Algebras

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1 Algebras for an endofunctor

An *algebra* for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is a pair (A, f) where A is an object of $\mathcal{C}$ and $f : FA \to A$ a morphism of $\mathcal{C}$. A morphism $h : (A, f) \to (B, g)$ between two such algebras consists of a morphism $h : A \to B$ such that

$$\begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "f"'] & FB \arrow[d, "g"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

In the following, we mostly consider algebras in **Set**.

1. Define inductively the functions
   - `length : 'a list -> int` giving the length of a list,
   - `map : ('a -> 'b) -> 'a list -> 'b list` applying a function to all elements of a list,
   - `double : 'a list -> 'a list` which duplicates every successive element, for instance `double [1;2;3] = [1;1;2;2;3;3]`.
2. Suppose given a type `'a ilist` of infinite lists with elements of type `'a`. Define coinductively
   - `odd : 'a ilist -> 'a ilist` keeping elements of a list at odd positions,
   - `merge : 'a ilist -> 'a ilist -> 'a ilist` taking alternatively elements from one of two lists.
3. Show that $[0, S] : 1 + \mathbb{N} \to \mathbb{N}$ is an initial algebra for the endofunctor $T(X) = 1 + X$ of **Set**.
4. Use this fact to define the function $f : \mathbb{N} \to \mathbb{Q}$ such that $f(n) = 2^{-n}$.
5. Show that two initial algebras of an endofunctor are isomorphic (via morphisms of algebras).
6. Show that an initial algebra $f : FA \to A$ of an endofunctor F is an isomorphism.
7. Solve the equation $x = 1 + ax$ and develop the solution in power series.
8. Show that the set $A^* = \biguplus_{n \in \mathbb{N}} A^n$, which can be seen as the set of lists of elements of A, is an initial algebra for $T(X) = 1 + A \times X$.
9. Use this fact to define the length function $\ell : A^* \to \mathbb{N}$ and the double function $d : A^* \to A^*$. Show that $\ell \circ d(l) = 2\ell(l)$ for every $l \in A^*$.
10. Explain briefly how we could interpret simple inductive types of OCaml by using initial algebras.
11. What is the initial algebra for $T(X) = 1 + X \times X$? For $T(X) = X^*$? For $T(X) = A \times X$?

2 Coalgebras for an endofunctor

A *coalgebra* for $F : \mathcal{C} \to \mathcal{C}$ is a pair (A, f) with $f : A \to FA$. Morphisms are defined similarly as previously.

1. Show that the set $A^{\mathbb{N}}$ of *streams* is a final coalgebra for the endofunctor $T(X) = A \times X$.
2. Use this to define,
   - given $a \in A$, the constant stream equal to a,
   - the function $\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ which to n associates the stream $(n, n+1, n+2, \ldots)$,
   - the function $A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$ which merges two streams,
   - the functions $A^{\mathbb{N}} \to A^{\mathbb{N}}$ keeping even and odd elements.
3. Show that final coalgebras are unique up to isomorphism and are isomorphisms.
4. Show that $merge(even(l), odd(l)) = l$ for every $l \in A^{\mathbb{N}}$.

A *bisimulation* on $A^{\mathbb{N}}$ is a relation $R \subseteq A^{\mathbb{N}} \times A^{\mathbb{N}}$ such that $R(x :: l, x' :: l')$ implies $x = x'$ and $R(l, l')$. The *coinductive proof principle* says that if $R(l, l')$ for some bisimulation R then $l = l'$.

5. Assuming this principle, show again the result of previous question.
6. Show the coinductive proof principle (hint: show that R has a coalgebra structure).
7. Generalize the coinductive proof principle to an arbitrary endofunctor.
8. What is the final coalgebra of $T(X) = 1 + A \times X$? of $T(X) = 1 + X$?
9. Show that automatas can be seen as coalgebras.

References

[1] B. Jacobs and J. Rutten. An introduction to (co)algebra and (co)induction. 2011.