TD4 – λ-calculus & CCC

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1 Cartesian closed categories

A category $\mathcal{C}$ is *cartesian closed* when it has finite products and for every object A of $\mathcal{C}$ the functor $A \times - : \mathcal{C} \to \mathcal{C}$ admits a right adjoint, written $A \Rightarrow -$, i.e. there exists a bijection $\mathcal{C}(A \times B, C) \cong \mathcal{C}(B, A \Rightarrow C)$ natural in B and C.

1. Show that the category **Set** is cartesian closed.
2. Describe the unit and the counit of the adjunction defining the closure.
3. What are the naturality conditions of the unit and the counit in a CCC?
4. What are the triangular laws between the unit and the counit in a CCC?
5. Check that these laws are satisfied in the case of **Set**.
6. [Optional] Show that the category **Cat** is cartesian closed.

Here, we call *signature* a graph $(\mathcal{T}, \mathcal{F})$: the vertices $\mathcal{T}$ of the graph are thought of as generators for types and the edges $\mathcal{F}$ as generators for terms.

7. Construct a forgetful functor **CCC** $\to$ **Graph**.
8. Explain what a left adjoint to this functor should look like. What universal property would it satisfy?

2 The simply typed λ-calculus

We suppose fixed a signature $\Sigma = (\mathcal{T}, \mathcal{F})$. The syntax of λ-terms with products is:

$$M \quad ::= \quad f \in \mathcal{F} \quad | \quad x \quad | \quad \lambda x.M \quad | \quad MM \quad | \quad \langle M, M \rangle \quad | \quad \pi_1 \quad | \quad \pi_2 \quad | \quad \langle \rangle$$

and the syntax of types:

$$A \quad ::= \quad A \in \mathcal{T} \quad | \quad A \Rightarrow A \quad | \quad 1 \quad | \quad A \times A$$

The rules are shown on next page.

1. Define the substitution operation $M[N/x]$ by induction on the structure of the term M. Show that typing is preserved under β-reduction.
2. Let $\mathcal{C}$ be a cartesian closed category. Define an interpretation of every sequent
$$x_1 : A_1, \ldots, x_n : A_n \vdash M : A$$
as a morphism
$$[\![M]\!] : [\![A_1]\!] \times \ldots \times [\![A_n]\!] \to [\![A]\!]$$
of $\mathcal{C}$. We suppose fixed the interpretation of the signature.
3. *Soundness.* Show that the equations of λ-calculus are valid in the interpretation:
$$\Gamma \vdash M = N : A \qquad \text{implies} \qquad [\![M]\!] = [\![N]\!]$$
4. Conversely, explain how to build a category Λ containing the signature as morphisms, whose objects are types and morphisms are simply typed λ-terms. Check that this category is cartesian closed.
5. *Completeness.* Show that $[\![M]\!] = [\![N]\!]$ in every CCC implies $\Gamma \vdash M = N : A$.
6. Show that the category Λ is the free CCC on the signature.

The simply typed λ-calculus

We suppose fixed a signature $\Sigma = (\mathcal{T}, \mathcal{F})$. A context Γ is a list of pairs $x_i : A_i$.

- terms:
$$M \quad ::= \quad f \in \mathcal{F} \quad | \quad x \quad | \quad \lambda x.M \quad | \quad MM \quad | \quad \langle\rangle \quad | \quad \langle M, M \rangle$$
- types:
$$A \quad ::= \quad A \in \mathcal{T} \quad | \quad A \Rightarrow A \quad | \quad 1 \quad | \quad A \times A$$
- typing rules:
  - basic rules:
$$\frac{}{x : A \vdash x : A}(\mathrm{id}) \qquad \frac{f : A \to B \in \mathcal{F}}{\Gamma \vdash f : A \Rightarrow B}$$
  - structural rules:
$$\frac{\Gamma \vdash M : A \qquad x \notin \mathrm{FV}(\Gamma)}{\Gamma, x : B \vdash M : A}(\text{weakening})$$
$$\frac{\Gamma, x : A, y : A \vdash M : A}{\Gamma, x : A \vdash M[x/y] : A}(\text{contraction})$$
$$\frac{\Gamma, x : A, y : B, \Delta \vdash M : C}{\Gamma, y : B, x : A, \Delta \vdash M : C}(\text{exchange})$$
  - pure λ-calculus:
$$\frac{x : A, \Gamma \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}(\text{abs}) \qquad \frac{\Gamma \vdash N : A \qquad \Delta \vdash M : A \Rightarrow B}{\Gamma, \Delta \vdash MN : B}(\text{app})$$
  - products:
$$\frac{}{\Gamma \vdash \langle\rangle : 1}(\text{unit}) \qquad \frac{\Gamma \vdash M : A \qquad \Delta \vdash N : B}{\Gamma, \Delta \vdash \langle M, N \rangle : A \times B}(\text{pair})$$
$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \pi_1 M : A}(\text{proj}_1) \qquad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \pi_2 M : B}(\text{proj}_2)$$
- equations:
  - equality:
$$\frac{\Gamma \vdash M : A}{\Gamma \vdash M = M : A}(\text{refl}) \qquad \frac{\Gamma \vdash M = N : A}{\Gamma \vdash N = M : A}(\text{sym})$$
$$\frac{\Gamma \vdash M = N : A \qquad \Gamma \vdash N = P : A}{\Gamma \vdash M = P : A}(\text{trans})$$
$$\frac{x : A, \Gamma \vdash M : B \qquad \Gamma \vdash N = N' : A}{\Gamma \vdash M[N/x] = M[N'/x] : B}(\text{cong})$$
  - pure λ-calculus:
$$\frac{\Gamma \vdash N : A \qquad x : A, \Delta \vdash M : B}{\Gamma, \Delta \vdash (\lambda x.M)N = M[N/x] : B}(\beta) \qquad \frac{\Gamma \vdash M : A \Rightarrow B}{\Gamma \vdash M = \lambda x.Mx : A \Rightarrow B}(\eta)$$
  - products:
$$\frac{\Gamma \vdash M : 1}{\Gamma \vdash M = \langle\rangle : 1} \qquad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash M = \langle \pi_1 M, \pi_2 M \rangle : A \times B}$$
$$\frac{\Gamma \vdash M : A \qquad \Delta \vdash N : B}{\Gamma, \Delta \vdash \pi_1 \langle M, N \rangle = M : A} \qquad \frac{\Gamma \vdash M : A \qquad \Delta \vdash N : B}{\Gamma, \Delta \vdash \pi_2 \langle M, N \rangle = N : B}$$