Algebras for a monad

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1 Adjunctions for a monad

A *monad* (T, μ, η) is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ equipped with two natural transformations $\mu : T \circ T \to T$ and $\eta : \mathrm{id}_{\mathcal{C}} \to T$ such that, for every $A \in \mathcal{C}$,

$$\begin{tikzcd} TTTA \arrow[r, "T\mu_A"] \arrow[d, "\mu_{TA}"'] & TTA \arrow[d, "\mu_A"] \\ TTA \arrow[r, "\mu_A"'] & TA \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} TA \arrow[r, "T\eta_A"] \arrow[dr, "\mathrm{id}_{TA}"'] & TTA \arrow[d, "\mu_A"] & TA \arrow[l, "\eta_{TA}"'] \arrow[dl, "\mathrm{id}_{TA}"] \\ & TA & \end{tikzcd}$$

An *algebra* for a monad (T, μ, η) on a category $\mathcal{C}$ is a pair (A, a) with $a : TA \to A$ such that

$$\begin{tikzcd} TTA \arrow[r, "Ta"] \arrow[d, "\mu_A"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & TA \arrow[d, "a"] \\ & A \end{tikzcd}$$

A morphism of T-algebras $f : (A, a) \to (B, b)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Given a category $\mathcal{C}$ and T a monad on $\mathcal{C}$, we write $\mathcal{C}^T$ for the category of T-algebras.

1. Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ has a left adjoint.
   What is the induced monad T on $\mathbf{Set}$? What is its category of algebras?

2. What are the algebras of the finite powerset monad on $\mathbf{Set}$?

3. Given a right adjoint functor $U : \mathcal{D} \to \mathcal{C}$, show that the category $\mathcal{D}$ is not always isomorphic to the category of algebras for the induced monad (hint: consider the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$).

4. Given a monad $T : \mathcal{C} \to \mathcal{C}$, show that the forgetful functor $\mathcal{C}^T \to \mathcal{C}$ has a left adjoint, and that the induced monad is T.

5. Construct the Kleisli category $\mathcal{C}_T$ associated to a monad. Show that the "forgetful functor" $\mathcal{C}_T \to \mathcal{C}$ has a left adjoint and that the induced monad is T.

6. [Optional] Fix a monad T on $\mathcal{C}$ and consider the category whose objects are triples $(\mathcal{D}, F, G)$ with $F : \mathcal{C} \to \mathcal{D}$ left adjoint to $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = T$, and whose morphisms $H : (\mathcal{D}, F, G) \to (\mathcal{D}', F', G')$ are functors $H : \mathcal{D} \to \mathcal{D}'$ such that $H \circ F = F'$ and $G' \circ H = G$. Show that the adjunctions associated to $\mathcal{C}^T$ and $\mathcal{C}_T$ are respectively terminal and initial in this category.

2 The Kleisli category as the category of free algebras

Our goal here is to show that the Kleisli category $\mathcal{C}_T$ associated to a monad T on $\mathcal{C}$ is the category of free algebras. A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* (resp. *faithful*) when for every pair of objects A and B, the function

$$F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(A, B)$$

is surjective (resp. injective).

7. Show that the "free algebra" functor $F : \mathcal{C}_T \to \mathcal{C}^T$ is full and faithful.

An *equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ consists in functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq \mathrm{id}_{\mathcal{C}}$ and $F \circ G \simeq \mathrm{id}_D$.

8. Show that an equivalence of categories is the same as a functor $F : \mathcal{C} \to \mathcal{D}$ which is essentially surjective (every object of $\mathcal{D}$ is isomorphic to one in the image of F) and full and faithful.

9. Show that the category $\mathcal{C}_T$ is equivalent to the full subcategory of $\mathcal{C}^T$ whose objects are free algebras.