbb{1} \qquad\qquad \Delta \; : \; \mathscr{C} \; \longrightarrow \; \mathscr{C} \times \mathscr{C}$$

have a right adjoint.