### Travaux Dirigés

### An equivalent formulation of adjunctions Application to cartesian and to cartesian closed categories

 $\lambda$ -calculs et catégories (14 novembre 2016)

# 1 An equivalent formulation of adjunctions

§1. Suppose given a functor

$$R : \mathscr{B} \longrightarrow \mathscr{A}$$

between two categories  $\mathscr{A}$  and  $\mathscr{B}$ . Show that every map

$$\eta : A \longrightarrow R(LA)$$

from an object A of the category  $\mathscr A$  into an object noted LA of the category  $\mathscr B$  induces a family of functions

$$\varphi_B : \mathscr{B}(LA, B) \longrightarrow \mathscr{A}(A, RB)$$

parametrized by the objects B of the category  $\mathcal{B}$ .

§2. Show that the family  $\varphi_B$  is natural in B in the sense that it defines a natural transformation

$$\varphi : \mathscr{B}(LA, -) \Rightarrow \mathscr{A}(A, R-)$$

between the set-valued functors

$$\mathscr{B}(LA, -) = B \mapsto \mathscr{B}(LA, B)$$
  $\mathscr{A}(A, R-) = B \mapsto \mathscr{A}(A, RB)$ 

from the category  $\mathcal{B}$  to the category Set of sets and functions.

§3. One says that a pair  $(LA,\eta)$  consisting of an object LA of the category  ${\mathscr B}$  and of a map

$$\eta : A \longrightarrow R(LA)$$

represents the set-valued functor

$$\mathscr{A}(A,R-)$$
 :  $\mathscr{B}$   $\longrightarrow$  **Set** (1)

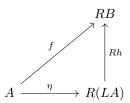
when every function  $\varphi_B$  defined in §1 is a bijection. Show that  $(LA, \eta)$  represents the set-valued functor  $\mathscr{A}(A, R-)$  precisely when the following property holds: for every object B and for every map

$$f : A \longrightarrow RB$$

there exists a unique map

$$h : LA \longrightarrow B$$

such that the diagram below commutes:



§4. We suppose from now on that every object A of the category  $\mathscr{A}$ , there exists a pair  $(LA,\eta_A)$  which represents the set-valued functor  $\mathscr{A}(A,R-)$ . For every map  $f:A_1\to A_2$  of the category  $\mathscr{A}$ , construct a map

$$Lf : LA_1 \longrightarrow LA_2$$

of the category  $\mathcal B$  such that the diagram below commutes:

$$A_{2} \xrightarrow{\eta_{A_{2}}} RLA_{2}$$

$$\uparrow \qquad \qquad \uparrow_{RLf}$$

$$A_{1} \xrightarrow{\eta_{A_{1}}} RLA_{1}$$

§5. Use the construction in §4. to define a functor

$$L : \mathscr{A} \longrightarrow \mathscr{B}$$

and a family of bijections

$$\varphi_{A,B}$$
 :  $\mathscr{B}(LA,B)$   $\cong$   $\mathscr{A}(A,RB)$ 

and show that this family  $\varphi$  is natural in A and B.

- §6. Conclude that given a functor  $R: \mathcal{B} \to \mathcal{A}$ , the existence of a pair  $(LA, \eta_A)$  representing the set-valued functor  $\mathcal{A}(A, R-)$  for every object A of the category  $\mathcal{A}$  implies the existence of a left adjoint functor  $L: \mathcal{A} \to \mathcal{B}$ .
- §7. Conversely, show that whenever we have a pair of adjoint functors

$$L:\mathscr{A} \xleftarrow{L} \mathscr{B}:R$$

every object A of the category  $\mathscr A$  comes equipped with a pair  $(LA,\eta_A)$  which represents the set-valued functor

$$\mathscr{A}(A,R-) = B \mapsto \mathscr{A}(A,RB) : \mathscr{B} \longrightarrow \mathbf{Set}.$$

§8. Apply the results of §6 to establish that the forgetful functor  $R:\mathbf{Mon}\to\mathbf{Set}$  from the category  $\mathscr{B}=\mathbf{Mon}$  of monoids and homomorphisms to the category  $\mathscr{A}=\mathbf{Set}$  of sets and functions has the free monoid functor

$$L = A \mapsto A^*$$
 : Set  $\to$  Mon

as left adjoint.

# Application to cartesian closed categories

§1. Show that every adjoint pair

$$L: \mathscr{A} \rightleftarrows \mathscr{B}: R$$

where L is left adjoint to R induces an adjoint pair

$$R^{op}: \mathscr{B}^{op} \longleftarrow \mathscr{A}^{op}: L^{op}$$

where the functor  $L^{op}$  is right adjoint to  $R^{op}$ .

§2. From this and exercise 1, deduce that a functor  $L: \mathscr{A} \to \mathscr{B}$  has a right adjoint precisely when for every object B of the category  $\mathscr C$  there exists a pair  $(RB, \varepsilon_B)$  consisting of an object RB of the category  $\mathscr A$  and of a map

$$\varepsilon_B$$
 :  $L(RB)$   $\longrightarrow$   $B$ 

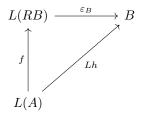
such that the following property holds: for every object A of the category  $\mathcal A$  and for every map

$$f: LA \longrightarrow B$$

there exists a unique map

$$h : A \longrightarrow RB$$

such that the diagram below commutes:



Terminology: one says in that case that the pair  $(RB, \varepsilon_B)$  represents the functor

$$\mathscr{B}(L-,B) = A \mapsto \mathscr{B}(LA,B) : \mathscr{A}^{op} \longrightarrow \mathbf{Set}.$$

§3. Aply this alternative formulation of adjunctions to the functor

$$L = B \mapsto A \times B : \mathscr{C} \longrightarrow \mathscr{C}$$

associated to an object A of a cartesian category  $\mathscr C$  with

- the object

and show that one recovers in this way the equivalence between the two formulations of cartesian closed category given in the course.

# 3 Application to cartesian categories

As we explained during the course, the category  $\mathbb{1}$  with one object \* and one map (=the identity map) is terminal in the category Cat. This means that for every category  $\mathscr{C}$ , there exists a unique functor

$$! : \mathscr{C} \longrightarrow \mathbb{1}. \tag{2}$$

At the same time, every object A of the category  $\mathscr C$  gives rise to a functor, also noted

$$A : \mathbb{1} \longrightarrow \mathscr{C}$$
 (3)

which transports the unique object \* of the category  $\mathbb{1}$  to the object A.

- §1. Show that an object A is terminal in the category  $\mathscr{C}$  if and only if the associated functor (3) is right adjoint to the canonical functor (2).
- §2. Show that an object A is initial in the category  $\mathscr{C}$  if and only if the associated functor (3) is left adjoint to the canonical functor (2).
- §3. Show that the operation  $A \mapsto (A, A)$  which transports every object A of the category  $\mathscr C$  to the object (A, A) of the category  $\mathscr C \times \mathscr C$  defines a functor

$$\Delta$$
 :  $\mathscr{C}$   $\longrightarrow$   $\mathscr{C} \times \mathscr{C}$ .

This functor  $\Delta$  is called the diagonal functor of the category  $\mathscr{C}$ .

§4. Suppose given a pair of objects A, B in a category  $\mathscr{C}$ . Show that a triple  $(A \times B, \pi_1, \pi_2)$  consisting of an object  $A \times B$  and of two maps

$$\pi_1: A \times B \to A$$
  $\pi_2: A \times B \to B$ 

defines a cartesian product of A and B precisely when the pair  $(A \times B, \pi)$  consisting of the object  $A \times B$  and of the map in the category  $\mathscr{C}^2 = \mathscr{C} \times \mathscr{C}$ 

$$\pi = (\pi_1, \pi_2) : \Delta(A \times B) \longrightarrow (A, B)$$

represents the functor

$$\mathscr{C}^2(\Delta -, (A, B))$$
 :  $\mathscr{C}^{op} \longrightarrow \mathbf{Set}$ .

Here, we write 2 for the category with two objects a, b and two maps (= identity maps for a and b).

§5. From this and the exercise 2, deduce that a category  $\mathscr C$  is cartesian precisely when the two canonical functors

$$!\,:\,\mathscr{C}\,\longrightarrow\,\mathbb{1}\qquad\qquad \Delta\,:\,\mathscr{C}\,\longrightarrow\,\mathscr{C}\times\mathscr{C}$$

have a right adjoint.