

TD4 – Algebras

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1 Algebras for an endofunctor

An *algebra* for an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$ is a pair (A, f) where A is an object of \mathcal{C} and $f : FA \rightarrow A$ a morphism of \mathcal{C} . A morphism $h : (A, f) \rightarrow (B, g)$ between two such algebras consists of a morphism $h : A \rightarrow B$ such that

$$\begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ f \downarrow & & \downarrow g \\ A & \xrightarrow{h} & B \end{array}$$

In the following, we mostly consider algebras in **Set**.

1. Define inductively the functions `length : 'a list -> int` (giving the length of a list), `map : ('a -> 'b) -> 'a list -> 'b list` (applying a function to all elements of a list), `double : 'a list -> 'a list` (which duplicates every successive element, for instance `double [1;2;3] = [1;1;2;2;3;3]`).
2. We suppose given a type `'a ilist` of infinite lists with elements of type `'a`. Define coinductively `odd : 'a ilist -> 'a ilist` (keeping elements of a list at odd positions), `merge : 'a ilist -> 'a ilist -> 'a ilist` (taking alternatively elements from one of two lists).
3. Show that $[0, S] : 1 + \mathbb{N} \rightarrow \mathbb{N}$ is an initial algebra for the endofunctor $T(X) = 1 + X$ of **Set**.
4. Use this fact to define the function $f : \mathbb{N} \rightarrow \mathbb{Q}$ such that $f(n) = 2^{-n}$.
5. Show that two initial algebras of an endofunctor are isomorphic (via morphisms of algebras).
6. Show that an initial algebra $f : FA \rightarrow A$ of an endofunctor F is an isomorphism.
7. Show that the set $A^* = \bigsqcup_{n \in \mathbb{N}} A^n$, which can be seen as the set of lists of elements of A , is an initial algebra for $T(X) = 1 + A \times X$.
8. Use this fact to define the length function $\ell : A^* \rightarrow \mathbb{N}$ and the double function $d : A^* \rightarrow A^*$. Show that $\ell \circ d(l) = 2\ell(l)$ for every $l \in A^*$.
9. Explain briefly how we could interpret simple inductive types of OCaml by using initial algebras.
10. What is the initial algebra for $T(X) = 1 + X \times X$? For $T(X) = X^*$?

2 Coalgebras for an endofunctor

A *coalgebra* for $F : \mathcal{C} \rightarrow \mathcal{C}$ is a pair (A, f) with $f : A \rightarrow FA$. Morphisms are defined similarly as previously.

1. Show that the set $A^{\mathbb{N}}$ of *streams* is a final coalgebra for the endofunctor $T(X) = A \times X$.
2. Use this to define, given $a \in A$, the constant stream equal to a . Define the function $\mathbb{N} \rightarrow \mathbb{N}^{\mathbb{N}}$ which to n associates the stream $(n, n + 1, n + 2, \dots)$. Define the function $A^{\mathbb{N}} \times A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ which merges two streams. Define the functions $A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ keeping even and odd elements.
3. Show that final coalgebras are unique up to isomorphism and are isomorphisms.
4. Show that $\text{merge}(\text{even}(l), \text{odd}(l)) = l$ for every $l \in A^{\mathbb{N}}$.
5. A *bisimulation* on $A^{\mathbb{N}}$ is a relation $R \subseteq A^{\mathbb{N}} \times A^{\mathbb{N}}$ such that $R(x :: l, x' :: l')$ implies $x = x'$ and $R(l, l')$. The *coinductive proof principle* says that if $R(l, l')$ for some bisimulation R then $l = l'$. Assuming this principle, show again the result of previous question.
6. Show the coinductive proof principle.
7. What is the final coalgebra of $T(X) = 1 + A \times X$? of $T(X) = 1 + X$?

3 Algebras for a monad

An *algebra* for a monad (T, μ, η) is a pair (A, f) with $f : TA \rightarrow A$ such that

$$\begin{array}{ccc}
 TTA & \xrightarrow{Tf} & TA \\
 \mu_A \downarrow & & \downarrow f \\
 TA & \xrightarrow{f} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\eta_A} & TA \\
 \text{id}_A \searrow & & \downarrow f \\
 & & A
 \end{array}$$

A morphism of T -algebras is defined as a morphism between algebras for a functor. Given a category \mathcal{C} and T a monad on \mathcal{C} , we write \mathcal{C}^T for the category of T -algebras.

1. Consider the monad T on **Set** induced by the adjunction whose right adjoint is the forgetful functor $U : \mathbf{Mon} \rightarrow \mathbf{Set}$ where **Mon** is the category of monoids and morphisms of monoids. Describe this monad. What is an algebra for this monad?
2. Show that the forgetful functor $\mathcal{C}^T \rightarrow \mathcal{C}$ has a left adjoint.
3. Fix a monad T on \mathcal{C} and consider the category whose objects are triples (\mathcal{D}, F, G) with $F : \mathcal{C} \rightarrow \mathcal{D}$ left adjoint to $G : \mathcal{D} \rightarrow \mathcal{C}$, and whose morphisms $H : (\mathcal{D}, F, G) \rightarrow (\mathcal{D}', F', G')$ are functors $H : \mathcal{D} \rightarrow \mathcal{D}'$ such that $H \circ F = F'$ and $G' \circ H = G$. Show that the above adjunction is a terminal object in this category.
4. Show that the Kleisli category \mathcal{C}_T is an initial object in this category.