TD6 – Presheaves as cocompletion

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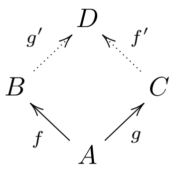
1 Pushouts and coequalizers

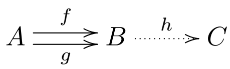
A *pushout* of two coinitial morphisms $f : A \to B$ and $g : A \to C$ consists of an object D together with two morphisms $g' : C \to D$ and $f' : B \to D$ such that $g' \circ f = f' \circ g$



and for every pair of morphisms $g' : B \to D''$ and $f'' : C \to D''$ there exists a unique $h : D \to D'$ such that $h \circ g' = g''$ and $h \circ f' = f''$.

1. What is a pushout in **Top**? In **Set**?

A *coequalizer* of two morphisms $f, g : A \to B$ consists of a morphism $h : B \to C$ such that $h \circ f = h \circ g$

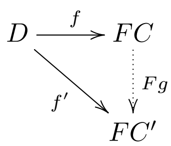


and for every morphism $h' : B \to C'$ such that $h' \circ f = h' \circ g$ there exists a unique $i : C \to C'$ such that $i \circ h = h'$.

2. What is a coequalizer in **Top**? In **Set**? How can we encode the quotient of a set by an equivalence relation as a coequalizer?
3. Show that a category with coproducts and coequalizers has pushouts.

2 Colimits

Suppose given a functor $F : \mathcal{C} \to \mathcal{D}$ and D and object of $\mathcal{D}$. An *universal arrow* from D to F is given by a pair (C, f) where C is an object of $\mathcal{C}$ and $f : D \to FC$ is a morphism in $\mathcal{D}$ such that for every other such pair (C', f') with $f' : D \to FC'$, there exists a unique morphism $g : C \to C'$ of $\mathcal{C}$ such that $Fg \circ f = f'$.



1. Suppose that $U : \mathcal{D} \to \mathcal{C}$ is a functor admitting a left adjoint $F : \mathcal{C} \to \mathcal{D}$. Show that for every object C of $\mathcal{C}$, (FC, η_C) is a universal arrow from C to U. What does this mean in the case of the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$?

Suppose given two categories $\mathcal{J}$ and $\mathcal{C}$. The *diagonal functor* $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ is such that

- given $C \in \mathcal{C}$, $\Delta(C)$ sends every object of $\mathcal{J}$ to C and every morphism of $\mathcal{J}$ to id_C,
- given $f : C \to D \in \mathcal{C}$, $\Delta(f)$ is the natural transformation whose components are f.

The *colimit* of a functor $F : \mathcal{J} \to \mathcal{C}$ is a universal arrow from F to Δ.

2. What is the colimit of a functor F in the case where $\mathcal{J}$ is the category with two objects and their respective identities?
3. What is the colimit of a functor F in the case where $\mathcal{J}$ is the empty category?
4. Express the notion of pushout as a colimit.

5. Show that any graph can be obtained as the colimit of a functor $F : \mathcal{J} \to \mathbf{Graph}$ such that the image of an object is either G_0 (the graph with one vertex and no edge) or G_1 (the graph with two vertices and one edge between them).

6. Show that a left adjoint preserves colimits.

7. Show that in a cartesian closed category with finite colimits, we have

$$A \times (B + C) \cong (A \times B) + (A \times C) \qquad \text{and} \qquad A \Rightarrow (B \times C) \cong (A \Rightarrow B) \times (A \Rightarrow C)$$

3 Presheaf categories as free cocompletions

1. What are coproducts, pushouts and equalizers in the category of graphs?

2. Explain why every presheaf category is complete and cocomplete (assuming this for **Set**).

3. Describe a functor $I : \mathcal{G} \to \mathbf{Top}$ sending 0 to the point and 1 to the standard interval.

4. Use this functor in order to build a *nerve* functor $N_I : \mathbf{Top} \to \hat{\mathcal{G}}$ associating a graph to every topological space.

To any presheaf $P \in \hat{\mathcal{C}}$, we can associate a *category of elements* whose

- objects are pairs (A, a) with $A \in \mathcal{C}$ and $a \in P(A)$,
- and morphisms $f : (A, a) \to (B, b)$ are morphisms $f : A \to B$ of $\mathcal{C}$ such that $P(f)(b) = a$.

We write $\pi_P : \mathrm{El}(P) \to \mathcal{C}$ for the first projection functor. We define the *geometric realization* functor by

$$R_I(P) \quad = \quad \operatorname{colim}(\mathrm{El}(P) \xrightarrow{\pi_P} \mathcal{G} \xrightarrow{I} \mathbf{Top})$$

5. Compute the geometric realization of the graph $\cdot \rightrightarrows \cdot \longrightarrow \cdot$.

6. Show that R_I is left adjoint to N_I.

7. Notice that the above proofs could be generalized to any functor $I : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ cocomplete and deduce that any presheaf $P \in \hat{\mathcal{C}}$ is canonically a colimit of representables:

$$P \quad = \quad \operatorname{colim}(\mathrm{El}(P) \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{Y} \hat{\mathcal{C}})$$

   We admit the following result: given a adjunction, the right adjoint is full and faithful if and only if the counit is an isomorphism.

8. Show that $\hat{\mathcal{C}}$ is the free cocompletion of $\mathcal{C}$: given a functor $F : \mathcal{C} \to \mathcal{D}$, there exists a unique cocontinuous functor $G : \hat{\mathcal{C}} \to \mathcal{D}$ such that $G \circ Y = F$.

9. Define the geometric realization of an n-graph.